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A discrete stress–strength interference model based on universal generating function

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Abstract

Continuous stress–strength interference (SSI) model regards stress and strength as continuous random variables with known probability density function. This, to some extent, results in a limitation of its application. In this paper, stress and strength are treated as discrete random variables, and a discrete SSI model is presented by using the universal generating function (UGF) method. Finally, case studies demonstrate the validity of the discrete model in a variety of circumstances, in which stress and strength can be represented by continuous random variables, discrete random variables, or two groups of experimental data.

Keywords: Discrete model; Stress–strength interference; Universal generating function; Component reliability

1. Introduction

The stress–strength interference (SSI) model has been widely used for reliability design of mechanical component. In this model, if stress on a component and strength of a component are denoted by S_1 and S_2, respectively, the component reliability denoted by R is then defined as

$$R = Pr(S_2 > S_1). \tag{1}$$

Eq. (1) is the most basic expression of the SSI model, which means that the component reliability is taken as the probability that the strength is larger than the stress. Furthermore, if both stress and strength are treated as continuous random variables (r.v.) and their probability density functions (p.d.f.), denoted by $f_1(S_1)$ and $f_2(S_2)$ respectively, Eq. (1) can be rewritten as the following formulas:

$$R = \int_{-\infty}^{\infty} f_1(S_1)\left[\int_{S_1}^{\infty} f_2(S_2)\,\mathrm{d}S_2\right]\mathrm{d}S_1 \tag{2a}$$

or

$$R = \int_{-\infty}^{\infty} f_2(S_2)\left[\int_{-\infty}^{S_2} f_1(S_1)\,\mathrm{d}S_1\right]\mathrm{d}S_2. \tag{2b}$$

For the sake of clarity, Eq. (2) can be called the continuous SSI model.

Theoretically, we can calculate the reliability or unreliability of a component analytically or numerically on the basis of Eq. (2) when the p.d.f. of stress and strength are available. However, in engineering practice, it is often difficult to know the exact distribution of stress and strength. In most cases, what we can obtain is the finite experimental data regarding stress and strength only. Consequently, it is necessary to study the approximate methods when calculating component reliability, and in this regard, some efforts have been made by many researchers.

Kapur [1] devised an approach for determining the bounds on exact unreliability, and this approach required only information regarding the subinterval probabilities

Abbreviations (the singular and plural of an abbreviation is always spelled the same): r.v., random variable; p.d.f., probability density function; p.m.f., probability mass function; SSI, stress–strength interference; UGF, universal generating function

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within an interference region. To improve the accuracy of calculation, Park and Clark [2] modified Kapur's formulation on the quadratic programming problem and presented a solution to this problem. Shen [3] proposed another empirical approach to computing the unreliability bounds based upon the subinterval probabilities of the stress and strength in the interference region. Wang and Liu [4] presented a multiple-line-segment method of implementing the SSI model when only discrete interval probabilities of stress and strength inside an interference region are available. Guo and Li [5] presented an algorithm for computing the unreliability bounds based on an improved Monte Carlo method. Wang and Liu [6] presented an approach to calculate fuzzy unreliability of a component/ system. In this approach the p.d.f. of stress and strength were approximated by piecewise fuzzy line-segments that were expressed by linear fuzzy polynomials, and the discrete interval probabilities were treated as fuzzy numbers. Additionally, summarizing the research results from diverse disciplines, Kotz et al. [7] generalized the stress–strength model and provided the computation methods on the basis of maximum likelihood estimation.

In this paper, unlike those in the continuous SSI model, stress and strength are treated as discrete r.v. and their probability mass functions (p.m.f.) are represented by universal generating functions (UGF). According to the basic definition of component reliability expressed by Eq. (1), a discrete SSI model is established. This model can be utilized for calculating component reliability under the following circumstances:

(1) stress and strength are discrete r.v.,
(2) stress and strength are continuous r.v.,
(3) the distributions of stress and strength are unavailable but their frequency distributions are known based on experiment data.

This paper begins with a description of the UGF method that is the modeling tool of discrete SSI system and proceeds with a building model of SSI system by employing UGF method. Finally, two cases are studied to demonstrate the effectiveness and advantage of the discrete SSI model.

2. Brief description of UGF method

In this section, emphasis is put on the basic process but not the fundamental mathematics of UGF method. The concept of UGF was first introduced by Ushakov [8]. Then in a series of research work by Lisnianski and Levitin, the UGF method has been applied to reliability analysis and optimization of multi-state system [9,10].

