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Posbist Reliability Theory of k-out-of-n:G System

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A preliminary analysis of the posbist reliability of typical system structures was provided by Cai et al. [*Fuzzy Sets and Systems* 42: 145–172 (1991) and *Microelectronics and Reliability* 35(1): 49–56 (1995)]. In this paper, we provide a detailed analysis of the posbist reliability of k-out-of-n:G system structures. Expressions of the posbist reliability of k-out-of-n:G system is developed. It is presented that all methods for generating membership functions can be used in principle for constructing relevant possibility distributions. Several methods for constructing the possibility distributions are discussed in details and a method for generating the $L-R$ type possibility distributions is provided for posbist reliability analysis of fatigue lifetime data of mechanical parts. Two numerical examples are given to illustrate some of these methods.

Keywords: Posbist reliability, Gaussian fuzzy variable, system lifetime, fuzzy reliability, possibility distributions, membership functions.

1 INTRODUCTION

The conventional reliability theory is built on the probability and the binary-state assumptions [1]. It has been successfully used for solving various reliability problems. However, it is not suitable when the failure probabilities concerned are very small or when there is a lack of sufficient data. Among other researchers, we mention the works by Tanaka *et al*. [2], Singer [3], Onisawa [4], Cappelle and Kerre [5], Cremona and Gao [6], Utkin and Gurov [7], Cai et al [1, 8, 9], Huang [10–12], and Huang et al [13–16]. All these researchers have attempted to define reliability in terms other than the probabilistic definition. The fuzzy set concept represents a new paradigm of accounting for uncertainty. Two new assumptions in these definitions include the fuzzy-state assumption and the possibility assumption. The fuzzy state assumption indicates that the state of a piece of equipment can be represented by a fuzzy variable. The possibility assumption indicates that the reliability of a piece of equipment has to be measured subjectively. The posbist reliability theory is based on the possibility assumption and the binary-state assumption.

In this paper, we provide an analysis of the posbist reliability theory and illustrate its applications in system reliability analysis. The lifetime of the system is treated as a fuzzy variable defined on the possibility space (U, Φ, P_{oss}) and the universe of discourse is expanded to $(-\infty, +\infty)$. As suggested by Dubios and Prade [17], we approximate the possibility distribution function (i.e., the membership function) $\mu_X(x)$ by two functions $L(x)$ and $R(x)$ with a point of intersection at max $\mu_X(x) = 1$, i.e., the $L - R$ type possibility distribution function is adopted. The lifetime of the system is assumed to be a Gaussian fuzzy variable, which is a special $L - R$ type fuzzy variable. Under these conditions, the posbist reliability analysis of k-out-of-n:G systems is provided in details.

Because the concept of membership function is closely related to the concept of possibility distributions [18], we believe that in principle, all methods developed for generating membership functions can be used to construct relevant possibility distributions. We will further discuss the properties of the methods for constructing possibility distributions. A method used to generate the $L-R$ type possibility distribution is applied to the posbist reliability analysis of fatigue of mechanical parts. Finally, two examples are given to illustrate the application of this method to generating the possibility distribution of the fatigue lifetime of crankshaft in diesel engine.

2 THE POSBIST RELIABILITY THEORY

The assumptions of the posbist reliability theory include (1) the system failure behavior is fully characterized in the context of the possibility theory and (2) at any instant of time the system is in one of two crisp states: perfectly functioning or completely failed [8].

2.1 Basic concept in the possibility context

The concept of the posbist reliability theory was introduced in details in [8]. For ease of reference, we list several basic definitions related to this theory.

