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Fuzzy Sets and Systems 157 (2006) 1674 – 1686
FUZZY sets and systems
www.elsevier.com/locate/fss

Bayesian reliability analysis for fuzzy lifetime data

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Received 15 July 2003; received in revised form 29 July 2005; accepted 16 November 2005
Available online 6 December 2005

Abstract

Lifetime data are important in reliability analysis. Classical reliability estimation is based on precise lifetime data. It is usually assumed that observed lifetime data are precise real numbers. However, some collected lifetime data might be imprecise and are represented in the form of fuzzy numbers. Thus, it is necessary to generalize classical statistical estimation methods for real numbers to fuzzy numbers. Bayesian methods have proved to be very useful when the sample size is small. There is little study on Bayesian reliability estimation based on fuzzy lifetime data. Most of the reported works in this area is limited to single parameter lifetime distributions. In this paper, we propose a new method to determine the membership function of the estimates of the parameters and the reliability function of multi-parameter lifetime distributions. An artificial neural network is used to approximate the calculation process of parameter estimation and reliability prediction. The genetic algorithm is used to find the boundary values of the membership function of the estimate of interest at any cut level. This method can be used to determine the membership functions of the Bayesian estimates of multi-parameter distributions. The effectiveness of the proposed method is illustrated with normal and Weibull distributions.

Keywords: Bayesian estimation; Fuzzy lifetime; Membership function; Parameter estimation; Fuzzy reliability; Neural network; Genetic algorithm

1. Introduction

Estimation of the reliability of a product requires its lifetime data. Many reliability analysis methods are based on the availability of a large amount of lifetime data. In these methods, the parameters of the lifetime distribution are assumed to be constant but unknown and sample statistics are used as the estimators of these parameters. This requires a relatively large amount of lifetime data. Such methods have been implemented in a variety of fields and have solved many practical problems successfully. However, with the progress of modern industrial technologies, product development cycle has become shorter and shorter while the lifetime of products has become longer and longer. As a result, it is time consuming, expensive, and sometimes impossible to obtain many observations [24,30,37]. In many engineering applications, there may be very few available data points, sometimes only one or two observations. In these cases it is impossible to estimate lifetime distribution parameters with conventional reliability analysis methods. The Bayesian approach has been developed to handle such difficulties. In this approach, the parameters

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doi:10.1016/j.fss.2005.11.009

of the lifetime distributions are considered to be random variables themselves. This enables an engineer to systematically combine subjective judgment based on intuition, experience, or indirect information with observed data to obtain a balanced estimate and to update the estimate as more information and data become available. This method takes advantage of all available prior information (combining experience, subjective judgment, and available lifetime data). Thus, it can be used even when there are only a few lifetime data points. Many research results on the classic Bayesian reliability analysis have been reported and many practical problems have been solved successfully with this method [1,28].

In engineering applications, randomness is not the only aspect of uncertainty. In many fields of application, owing to the fuzziness of environment and the negligence of observers, it is sometimes impossible to obtain exact observations of lifetime. The obtained lifetime data may be "polluted" and imprecise most of the time. In addition, restricted by human and other resources in experiment, especially for new equipments, exceptionally long-life equipments, and non-mass-production products, for which there is no comparative reliability information available, more often than not, the lifetime is based upon subjective evaluation or rough estimate. That leads to the fuzziness of lifetime data. Moreover, lifetime of a system may be difficult to measure due to the complexity of the system. This is frequently the case for a complex computer system with many components such as a pipelined computer, an array computer, and a multi-processor computer, in which not all of the components are required to support the system for all the time. This implies that different components in the system experience different lifetimes. Since each component in a system makes contribution to the system reliability, one may argue how to determine the system lifetime and then view it as a fuzzy one due to the complexity of the system [23,5]. For example, "The lifetime of a bearing is around 8.17×10^6 revolutions" and "The lifetime of some shaft be between 1500 and 2000 h, but near to 2000 h" etc. are fuzzy concepts relating to lifetime. Thus, one has to represent lifetime data with fuzzy measures in reliability analysis. Among others, we mention Tanaka et al. [37], Singer [34], Onisawa [29], Cappelle and Kerre [9], Cremona and Gao [13], Utkin and Gurov [39], and Cai et al. [4,6–8], who have all attempted to define reliability in terms of fuzzy set theory. Applications of fuzzy set theory in reliability include fault tree analysis, failure modes and effects analysis, optimization of probist reliability, life testing, structural reliability, software reliability, human reliability, as well as profust, posbist, and posfust reliability theories [4]. Many papers on generalization of classical statistical methods to analysis of fuzzy data have been published [16,17,19,22,26,36,42]. The problem of hypothesis testing with fuzzy data was investigated by Grzegorzewski and Hryniewicz [20]. Early studies of Bayesian decision analysis for imprecise data were done by Casals et al. [10,11], and Gil [18]. In many of these papers the authors used the notion of fuzzy data by Zadeh [43] and Tanaka et al. [38] to describe imprecise observations.