2.1. UGF of discrete random variable

Suppose that a discrete r.v. X has a p.m.f. characterized by the vector $\boldsymbol{x}$ consisting of the possible values of X and the vector $\boldsymbol{p}$ consisting of the corresponding probabilities, which can be formulated by the following expressions:

$$\boldsymbol{x} = (x_1,\ x_2,\ \ldots,\ x_k),$$

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_k),$$

$$p_i = Pr(X = x_i), \quad i = 1, 2, \ldots, k.$$

Based on the basic principle of UGF method, the p.m.f. of discrete r.v. X can be represented by a polynomial function of variable z, $u_X(z)$, that relates the possible values of X to the corresponding probabilities as

$$u_X(z) = p_1 z^{x_1} + p_2 z^{x_2} + \cdots + p_k z^{x_k} = \sum_{i=1}^{k} p_i z^{x_i}. \tag{3}$$

It should be mentioned that, for an arbitrary discrete r.v., its UGF is uniquely determined by its p.m.f. This means that a one-to-one correspondence exists between the p.m.f. and UGF of a discrete r.v.

2.2. UGF of function of discrete random variables

Consider n independent discrete r.v. $X_1,\ X_2,\ \ldots,\ X_n$. Let the UGF of each r.v. be $u_{X_1}(z),\ u_{X_2}(z),\ \ldots,\ u_{X_n}(z)$, respectively, and $f(X_1,\ X_2,\ \ldots,\ X_n)$, an arbitrary function of variables $X_1,\ X_2,\ \ldots,\ X_n$. Then, by employing composition operator $\otimes$, the UGF of function $f(X_1,\ X_2,\ \ldots,\ X_n)$, $u_f(z)$, can be obtained as follows:

$$u_f(z) = \otimes(u_{X_1}(z),\ u_{X_2}(z),\ \ldots,\ u_{X_n}(z)). \tag{4}$$

2.3. Definition and properties of composition operator $\otimes$

Without loss of generality, we still consider n independent discrete r.v. $X_1,\ X_2,\ \ldots,\ X_n$ and an arbitrary function $f(X_1,\ X_2,\ \ldots,\ X_n)$. Suppose that the number of possible values of each r.v. are $k_1, k_2, \ldots, k_n$, respectively. According to Eq. (3), the UGF of individual r.v. can be obtained as follows:

$$u_{X_1}(z) = \sum_{j_1=1}^{k_1} p_{1j_1} z^{x_{1j_1}},$$

$$u_{X_2}(z) = \sum_{j_2=1}^{k_2} p_{2j_2} z^{x_{2j_2}},$$

$$\ldots$$

$$u_{X_n}(z) = \sum_{j_n=1}^{k_n} p_{nj_n} z^{x_{nj_n}}.$$

To obtain the UGF of function $f(X_1,\ X_2,\ \ldots,\ X_n)$, composition operator $\otimes$ is defined as

$$\otimes\left(\sum_{j_1=1}^{k_1} p_{1j_1} z^{x_{1j_1}}, \sum_{j_2=1}^{k_2} p_{2j_2} z^{x_{2j_2}}, \ldots, \sum_{j_n=1}^{k_n} p_{nj_n} z^{x_{nj_n}}\right) = \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\cdots\sum_{j_n=1}^{k_n}\left(\prod_{i=1}^{n} p_{ij_i} z^{f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n})}\right). \tag{5}$$

Indeed, composition operator $\otimes$ represents an operation rule, which strictly depends on the expression of function $f(X_1, X_2, \ldots, X_n)$. Based on this rule, the UGF of random function $f(X_1, X_2, \ldots, X_n)$ can be obtained, which also has a form of polynomial function and corresponds to the p.m.f. of function $f(X_1, X_2, \ldots, X_n)$.

It should be noted that UGFs are not regular polynomials in spite of resembling the polynomials. However, UGFs inherit the essential properties of regular polynomials. For example, in the operation of UGF, like terms can be collected, and commutative law and associative law are applicable:

$$\begin{aligned} u_f(z) &= \otimes\,(u_{X_1}(z), \ldots, u_{X_i}(z), u_{X_{i+1}}(z), \ldots, u_{X_n}(z)) \\ &= \otimes\,(u_{X_1}(z), \ldots, u_{X_{i+1}}(z), u_{X_i}(z), \ldots, u_{X_n}(z)), \end{aligned}$$

$$\begin{aligned} u_f(z) &= \otimes\,(u_{X_1}(z), \ldots, u_{X_i}(z), u_{X_{i+1}}(z), \ldots, u_{X_n}(z)) \\ &= \otimes\,(\otimes(u_{X_1}(z), \ldots, u_{X_i}(z)), \otimes(u_{X_{i+1}}(z), \ldots, u_{X_n}(z))). \end{aligned}$$

Once the UGF of random function $f(X_1, X_2, \ldots, X_n)$ is obtained, we can regard it as a new r.v. and analyze its statistic characteristics.