Definition 1 [8]. A fuzzy variable X is a real valued function defined on a possibility space (U, Φ, P_{oss})

$$X : U \to R = (-\infty, +\infty) .$$

Its membership function μ_X is a mapping from R to the unit interval [0, 1] with

$$\mu_X(x) = P_{oss}(X = x), x \in R.$$

Thus, a fuzzy set X is defined as

$$\tilde{X} = \{x, \mu_X(x)\}.$$

Based on X, the distribution function of X is given by

$$\pi_X(x) = \mu_X(x).$$

Definition 2 [8]. The possibility distribution function of a fuzzy variable X, denoted by π_X, is a mapping from R to the unit interval [0, 1] such that

$$\pi_X(x) = \mu_X(x)$$

$$= P_{oss}(X = x), x \in R$$

Definition 3 [8]. Given a possibility space (U, Φ, P_{oss}), the sets $A_1, A_2, \ldots, A_n \subset \Phi$ are said to be mutually unrelated if for any permutation of the set $\{1, 2, \ldots, n\}$, denoted by $\{i_1, i_2, \ldots, i_k\}$ $(1 \leq k \leq n)$, the following equation holds:

$$P_{oss}\left(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}\right) = \min\left(P_{oss}\left(A_{i_1}\right), P_{oss}\left(A_{i_2}\right), \cdots, P_{oss}\left(A_{i_k}\right)\right).$$

Definition 4 [8]. Given a possibility space (U, Φ, P_{oss}), the fuzzy variables $X_1, X_2, \ldots, X_n$ are said to be mutually unrelated if for any permutation of the set $\{1, 2, \ldots, n\}$, denoted by $\{i_1, i_2, \ldots, i_k\}$ $(1 \leq k \leq n)$, the sets

$$\left\{X_{i_1} = x_1\right\}, \left\{X_{i_2} = x_2\right\}, \ldots, \left\{X_{i_k} = x_k\right\},$$

where $(x_1, x_2, \ldots, x_k \in R)$, are mutually unrelated.

2.2 Lifetime of the system

When the conventional binary-state assumption is adopted, the failure of a system is defined precisely. However in practice, the instant of time when a system failure occurs may be uncertain and we may be unable to determine it accurately. In this case, it has to be characterized in the context of a possibility measure. According to the existence theorem of the possibility space [19], we can reasonably assume that there exists a single possibility space (U, Φ, P_{oss}) to characterize all the uncertainties of the times of failure of the system and its components. Accordingly, the lifetimes of the system and its components are treated as Nahmias' fuzzy variables defined on the common possibility space.

Definition 5 [8]. Given a possibility space (U, Φ, P_{oss}), the lifetime of a system is a non-negative real-valued fuzzy variable

$$X : U \to R^+ = (0, +\infty)$$

with possibility distribution function

$$\mu_X(x) = P_{oss}(X = x), x \in R^+.$$

The posbist reliability of a system is then defined as the possibility that the system performs its assigned functions satisfactorily during a predefined exposure period under a given environment [8], that is,

$$R(t) = P_{oss}(X > t) = \sup_{u>t} P_{oss}(X = u) = \sup_{u>t} \mu_X(u), t \in R^+. \quad (1)$$

To simplify calculations when dealing with real-life problems, we may expand the universe of discourse of the lifetime of a system from $(0, +\infty)$ to $(-\infty, +\infty)$, i.e.,

$$X : U \to R = (-\infty, +\infty).$$

In the following sections, we will show that this expansion makes the proofs originally given in [8, 9] much simplified and the complexity of calculation is greatly reduced without affecting the nature of the problems to be solved.

3 POSBIST RELIABILITY ANALYSIS OF k-out-of-n:G SYSTEM

Suppose that X is the lifetime of the system. Assume that the components of the system under consideration are mutually unrelated. That is to say, the lifetimes of the components, denoted by $X_1, X_2, \ldots, X_n$, are mutually unrelated. Furthermore, we assume that X_i is a Gaussian fuzzy variable. Its possibility distribution function is given by the following equation and illustrated in Figure 1.

$$\mu_{X_i}(x) = \begin{cases} \exp\left(-\left(\frac{m_i - x}{b_i}\right)^2\right), & x \le m_i \\ \exp\left(-\left(\frac{x - m_i}{b_i}\right)^2\right), & x > m_i \end{cases},$$

$$x \in R,\ m_i, b_i > 0,\ i = 1, 2, \ldots, n \quad (2)$$

An n-component system is called a k-out-of-n:G system if it meets the condition that it works (or in “good” state) if and only if at least k of the n components work (or in “good” state). The system fails when the number of failed components is more than $n - k$. The k-out-of-n:G system is a very popular type of redundancy in fault-tolerant system and is widely applied in both industrial and military systems. For example, a four-engine civil airplane works when