Viertl [40], Viertl and Gurker [41], Hryniewicz [21], and Chou and Yuan [12] also used fuzzy set theory in Bayesian reliability analysis. However, their research results are mainly on single parameter distributions such as the exponential, binomial, and Poisson distributions. To our best knowledge, there are no reports on estimating the parameters and the reliability function of multi-parameter distributions using the fuzzy Bayesian approach. The main reason is that the explicit membership functions of the parameters and the reliability function of a multi-parameter distribution are difficult to determine. In this paper, we propose a numerical method to determine the membership functions of the parameters and the reliability function of multi-parameter distributions using the fuzzy Bayesian approach. The proposed method determines the membership functions using neural networks and the genetic algorithm. The normal distribution and the Weibull distribution are used to illustrate the proposed method for estimating the parameters and the reliability function of these two distributions.

2. The Bayesian approach

For a continuous random variable X, $X \sim f(\cdot|\theta)$, $\theta \in \Theta$, with continuous parameter space Θ, a priori density $\pi(\cdot)$ of the parameter θ, and observation space M_x of X, the Bayes' theorem for exact data observations, $x_1, x_2, \ldots, x_n$, is

$$\pi(\theta|x_1, x_2, \ldots, x_n) = \frac{l(\theta; x_1, x_2, \ldots, x_n)\pi(\theta)}{\int_\Theta l(\theta; x_1, x_2, \ldots, x_n)\pi(\theta)\,\mathrm{d}\theta}, \quad \forall \theta \in \Theta, \tag{1}$$

where $l(\theta; x_1, x_2, \ldots, x_n)$ is the likelihood function of the observations with given parameter θ. When the lifetime data set is complete with exact observations, the likelihood function is given by

$$l(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\theta), \quad \forall\theta \in \Theta. \tag{2}$$

Mainly because of its use of subjective prior beliefs, the approach to statistical inference based on the Bayes' theorem has been controversial within the field of statistics for many years. It is important to note that the prior distribution cannot be specified arbitrarily and it should be based on prior information. Several effective methods have been reported to determine priori-distributions, such as non-informative priors, conjugate priors, Jeffreys priors, empirical Bayesian priors, maximum entropy priors, bootstrap priors, and random weight priors [3,25,32,35,44].

3. Bayesian parameter estimation for multi-parameter lifetime distributions

In this section, we summarize the research results reported in the literature on Bayesian parameter estimation and Bayesian reliability prediction using exact lifetime data. In Section 5, we will report our proposed method for Bayesian parameter estimation and Bayesian reliability prediction using fuzzy lifetime data.

Given the lifetime probability density function (PDF) $f(x|\theta)$, the sample data $D = (x_1, x_2, \ldots, x_n)$, the parameter space Θ, and the priori-distribution $\pi(\theta)$, we can determine the posteriori-distribution of parameter θ. According to the Bayes' theorem,

$$\pi(\theta|D) = \pi(\theta|x_1, x_2, \ldots, x_n) \propto \pi(\theta)l(\theta; x_1, x_2, \ldots, x_n). \tag{3}$$

After the posteriori-distribution of the parameter, $\pi(\theta|D)$, is determined, many reliability indices can be estimated. There are two important indices in Bayesian reliability analysis. One is the mathematical expectation of the parameter of the lifetime distribution:

$$\hat{\theta} = \int_0^{+\infty} \pi(\theta|D)\theta \, \mathrm{d}\theta. \tag{4}$$

The other is the updated reliability function

$$R(t|D) = \int_t^{\infty} \int_{\Theta} f(x|\theta)\pi(\theta|D) \, \mathrm{d}\theta \, \mathrm{d}x. \tag{5}$$

In the following subsections, we summarize the detailed equations for updating $\hat{\theta}$ and $R(t|D)$ for two specific multi-parameter distributions, namely, the normal and the Weibull distributions.

3.1. The normal distribution

The PDF of the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{6}$$

According to Jeffrey's method [3,25], the joint priori-distribution of the two parameters of the normal distribution is

$$\pi(\mu, \sigma) \propto \frac{1}{\sigma^2}. \tag{7}$$

The likelihood function of the observed lifetimes following the normal distribution is

$$l(x_1, x_2, \ldots, x_n; \mu, \sigma) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right) \tag{8}$$

With Eq. (3), the joint posterior distribution of the parameters of the normal distribution is

$$\pi(\mu,\sigma|D)=\frac{1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}. \tag{9}$$

The marginal posteriori-distributions of the parameters μ, σ are, respectively,

$$\pi(\mu|D)=\int_0^{+\infty}\pi(\mu,\sigma|D)\,\mathrm{d}\sigma=\frac{\int_0^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\sigma}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}, \tag{10}$$

$$\pi(\sigma|D)=\int_0^{+\infty}\pi(\mu,\sigma|D)\,\mathrm{d}\mu=\frac{\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}. \tag{11}$$