3. Discrete SSI model

Assume that stresses on a component and strength of a component are two independent discrete r.v. that are denoted by S_1 and S_2, respectively. If the p.m.f. of stress and strength are known as follows:

$$\begin{aligned} \mathbf{S}_1 &= (S_{11}, S_{12}, \ldots, S_{1k_1}), \quad \mathbf{p}_1 = (p_{11}, p_{12}, \ldots, p_{1k_1}), \\ \mathbf{S}_2 &= (S_{21}, S_{22}, \ldots, S_{2k_2}), \quad \mathbf{p}_2 = (p_{21}, p_{22}, \ldots, p_{2k_2}), \end{aligned}$$

where k_1 and k_2 are numbers of possible values that S_1 and S_2 can take on, respectively, then, according to Eq. (3), the UGF of stress and strength can be obtained as follows:

$$\begin{aligned} u_{S_1}(z) &= \sum_{j_1=1}^{k_1} p_{1j_1} z^{S_{1j_1}}, \\ u_{S_2}(z) &= \sum_{j_2=1}^{k_2} p_{2j_2} z^{S_{2j_2}}. \end{aligned}$$

We construct a function, $f(S_1, S_2)$, of the r.v. of stress and strength

$$f(S_1, S_2) = S_2 - S_1. \tag{6}$$

Based on the UGF method introduced in Section 2, the UGF of discrete function $f(S_1, S_2)$ can be obtained as follows:

$$\begin{aligned} u_f(z) &= \otimes\,(u_{S_1}(z), u_{S_2}(z)) \\ &= \otimes\left(\sum_{j_1=1}^{k_1} p_{1j_1} z^{S_{1j_1}}, \sum_{j_2=1}^{k_2} p_{2j_2} z^{S_{2j_2}}\right) \\ &= \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\left(\prod_{i=1}^{2} p_{ij_i} z^{f(S_{1j_1}, S_{2j_2})}\right) \\ &= \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\left(\prod_{i=1}^{2} p_{ij_i} z^{(S_{2j_2} - S_{1j_1})}\right). \end{aligned} \tag{7}$$

As a determinate case, the operator $\otimes$ in Eq. (7) is defined as subtraction of corresponding powers in the multiplying polynomials. It is not difficult to understand that the final form of UGF of function $f(S_1, S_2)$ is a polynomial function containing $K \leqslant k_1 \times k_2$ terms (the total number of terms can be less than $k_1 \times k_2$ after collecting the like terms). Therefore, Eq. (7) can be rewritten as

$$u_f(z) = \sum_{j=1}^{K} P_j z^{f_j}, \tag{8}$$

where f_j and P_j ($j = 1,2,\ldots,K$) are possible values of function $f(S_1, S_2)$ and corresponding probabilities, respectively.

As expressed by Eq. (1), the component reliability is defined as the probability that strength is larger than stress. Transforming Eq. (1), we can obtain

$$R = Pr(S_2 - S_1 > 0). \tag{9}$$

Substituting Eq. (6) into Eq. (9) yields

$$R = Pr(f(S_1, S_2) > 0), \tag{10}$$

where $f(S_1, S_2)$, essentially, is a new discrete r.v. and its distribution properties can be denoted by its UGF.

To calculate the probability expressed by Eq. (10) on the basis of UGF of function $f(S_1, S_2)$, we can define a binary-valued function with domain on the set of possible values of function $f(S_1, S_2)$ as

$$\alpha(f_j) = \begin{cases} 1, & f_j > 0, \\ 0, & f_j \leqslant 0. \end{cases}$$

Then, based on Eq. (10), the component reliability can be calculated as

$$\begin{aligned} R &= Pr(f(S_1, S_2) > 0) \\ &= \sum_{j=1}^{K} P_j \alpha(f_j). \end{aligned} \tag{11}$$

For the sake of comparison, Eq. (11) can be called the discrete SSI model.

4. Case studies

As mentioned in Section 1, the discrete SSI model presented in Section 3 can be used to evaluate the reliability of a component under several cases. It is obvious that, when stress and strength are discrete r.v., the component reliability can be directly obtained according to Eqs. (7), (8), and (11). In this section, two cases are taken into account. Case 1 denotes that stress and strength are continuous r.v. with known p.d.f., and Case 2 denotes that only the frequency distribution of stress and strength are available based on data statistical analysis.