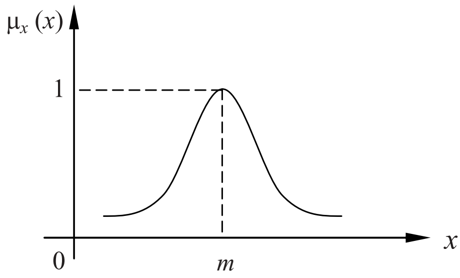

FIGURE 1
Possibility distribution function of X_i.

less than three engines failed, and it is regarded as a 2-out-of-4:G system.

The lifetime X of a k-out-of-n:G system is equal to the lifetime X_d of component d whose lifetime is the kth longest lifetime among all the n components and can be expressed as

$$X = \operatorname*{order}_{k}(X_1, X_2, X_3, \ldots, X_n) = X_d \tag{3}$$

where $\text{order}_k(\cdot)$ denotes the kth variable after descending ordering operation on a series of variables. Indeed, Eq.(3) is much more complicated than previous scenarios, and it involves many different combination of variable to meet all the possible situations. We present a 2-out-of-3:G system as follows to illustrate the general solution procedure.

Without loss of generality, let the system lifetime be represented by X. The system consists of three mutually unrelated components with the lifetime denoted as X_1, X_2, X_3, and assume that they are normally distributed, strictly convex fuzzy variables with continuous possibility distribution function $\mu_{X_1}(x),\ \mu_{X_2}(x)$ and $\mu_{X_3}(x)$, respectively. Then, there exists a unique triad $(a_1, a_2, a_3), a_1, a_2, a_3 \in R^+, a_1 \leq a_2 \leq a_3$, such that

$$\mu_X(x) = \begin{cases} \max\Big(\min\big(\mu_{X_1}(x), \mu_{X_2}(x)\big), \\ \quad \min\big(\mu_{X_2}(x), \mu_{X_3}(x)\big), \\ \quad \min\big(\mu_{X_1}(x), \mu_{X_3}(x)\big)\Big), & 0 < x \leq a_1 \\ \max\big(\mu_{X_2}(x), \mu_{X_3}(x)\big), & a_1 < x \leq a_2 \\ \max\big(\mu_{X_1}(x), \mu_{X_2}(x)\big), & a_2 < x \leq a_3 \\ \max\Big(\min\big(\mu_{X_1}(x), \mu_{X_2}(x)\big), \\ \quad \min\big(\mu_{X_2}(x), \mu_{X_3}(x)\big), \\ \quad \min\big(\mu_{X_1}(x), \mu_{X_3}(x)\big)\Big), & x > a_3 \end{cases} \tag{4}$$

Proof. As X_1, X_2, X_3 are mutually unrelated and normally distributed, strictly convex fuzzy variables with continuous possibility distribution function $\pi_{X_i}(x) = \mu_{X_i}(x)$. According to Theorem 4.3 in Ref. [8], there exists a unique real number, say $a_i \in R$, such that

$$\sup_{u \geq x} \mu_{X_i}(u) = \begin{cases} 1, x \leq a_i \\ \mu_{X_i}(x), x > a_i \end{cases}, \quad (i = 1, 2, 3) \tag{5}$$

Without loss of generality, we assume that $a_1 \leq a_2 \leq a_3$, and proofs of Eq. (4) is presented as follows.

$$\mu_X(x) = P_{oss}(X = x) = P_{oss}\left(X = \underset{2}{order}(X_1, X_2, X_3) = t\right), \tag{6}$$