The expected values of the updated parameters μ and σ are, respectively,

$$\hat{\mu}=\int_{-\infty}^{+\infty}\pi(\mu|D)\mu\,\mathrm{d}\mu=\frac{\int_{-\infty}^{+\infty}\mu\int_0^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\sigma\,\mathrm{d}\mu}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}, \tag{12}$$

$$\hat{\sigma}=\int_0^{+\infty}\pi(\sigma|D)\sigma\,\mathrm{d}\sigma=\frac{\int_0^{+\infty}\sigma\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}. \tag{13}$$

The updated reliability function of the normal distribution becomes

$$R(t|D)=\frac{\int_t^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sqrt{2\pi}\sigma\exp(-(x-\mu)^2/2\sigma^2)1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma\,\mathrm{d}x}{\int_0^{+\infty}\int_{-\infty}^{+\infty}1/\sigma^{n+2}\exp(-1/2\sigma^2\sum_{i=1}^n(x_i-\mu)^2)\,\mathrm{d}\mu\,\mathrm{d}\sigma}. \tag{14}$$

3.2. The Weibull distribution

The PDF of the Weibull distribution with two parameters m and η is

$$f(x|m,\eta)=\frac{m}{\eta}\left(\frac{x}{\eta}\right)^{m-1}\exp\left(-\left(\frac{x}{\eta}\right)^m\right),\quad 0<m,\eta<+\infty. \tag{15}$$

According to the non-informative method [13], the priori-distribution of the parameters of the Weibull distribution is

$$\pi(m,\eta)\propto\frac{1}{m\eta}. \tag{16}$$

The likelihood function of the observations from a Weibull distribution is

$$l(m,\eta;x_1,x_2,\ldots,x_n)=\left(\frac{1}{\eta}\right)^{mn}m^n\prod_{i=1}^n x_i^{m-1}\exp\left(-\left(\frac{1}{\eta}\right)^m x_i^m\right). \tag{17}$$

The joint posteriori-distribution of the parameters of the Weibull distribution is

$$\pi(m,\eta|x_1,x_2,\ldots,x_n)=\frac{(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)}{\int_0^{+\infty}\int_0^{+\infty}(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}. \tag{18}$$

The marginal posteriori-distributions of the parameters m and η are, respectively,

$$\pi(m|D)=\int_0^{+\infty}\pi(m,\eta)\,\mathrm{d}\eta=\frac{\int_0^{+\infty}(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)\,\mathrm{d}\eta}{\int_0^{+\infty}\int_0^{+\infty}(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}, \tag{19}$$

$$\pi(\eta|D)=\int_0^{+\infty}\pi(m,\eta)\,\mathrm{d}m=\frac{\int_0^{+\infty}(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m}{\int_0^{+\infty}\int_0^{+\infty}(1/\eta)^{mn+1}m^{n-1}\prod_{i=1}^n x_i^{m-1}\exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}. \tag{20}$$

The expected values of the parameters m and η are, respectively,

$$\hat{m} = \int_0^{+\infty} \pi(m|D)m\,\mathrm{d}m = \frac{\int_0^{+\infty} m \int_0^{+\infty} (1/\eta)^{mn+1} m^{n-1} \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m x_i^m)\,\mathrm{d}\eta\,\mathrm{d}m}{\int_0^{+\infty} \int_0^{+\infty} (1/\eta)^{mn+1} m^{n-1} \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}, \tag{21}$$

$$\hat{\eta} = \int_0^{+\infty} \pi(\eta|D)\eta\,\mathrm{d}\eta = \frac{\int_0^{+\infty} \eta \int_0^{+\infty} (1/\eta)^{mn+1} m^{n-1} \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}{\int_0^{+\infty} \int_0^{+\infty} (1/\eta)^{mn+1} m^{n-1} \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m x_i^m)\,\mathrm{d}m\,\mathrm{d}\eta}. \tag{22}$$

The updated reliability function of the Weibull distribution then becomes

$$R(t|D) = \frac{\int_t^{+\infty} \int_0^{+\infty} \int_0^{+\infty} x^{m-1} (1/\eta)^{mn+m+1} m^n \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m (x_i^m + x^m))\,\mathrm{d}\eta\,\mathrm{d}m\,\mathrm{d}x}{\int_0^{+\infty} \int_0^{+\infty} (1/\eta)^{mn+1} m^{n-1} \prod_{i=1}^n x_i^{m-1} \exp(-(1/\eta)^m x_i^m)\,\mathrm{d}\eta\,\mathrm{d}m}. \tag{23}$$

As illustrated in this section, though the lifetime data are non-fuzzy, the equations for obtaining updated estimates of the parameters and the reliability function of multi-parameter distributions are quite complicated. They involve multiple integrals. Numerical methods may have to be used in evaluating these equations.