Case 1. Consider the example from [6]: stress on a component, S_1, is exponentially distributed with mean $\mu_1 = 50\,\text{MPa}$, and strength of the component, S_2, is *s*-normally distributed with mean $\mu_2 = 100\,\text{MPa}$ and standard deviation $\sigma_2 = 10\,\text{MPa}$. The exact value of reliability of this component is equal to 0.86194.

In this case, the basic idea of calculating component reliability is to translate approximately continuous r.v. with known p.d.f. into discrete r.v. with known p.m.f. Firstly, based on the operating environment of the component, we can determine an approximate range of possible values of stress and strength, which can be denoted by intervals $\langle S_{1\min}, S_{1\max}\rangle$ and $\langle S_{2\min}, S_{2\max}\rangle$, respectively. Then, the intervals $\langle S_{1\min}, S_{1\max}\rangle$ and $\langle S_{2\min}, S_{2\max}\rangle$ are divided into m and n subintervals, respectively ($m = n$ is allowable). The midpoint values of each subinterval are treated as possible values of r.v. and the area values of each subinterval are treated as the corresponding probabilities. Thus, we can obtain two discrete r.v. of stress and strength with known p.m.f. Finally, the discrete r.v. of stress and strength are represented by their UGF, while the component reliability can be calculated using the discrete SSI model presented in Section 3.

Let the range of stress and strength be $\langle 0, 6\mu_1\rangle = \langle 0, 300\rangle\,\text{MPa}$ and $\langle \mu_2-3\sigma_2, \mu_2+3\sigma_2\rangle = \langle 70, 130\rangle\,\text{MPa}$, respectively, and divide the two intervals into six subintervals each. The result of interval partition is depicted by Fig. 1. In this case, the area values of subintervals are calculated by using MATLAB 7.0.1. Based on the midpoint values and area values of all subintervals, we can obtain the discrete r.v. of stress and strength as follows, respectively:

$$S_1 = (25, 75, 125\ 175, 225, 275),$$
$$p_1 = (0.6321, 0.2325, 0.0855, 0.0315, 0.0116, 0.0043),$$
$$S_2 = (75, 85, 95, 105, 115, 125),$$
$$p_2 = (0.0214, 0.1359, 0.3413, 0.3413, 0.1359, 0.0214).$$

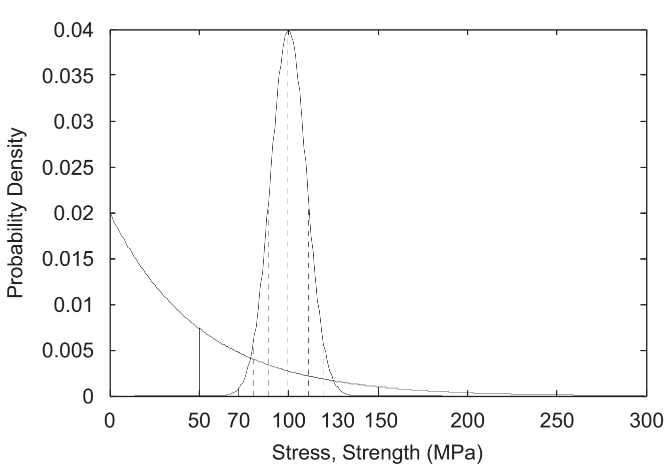

Fig. 1. P.d.f. curves and interval partition.

According to the procedure presented in Section 3, the reliability of this component is obtained as $R = 0.86250$. The relative error is 0.065% compared with the exact value of reliability.

In order to illustrate the influence of interval partition on calculation accuracy, one can divide the stress interval into 12 subintervals while keeping the number of strength subinterval unchanged. Thus, we can obtain another description of stress:

$$S_1 = (12.5, 37.5, 62.5, 87.5, 112.5, 137.5, 162.5, 187.5, 212.5, 237.5, 262.5, 287.5),$$
$$p_1 = (0.3953, 0.2387, 0.1447, 0.0878, 0.0533, 0.0323, 0.0196, 0.0119, 0.0072, 0.0044, 0.0027, 0.0016).$$

Similarly, the reliability of this component can be calculated as $R = 0.86163$, and the relative error is equal to 0.036%. This result indicates that reducing the length of subinterval can improve the computational accuracy when the range of stress and strength is fixed.

Case 2. Suppose that two groups of data, about stress on a component and strength of a component, are obtained from corresponding experiments. If the continuous SSI model is employed to calculate the component reliability, stress and strength are supposed to be continuous r.v. and their p.d.f. are approximately obtained by using methods of distribution fitting and parameter estimation. However, when using the discrete SSI model in this case, attention cannot be directed to the actual distribution of stress and strength, and the methods of distribution fitting and parameter estimation are not wanted. We can employ two discrete r.v. to represent the statistic information of stress and strength, and then calculate the component reliability directly.