There exist six mutually exclusive situations, and we have

$$\begin{aligned} &P_{oss}(X = x) \\ &\quad = P_{oss}\{(X_1 = x, X_2 \leq x, X_3 > x) \cup (X_1 = x, X_2 > x, X_3 \leq x) \\ &\quad\quad \cup (X_1 \leq x, X_2 = x, X_3 > x) \cup (X_1 > x, X_2 = x, X_3 \leq x) \\ &\quad\quad \cup (X_1 \leq x, X_2 > x, X_3 = x) \cup (X_1 > x, X_2 \leq x, X_3 = x)\} \\ &\quad = \max\{\min(X_1 = x, X_2 \leq x, X_3 > x),\ \min(X_1 = x, X_2 > x, X_3 \leq x), \\ &\quad\quad \min(X_1 \leq x, X_2 = x, X_3 > x),\ \min(X_1 > x, X_2 = x, X_3 \leq x), \\ &\quad\quad \min(X_1 \leq x, X_2 > x, X_3 = x),\ \min(X_1 > x, X_2 \leq x, X_3 = x)\} \end{aligned} \tag{7}$$

Further, we have

$$P_{oss}(X_i > x) = \sup_{u > x} \mu_{X_i}(u) = \begin{cases} 1, \quad x \leq a_i \\ \mu_{X_i}(x), x > a_i \end{cases}, \quad (i = 1, 2, 3), \tag{8}$$

$$P_{oss}(X_i \leq x) = \sup_{u \leq x} \mu_{X_i}(u) = \begin{cases} \mu_{X_i}(x), x \leq a_i \\ 1, \quad x > a_i \end{cases}, \quad (i = 1, 2, 3), \tag{9}$$

and

$$P_{oss}(X_i = x) = \mu_{X_i}(x), (i = 1, 2, 3) \tag{10}$$

Thus, Eq. (7) becomes

$$\begin{aligned} P_{oss}(X = x) = \max\Bigg\{ &\min\left(\mu_{X_1}(x), \sup_{u \leq x} \mu_{X_2}(u), \sup_{u > x} \mu_{X_3}(u)\right), \\ &\min\left(\mu_{X_1}(x), \sup_{u > x} \mu_{X_2}(u), \sup_{u \leq x} \mu_{X_3}(u)\right), \\ &\min\left(\sup_{u \leq x} \mu_{X_1}(u), \mu_{X_2}(x), \sup_{u > x} \mu_{X_3}(u)\right), \\ &\min\left(\sup_{u > x} \mu_{X_1}(u), \mu_{X_2}(x), \sup_{u \leq x} \mu_{X_3}(u)\right), \\ &\min\left(\sup_{u \leq x} \mu_{X_1}(u), \sup_{u > x} \mu_{X_2}(u), \mu_{X_3}(x)\right), \\ &\min\left(\sup_{u > x} \mu_{X_1}(u), \sup_{u \leq x} \mu_{X_2}(u), \mu_{X_3}(x)\right)\Bigg\} \end{aligned} \tag{11}$$

For $x \le a_1, a_1 < x \le a_2, a_2 < x \le a_3, x > a_3$, the corresponding posbist reliability can be obtained by substitute Eq. (8) and Eq. (9) into Eq. (11), and one will arrive at Eq. (4). For instance, considering the case $x \le a_1$, then we have

$$\begin{aligned} P_{oss}(X = x) = \max\{&\min(\mu_{X_1}(x), \mu_{X_2}(x), 1),\ \ \min(\mu_{X_1}(x), 1, \mu_{X_3}(u)), \\ &\min(\mu_{X_1}(u), \mu_{X_2}(x), 1),\ \ \min(1, \mu_{X_2}(x), \mu_{X_3}(u)), \\ &\min(\mu_{X_1}(u), 1, \mu_{X_3}(x)),\ \ \min(1, \mu_{X_2}(u), \mu_{X_3}(x))\} \\ = \max\{&\min(\mu_{X_1}(x), \mu_{X_2}(x)),\ \ \min(\mu_{X_2}(x), \mu_{X_3}(u)), \\ &\min(\mu_{X_1}(u), \mu_{X_3}(x))\} \end{aligned} \tag{12}$$