4. Fuzzy lifetime data

In order to model fuzzy observed lifetimes, a generalization of real numbers is necessary. A lifetime observation will be represented by a fuzzy number $\tilde{x}$. A fuzzy number is a subset, denoted by $\tilde{x}$, of the set of real numbers (denoted by $\Re$) and is characterized by the so called membership function $\mu_{\tilde{x}}(\cdot)$. Fuzzy numbers satisfy the following constraints [15]:

(1) $\mu_{\tilde{x}} : \Re \to [1, 0]$ is Borel-measurable;
(2) $\exists x_0 \in \Re : \mu_{\tilde{x}}(x_0) = 1$;
(3) The so-called λ-cuts $(0 < \lambda \leqslant 1)$, defined as $B_\lambda(\tilde{x}) = \{x \in \Re : \mu_{\tilde{x}}(x) \geqslant \lambda\}$, are all closed intervals, i.e., $B_\lambda(\tilde{x}) = [a_\lambda, b_\lambda], \forall \lambda \in (0, 1]$. This means that the membership function has to be a unimodal function with a maximum. For example, a strictly concave function is a unimodal function with a maximum.

With the definition of a fuzzy number given above, an exact (non-fuzzy) number can be treated as a special case of a fuzzy number. For a non-fuzzy real observation $x_0 \in \Re$, its corresponding membership function is $\mu_{x_0}(x_0) = 1$. For a non-fuzzy interval observation $[c, d]$, the corresponding membership function is $\mu_{[c,d]}(x) = 1$ for $c \leqslant x \leqslant d$. For a fuzzy lifetime observation, the L-R type membership functions (the triangular and trapezoidal membership functions are special cases of the L-R type membership functions) are commonly used.

Given fuzzy lifetimes $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$, we have their corresponding membership functions $\mu_{\tilde{x}_1}(\cdot), \mu_{\tilde{x}_2}(\cdot), \ldots, \mu_{\tilde{x}_n}(\cdot)$. We call $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ a fuzzy sample or a fuzzy vector where each $\tilde{x}_i$ for $i = 1, 2, \ldots, n$ can be considered to be a realization of the fuzzy number $\tilde{x}$. If the definition domain of $\tilde{x}$ is M, then the definition domain of $\tilde{\mathbf{x}}$ is M^n.

To the best of our knowledge, there are no reported studies for estimation of the parameters and the reliability function of multi-parameter distributions when the lifetime data are fuzzy. In the following section, we propose a new approach for this purpose.

5. Bayesian parameter estimation for multi-parameter distributions using fuzzy data

Given fuzzy lifetime data points, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$, with their corresponding membership functions, $\mu_{\tilde{x}_1}(\cdot), \mu_{\tilde{x}_2}(\cdot), \ldots, \mu_{\tilde{x}_n}(\cdot)$, the Bayesian point estimates of the parameters and the reliability function are fuzzy numbers whose fuzziness depends on the fuzziness of the n observed lifetime data points.

Consider a general fuzzy function $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ wherein the membership functions of the n fuzzy arguments are known. We need to find the membership function of $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$. When the explicit membership function of $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is difficult to determine with the extension principle of fuzzy set theory directly, we propose the following approach to generate the membership function numerically and guarantee that the membership function of $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is unimodal with a maximum.

Step 1: Let λ change from 0 to 1 with an increment size satisfying the precision requirement.

Step 2: For each fixed value of λ selected in Step 1, find the maximum value of the function $f(x_1, x_2, \ldots, x_n)$ such that $\mu_{\tilde{x}_i}(x) \geqslant \lambda$ for all $i = 1, 2, \ldots, n$. Denote this maximum value by $f_R(\lambda)$.

Step 3: For the same fixed value λ used in Step 1, find the minimum value of the function $f(x_1, x_2, \ldots, x_n)$ such that $\mu_{\tilde{x}_i}(x) \geqslant \lambda$ for all $i = 1, 2, \ldots, n$. Denote this minimum value by $f_L(\lambda)$.

Because of the unimodality requirement of a membership function, we will have

$$\mu_{\tilde{f}}(f) \geqslant \lambda \quad \text{for} \quad f_L(\lambda) \leqslant f \leqslant f_R(\lambda),$$

where $f_L(\lambda)$ and $f_R(\lambda)$ are, respectively, the lower and upper boundary values of the function $\tilde{f}$ at cut level λ.

Through the iterative procedure given above, we are able to obtain the membership function of the function $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ numerically. Reducing the step size of λ increases the accuracy of the membership function of $\tilde{f}$ at the expense of increased computation time.

From the equations given in Section 3, we can see that Bayesian parameter estimation and reliability prediction for non-fuzzy lifetime data following multi-parameter distributions require multiple integrals. Evaluation of such multiple integrals is time consuming. Now, we have to evaluate the membership functions of the parameters and the reliability function of such multi-parameter distributions using fuzzy lifetime data. It is obvious that determination of the membership functions of the estimated parameters and the reliability function requires repeated evaluations of such integrals. A more efficient method to evaluate these integrals is needed.