Assume that the number of data contained in each group is equal to100, and the data can be described by histograms after a simple processing. As given by Figs. 2 and 3, the class intervals of data and their corresponding relative frequencies are obtained. Similarly, the midpoint values of each class interval are treated as possible values of the r.v. of stress and strength, and relative frequencies of each class interval are treated as corresponding probabilities. Thus, two new r.v. of stress and strength with known p.m.f. are obtained as follows:

$$S_1 = (725, 755, 785, 815, 845, 875, 905),$$
$$p_1 = (0.01, 0.04, 0.24, 0.38, 0.25, 0.06, 0.02),$$
$$S_2 = (850, 900, 950, 1000, 1050, 1100, 1150),$$
$$p_2 = (0.02, 0.06, 0.25, 0.39, 0.22, 0.05, 0.01).$$

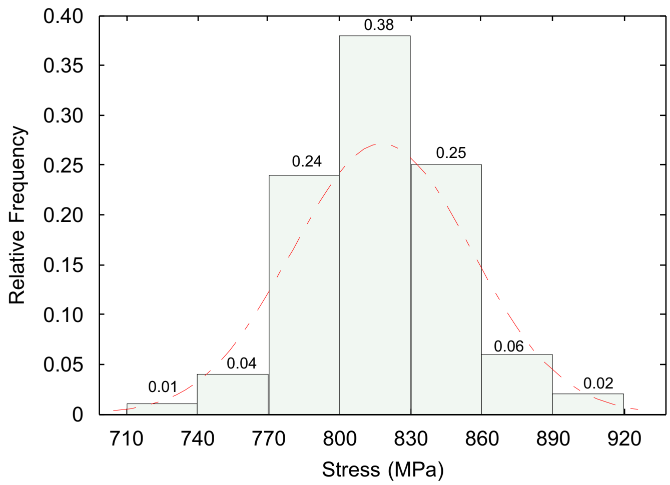

Fig. 2. Histogram of stress data.

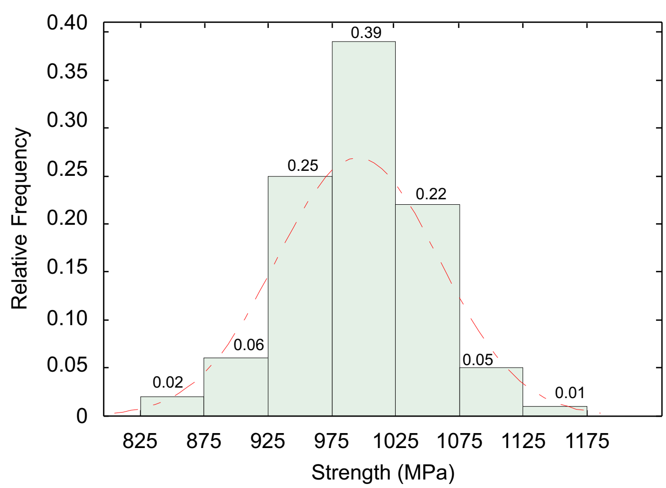

Fig. 3. Histogram of strength data.

By employing the discrete SSI model presented in Section 3, the reliability of this component is obtained as $R = 0.9972$.

As mentioned in Section 1, the method provided by [7] can be used to solve this problem when the probability distribution type of stress and strength are known. Suppose both stress and strength in this case are independent normal r.v. with unknown parameters. The two groups of data can be regarded as the observations of the stress and strength. Using the method of maximum likelihood estimation, the reliability of this component can be calculated as $R = 0.9962$. If this result is regarded as the exact value of reliability, the relative error resulting from the discrete SSI model is equal to 0.1%.

5. Conclusions

In this paper, the stress on a component and strength of a component are regarded as discrete r.v. with known p.m.f., and a discrete SSI model is established based on the UGF method. This model can be used to calculate the component reliability under different conditions, within which stress and strength can be represented by discrete r.v., continuous r.v., or two groups of data.

Study of case 1 demonstrates the validity of the discrete SSI model when stress and strength are continuous r.v. Furthermore, two numerical examples indicate that reducing the length of subinterval can improve the computational accuracy when the range of stress and strength is fixed.

Study of case 2 illustrates the usability of the discrete SSI model when only two groups of data on stress and strength are available. The calculated result of a numerical example indicates that a small error does exist when using the discrete SSI model. But, it should be mentioned that, when using the discrete SSI model, we need not know the actual distribution of stress and strength. And this is the essential advantage of the approach proposed here.

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