□

As a result, the posbist reliability of the 2-out-of-3:G system is given as

$$\begin{aligned} R_{k-outof-n}(t) &= \sup_{u>t} \mu_X(u) \\ &= \begin{cases} 1, & t \le a_2 \\ \max\{\mu_{X_1}(t), \mu_{X_2}(t)\}, & a_2 < t \le a_3 \\ \max(\min(\mu_{X_1}(t), \mu_{X_2}(t)), \min(\mu_{X_2}(t), \mu_{X_3}(t)), & \\ \quad \min(\mu_{X_1}(t), \mu_{X_3}(t))), & t > a_3 \end{cases} \end{aligned} \tag{13}$$

When X_1, X_2, X_3 are Gaussion fuzzy variables, Eq. (13) becomes

$$\begin{aligned} &R_{k-outof-n}(t) \\ &= \begin{cases} 1, & t \le m_2 \\ \max\left(\exp\left(-\left(\frac{t-m_1}{b_1}\right)^2\right), \exp\left(-\left(\frac{t-m_2}{b_2}\right)^2\right)\right), & m_2 < t \le m_3 \\ \max\left(\min\left(\exp\left(-\left(\frac{t-m_1}{b_1}\right)^2\right), \exp\left(-\left(\frac{t-m_2}{b_2}\right)^2\right)\right),\right. & \\ \quad \min\left(\exp\left(-\left(\frac{t-m_2}{b_2}\right)^2\right), \exp\left(-\left(\frac{t-m_3}{b_3}\right)^2\right)\right), & \\ \quad \left.\min\left(\exp\left(-\left(\frac{t-m_1}{b_1}\right)^2\right), \exp\left(-\left(\frac{t-m_3}{b_3}\right)^2\right)\right)\right), & t > m_3 \end{cases} \end{aligned} \tag{14}$$

Proof.

$$R_{k-outof-n}(t) = P_{oss}(X > t) = P_{oss}(X = \underset{2}{order}(X_1, X_2, X_3) > t) \tag{15}$$

There exist four mutually exclusive situations, and we have

$$\begin{aligned} P_{oss}(X > t) = &P_{oss}\{(X_1 \le t, X_2, X_3 > t) \cup (X_2 \le t, X_1, X_3 > t) \\ &\cup (X_3 \le t, X_1, X_2 > t) \cup (X_1, X_2, X_3 > t)\} \\ = &\max\{P_{oss}(X_1 \le t, X_2, X_3 > t), P_{oss}(X_2 \le t, X_1, X_3 > t), \\ &P_{oss}(X_3 \le t, X_1, X_2 > t), P_{oss}(X_1, X_2, X_3 > t)\} \end{aligned} \tag{16}$$

Because X_1, X_2, X_3 are mutually unrelated, and according to *Definition* 3 it follows that

$$\begin{aligned}&P_{oss}(X_1 \le t, X_2, X_3 > t)\\&\quad = \min(P_{oss}(X_1 \le t), P_{oss}(X_2 > t), P_{oss}(X_3 > t))\end{aligned} \tag{17}$$

$$\begin{aligned}&P_{oss}(X_2 \le t, X_1, X_3 > t)\\&\quad = \min(P_{oss}(X_2 \le t), P_{oss}(X_1 > t), P_{oss}(X_3 > t))\end{aligned} \tag{18}$$

$$\begin{aligned}&P_{oss}(X_3 \le t, X_1, X_2 > t)\\&\quad = \min(P_{oss}(X_3 \le t), P_{oss}(X_1 > t), P_{oss}(X_2 > t))\end{aligned} \tag{19}$$

$$\begin{aligned}&P_{oss}(X_1, X_2, X_3 > t)\\&\quad = \min(P_{oss}(X_1 > t), P_{oss}(X_2 > t), P_{oss}(X_3 > t))\end{aligned} \tag{20}$$