Artificial neural networks have been successfully used in many areas of science and engineering [27,31,33]. Cybenko [14] and Aourid and Do [2] show that a finite linear combination of sigmoidal functions used in neural networks can approximate any continuous function of n variables with support in the unit hypercube to any degree of accuracy. As a result, we propose to use a neural network to approximate $f(x_1, x_2, \ldots, x_n)$. A feedforward neural network with a single hidden layer is used. The backpropagation learning scheme is adopted.

To approximate the functions for point estimates of parameters, we need to approximate Eqs. (12), (13), (21), and (22). From these equations, we can see that input data for first and second neural network are $(x_1, x_2, \ldots, x_n)$ and the output data for first neural network are μ and σ or second neural network are m and η. Because the parameters may take real values, the transfer function of the output layer is chosen to be a linear function.

To find the Bayesian reliability function, we need to approximate Eqs. (14), and (23). We can see that input data of third neural network is $t(x_1, x_2, \ldots, x_n)$. The single output is the value of reliability R. Since the reliability value is between 0 and 1, we choose the transfer function of the output layer to be the log-sigmoidal function

$$\log \operatorname{sig}(n) = \frac{1}{1 + \exp(-n)}.$$

The transfer function used in the hidden layer is the tan-sigmoidal function

$$\operatorname{tansig}(n) = \frac{1 - \exp(1 - 2n)}{1 + \exp(1 - 2n)}.$$

Once the function $\tilde{f}$ has been approximated by a neural network, we still need to follow the proposed procedure to find the maximum and minimum values of $\tilde{f}$ subject to constraints for each selected λ value. This is a constrained optimization problem. Because the objective function is simulated by neural network, we cannot use the gradient projection method or the feasible-direction method to search for the optimal solution. In addition, it is difficult to find the global optimal solution by using direct search methods. As a result, we use genetic algorithms to solve the optimization problem in the process of determining the membership function of the function $\tilde{f}$.

To find the upper boundary value of the fuzzy reliability at cut level λ, we need to solve the following optimization problem:

$$\begin{aligned} &\text{Maximize} && R(t|x_1, x_2, \ldots, x_n) \\ &\text{Subject to:} && \mu_{\tilde{x}_i}(x) \geqslant \lambda, \quad i = 1, 2, \ldots, n, \\ & && 0 \leqslant \lambda \leqslant 1, \end{aligned} \tag{24}$$

where $\mu_{\tilde{x}_i}(x)$ is the membership function of lifetime data point $\tilde{x}_i$. The obtained maximum value of the reliability is denoted by $R_{\max}$. To find the lower boundary value of the fuzzy reliability at cut level λ, we need to solve the following optimization problem:

$$\begin{aligned} &\text{Minimize} && R(t|x_1, x_2, \ldots, x_n) \\ &\text{Subject to:} && \mu_{\tilde{x}_i}(x) \geqslant \lambda, \quad i = 1, 2, \cdots, n, \\ &&& 0 \leqslant \lambda \leqslant 1. \end{aligned} \tag{25}$$

The obtained minimum reliability value is denoted by $R_{\min}$. The membership function values of $\tilde{R}(t|\tilde{D})$ at $R_{\min}$ and $R_{\max}$ are both λ, that is,

$$\mu_{\tilde{R}(t|\tilde{D})}(R_{\max}) = \mu_{\tilde{R}(t|\tilde{D})}(R_{\min}) = \lambda. \tag{26}$$

To find the upper boundary value of the fuzzy estimate of the parameter θ at cut level λ, we need to solve the following optimization problem:

$$\begin{aligned} &\text{Maximize} && \hat{\theta}(x_1, x_2, \ldots, x_n) \\ &\text{Subject to:} && \mu_{\tilde{x}_i}(x) \geqslant \lambda, \quad i = 1, 2, \ldots, n, \\ &&& 0 \leqslant \lambda \leqslant 1. \end{aligned} \tag{27}$$

The obtained maximum value of θ is denoted by $\theta_{\max}$. To find the lower boundary value of the fuzzy estimate of the parameter θ at cut level λ, we need to solve the following optimization problem:

$$\begin{aligned} &\text{Minimize} && \hat{\theta}(x_1, x_2, \ldots, x_n) \\ &\text{Subject to:} && \mu_{\tilde{x}_i}(x) \geqslant \lambda, \quad i = 1, 2, \ldots, n, \\ &&& 0 \leqslant \lambda \leqslant 1. \end{aligned} \tag{28}$$

The obtained minimum value of θ is denoted by $\theta_{\min}$. The membership function values of $\tilde{\theta}$ at $\theta_{\min}$ and $\theta_{\max}$ are both λ, that is,

$$\mu_{\tilde{\hat{\theta}}}(\theta_{\max}) = \mu_{\tilde{\hat{\theta}}}(\theta_{\min}) = \lambda. \tag{29}$$

For the normal distribution, θ denotes μ or σ; for the Weibull distribution, θ denotes m or η.

Using of the genetic algorithms involves the following:

(1) *Representation*: A binary vector is used as a chromosome to represent the specific realizations of the n fuzzy lifetimes and the time instant. The length of a chromosome depends on the domain of the variables and the required precision. An initial population of solutions is created.
(2) *Fitness function*: The fitness function is used to measure the solutions in terms of their fitness. The objective function (for maximization problems) or the reciprocal of the objective function (for minimization problems) is taken as the fitness function.
(3) *Crossover*: The probability of crossover is selected based on the fitness of chromosomes. If the fitness is bigger, the chance of being selected is bigger. One-point crossover is used in our study.
(4) *Mutation*: The probability of mutation is initialized at the beginning. Mutation alters one or more genes with the probability equal to the probability of mutation. The method used for mutation is random point mutation. Mutation of a binary string is the process of changing one of the genes between 0 and 1.
(5) *Selection*: The expanded population is composed of chromosomes generated through crossover and mutation in addition to the original population chromosomes. The chromosomes with the higher fitness values are selected as the parent generation in the next iteration of the algorithm.
(6) *Stopping criterion*: The maximum number of iterations and the change in population fitness value are used in combination to determine when to stop the optimization process.

6. Case studies

In this section, we illustrate the proposed approach for Bayesian reliability analysis for fuzzy lifetime data using two multi-parameter distributions, namely, the normal and the Weibull distributions.

6.1. The normal distribution

The lifetime of a gear has often been modeled as a random variable following the normal distribution with PDF given in Eq. (6). Assume that we have five observed fuzzy lifetime data $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_5$, with membership functions given, respectively, by

$$\mu_{\tilde{x}_1}(x) = \begin{cases} (x-80)/5, & 80 \leqslant x \leqslant 85, \\ (90-x)/5, & 85 < x \leqslant 90, \end{cases} \qquad \mu_{\tilde{x}_2}(x) = \begin{cases} (x-90)/3, & 90 \leqslant x \leqslant 93, \\ (97-x)/4, & 93 < x \leqslant 97, \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} (x-92)/4, & 92 \leqslant x \leqslant 96, \\ (100-x)/4, & 96 < x \leqslant 100, \end{cases} \qquad \mu_{\tilde{x}_4}(x) = \begin{cases} (x-95)/4, & 95 \leqslant x \leqslant 99, \\ (104-x)/5, & 99 < x \leqslant 104, \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} (x-100)/5, & 100 \leqslant x \leqslant 105, \\ (110-x)/5, & 105 < x \leqslant 110. \end{cases}$$

To find the membership function of the point estimate of the parameter σ of the normal distribution, we first chose the transfer function of the output layer of second neural network to be a linear function. With Eq. (13), we obtained 66 training data points. Other parameter values that we used to train the neural network are:

Hidden nodes: 15;
Learning rate: 0.01;
Performance goal: 0.001;
Training epochs: 10 000.

Through training we obtained the interconnection weights and the bias weights of the neural networks.

The genetic algorithm was used to determine the membership function of the point estimate of the parameter, $\tilde{\hat{\sigma}}$. We used the following parameter values of the genetic algorithm:

Length of chromosome: 35;
Size of population: 15;
Fitness function: the parameter value when it is to be maximized and the reciprocal of the parameter value when it is to be minimized.
Probability of crossover: 0.4;
Probability of mutation: 0.01;
Iteration epochs: 200.

With the genetic algorithm, we calculated the parameter value at any cut level. The membership function of the fuzzy parameter estimation $\tilde{\hat{\sigma}}$ was obtained by connecting these estimation values at different membership grades, as shown in Fig. 1.

The fuzzy estimate of μ is calculated similarly. The obtained membership function of parameter μ is shown in Fig. 2.

According to Eq. (14), the reliability can be calculated given arbitrarily selected time t and sample data $x_1, x_2, \ldots, x_n$. Thus, data can be obtained for third neural network for approximation of the reliability function. Here we used the following parameter values in training the neural network:

Samples size: 32;
Number of hidden nodes: 10;
Learning rate: 0.01;
Performance goal: 0.001;
Training epochs: 50 000.

Through training we obtained the interconnection weights and bias weights.

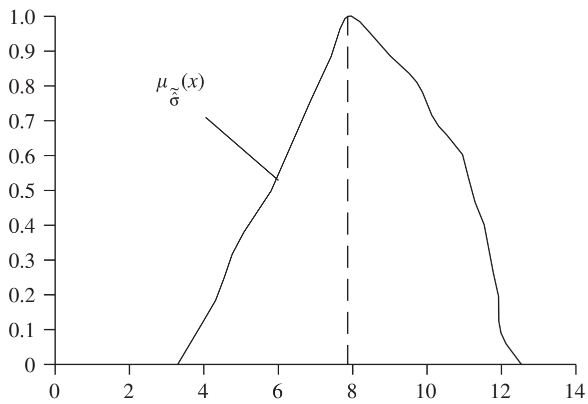

Fig. 1. Membership function of the parameter $\tilde{\hat{\sigma}}$.

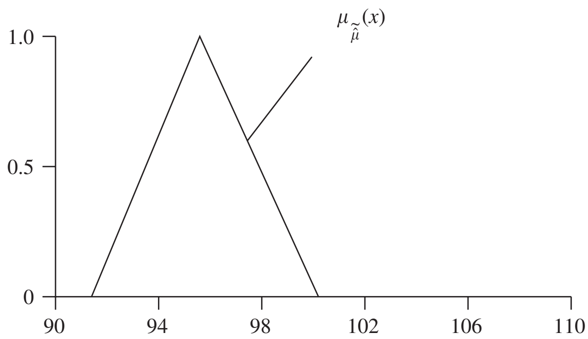

Fig. 2. Membership function of the parameter $\tilde{\hat{\mu}}$.

The genetic algorithm is then used to determine the membership function of the fuzzy reliability function. We used the following parameters of the genetic algorithm:

Length of chromosome: 35;
Size of population: 20;
Fitness function: the reliability function when it is to be maximized and the reciprocal of the reliability function when it is to be minimized.
Probability of crossover: 0.4;
Probability of mutation: 0.01;
Iteration epochs: 200.

With the genetic algorithm, the reliability at given time point t with a specified cut level can be calculated. The membership function of the fuzzy Bayesian reliability at time point t can be obtained by connecting these reliability values with different membership function values. This has been illustrated in Fig. 3.

6.2. The Weibull distribution

Bearings' lifetime has often been modeled with a 2-parameter Weibull distribution. The PDF of the Weibull distribution is given in Eq. (15). Suppose that we have obtained five fuzzy lifetime data points, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_5$, whose membership functions are, respectively:

$$\mu_{\tilde{x}_1}(x) = \begin{cases} (x-2.7)/0.24, & 2.7 < x \leqslant 2.94, \\ (3.2-x)/0.26, & 2.94 < x < 3.2, \end{cases}$$

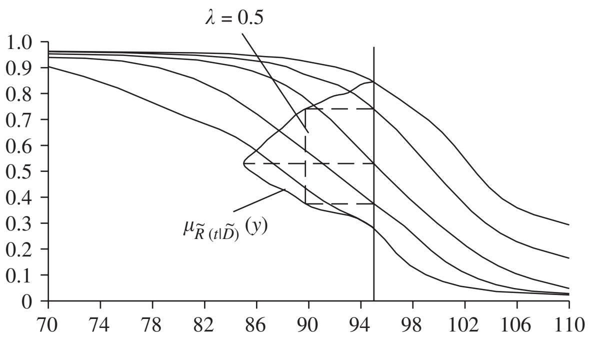

Fig. 3. Reliability values at the same time point with different degree of membership.

$$\mu_{\tilde{x}_2}(x) = \begin{cases} (x-3.0)/0.278, & 3.0 < x \leqslant 3.278, \\ (3.5-x)/0.222, & 3.278 < x < 3.5, \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} (x-3.1)/0.267, & 3.1 < x \leqslant 3.367, \\ (3.5-x)/0.133, & 3.367 < x < 3.5, \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} (x-3.5)/0.305, & 3.5 < x \leqslant 3.805, \\ (4.0-x)/0.195, & 3.805 < x < 4.0, \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} (x-4.0)/0.224, & 4.0 < x \leqslant 4.224, \\ (4.5-x)/0.276, & 4.224 < x < 4.5. \end{cases}$$

To estimate the parameter m, the transfer function of the output layer of first neural network was selected to be a linear function. With Eq. (21), we obtained 52 training data points. Other parameter values of the neural network used were:

Hidden nodes: 10;
Learning rate: 0.01;
Performance goal: 0.00001;
Training epochs: 10 000.

Through training we obtain the interconnection weights and the bias weights.

Then genetic algorithm was used to determine the membership function of fuzzy parameter estimation $\tilde{\hat{m}}$. We use the following parameter values of the genetic algorithm:

Length of chromosome: 30;
Size of population: 15;
Fitness function: the parameter value when it is to be maximized and the reciprocal of the parameter value when it is to be minimized.
Probability of crossover: 0.4;
Probability of mutation: 0.01;
Iteration epochs: 100.

With the genetic algorithm, the parameter estimation at any cut level can be calculated. The membership function of fuzzy parameter Bayesian estimation $\tilde{\hat{m}}$ can be obtained by connecting these estimates with different membership function values, as shown in Fig. 4.

The output layer transfer function of second neural network model was selected to be a linear function for estimation of the parameter η. With Eq. (22), we obtained 29 data points for training the neural network. Other parameter values for training the neural networks were:

Hidden nodes: 8;
Learning rate: 0.01;

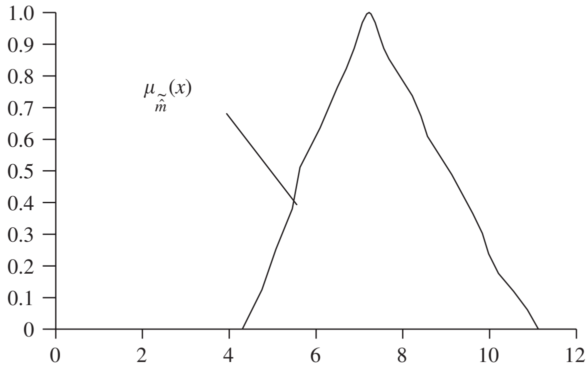

Fig. 4. Membership function of the parameter $\tilde{\hat{m}}$.

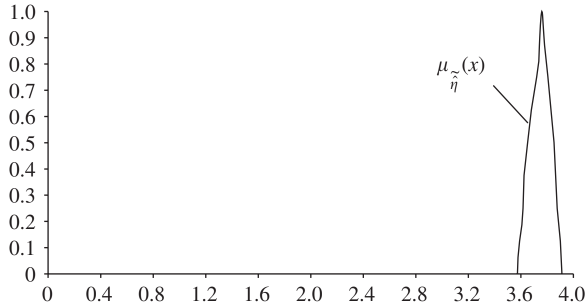

Fig. 5. Membership function of the parameter $\tilde{\hat{\eta}}$.

Performance goal: 0.00001;
Training epochs: 10 000.

Through training we obtained all the interconnection weights and the bias weights.

The genetic algorithm was then used to determine the membership function of the fuzzy parameter estimation $\tilde{\hat{\eta}}$. We used the following parameter values of the genetic algorithm:

Length of chromosome: 30;
Size of population: 20;
Fitness function: the parameter value when it is to be maximized and the reciprocal of the parameter value when it is to be minimized.
Probability of crossover: 0.4;
Probability of mutation: 0.01;
Iteration epochs: 100.

With the genetic algorithm, the parameter estimate at any cut level can be calculated. The membership function of the fuzzy parameter Bayesian estimation $\tilde{\hat{\eta}}$ can be obtained by connecting these estimates with different degrees of membership, as illustrated in Fig. 5.

With Eq. (23), we can calculate the reliability function at time t using precise data $x_1, x_2, \ldots, x_n$. This way, we can obtain a training data set for the neural network. The parameter values used in our training of third neural network are:

Samples size: 40;
Hidden nodes: 12;
Learning rate: 0.02;

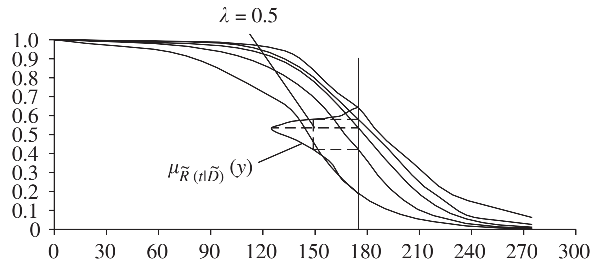

Fig. 6. Reliability values at different degrees of membership.

Performance goal: 0.0001;
Training epochs: 20 000.

Through training, we obtained the interconnection weights and the bias weights of the neural network.

The genetic algorithm was used to determine the membership function of fuzzy reliability $\tilde{R}(t|\tilde{D})$. The following parameter values of the genetic algorithm were used:

Length of chromosome: 30;
Size of population: 20;
Fitness function: the reliability function when it is to be maximized and the reciprocal of the reliability function when it is to be minimized.
Probability of crossover: 0.4;
Probability of mutation: 0.01;
Iteration epochs: 100.

With the genetic algorithm, we can also calculate the reliability values at time point t with different cut level values. The membership function of fuzzy Bayesian reliability estimation can be obtained by connecting these reliability values with different membership function values at the same time point t, as illustrated in Fig. 6.

7. Conclusion

This paper reports a method for determining the membership function of the fuzzy estimates of the parameters and the reliability functions of multi-parameter distributions. An artificial neural network is used to approximate the calculation process of fuzzy parameter estimation and reliability prediction. The genetic algorithm is used to find the boundary values of the membership functions at any cut level. The method provides a way to determine the membership functions of the parameter estimates and the reliability functions of multi-parameter distributions which are very difficult to obtain with the conventional methods. Though not illustrated in this paper, the proposed method can also be used to determine reliability indices other than parameter values and reliability functions, such as interval estimation of the parameters and highest a posteriori density (HPD) regions.

Acknowledgements

This research was partially supported by the National Natural Science Foundation of China under the contract number 50175010, the Excellent Young Teachers Program of the Ministry of Education of China under the contract number 1766, the National Excellent Doctoral Dissertation Special Foundation of China under the contract number 200232, and the Natural Sciences and Engineering Research Council of Canada. Constructive suggestions and comments from the referees and editors are very much appreciated.

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