Further, substituting Eq. (8) and (9) into Eq. (17)–(20) , we have

$$\begin{aligned}P_{oss}(X > t) = \max\Big\{ &\min\Big(\sup_{u\le t}\mu_{X_1}(u), \sup_{u>t}\mu_{X_2}(u), \sup_{u>t}\mu_{X_3}(u)\Big),\\ &\min\Big(\sup_{u\le t}\mu_{X_2}(u), \sup_{u>t}\mu_{X_1}(u), \sup_{u>t}\mu_{X_3}(u)\Big),\\ &\min\Big(\sup_{u\le t}\mu_{X_3}(u), \sup_{u>t}\mu_{X_1}(u), \sup_{u>t}\mu_{X_2}(u)\Big),\\ &\min\Big(\sup_{u>t}\mu_{X_1}(u), \sup_{u>t}\mu_{X_2}(u), \sup_{u>t}\mu_{X_3}(u)\Big)\Big\}\end{aligned} \tag{21}$$

For $t \le a_1, a_1 < t \le a_2, a_2 < t \le a_3, t > a_3$, the corresponding posbist reliability can be obtained by substituting Eq. (8) and Eq. (9) into Eq. (21), and we then get Eq. (14). For the case $a_2 < t \le a_3$, we have

$$\begin{aligned}&P_{oss}(X > t)\\&\quad = \max\{\min(1, \mu_{X_2}(u), 1), \quad \min(1, \mu_{X_1}(u), 1),\\&\qquad\qquad \min(\mu_{X_3}(u), \mu_{X_1}(u), \mu_{X_2}(u)), \min(\mu_{X_1}(u), \mu_{X_2}(u), 1)\}\\&\quad = \max\{\mu_{X_1}(u), \mu_{X_2}(u)\}\end{aligned} \tag{22}$$

Thus, substituting Eq. (2) into Eq. (22), we have

$$P_{oss}(X > t) = \max\left(\exp\left(-\left(\frac{t-m_1}{b_1}\right)^2\right), \exp\left(-\left(\frac{t-m_2}{b_2}\right)^2\right)\right),$$
$$\text{for } m_2 < t \le m_3 \tag{23}$$

□

We now consider an n-component system with $X_1, X_2, \ldots, X_n$ representing the mutually unrelated, normally distributed, and strictly convex fuzzy variables with the same continuous possibility distribution function for each identical component. We then have

$$\mu_{X_1}(x) = \mu_{X_2}(x) = \cdots = \mu_{X_n}(x)$$

Thus, the posbist reliability distribution when one requires more than $k-1$ components in the working state (regarded as a k-out-of-n:G system) is given by

$$
\begin{aligned}
R_{k-outof-n}(t) &= \sup_{u>t} \mu_X(u) \\
&= \sup_{u>t} \mu_{X_1}(u) \\
&= \begin{cases} 1, & t \le a_1 \\ \mu_{X_1}(t), & t > a_1 \end{cases}
\end{aligned} \tag{24}
$$

When X_i is a Gaussion fuzzy variable, we have

$$
R_{k-outof-n}(t) = \begin{cases} 1, & t \le m_1 \\ \exp\left(-\left(\frac{t-m_1}{b_1}\right)^2\right), & t > m_1 \end{cases} \tag{25}
$$

and Eq. (25) is independent of the value of k.

In summary, we can see that the posbist reliability of a k-out-of-n:G system consisting of identically and independently distributed components has the same membership function as each component.

4 THE METHODS FOR DEVELOPING POSSIBILITY DISTRIBUTIONS

4.1 Possibility distributions based on membership functions

As Zadeh [18] pointed out, a possibility distribution can be viewed as a fuzzy set which serves as an elastic constraint on the values that may be assigned to a variable. Therefore, the possibility distribution numerically equals to the corresponding membership function, i.e.,

$$
\pi_X(x) = \mu_A(x), \tag{26}
$$

where X is a fuzzy variable and A is the fuzzy set induced by X.

Note that although a possibility distribution and a fuzzy set have a common mathematical expression, the underlying concepts are different. The fuzzy set A is a fuzzy value that can be assigned to a certain variable. However, the possibility constraint A is a fuzzy set of nonfuzzy values that can possibly be assigned to X.

According to the above-mentioned viewpoint, we can use the methods for constructing membership functions to generate the corresponding possibility distributions. Using Eq. (26) if the membership function of a fuzzy set has been obtained, the possibility distribution of the fuzzy variable of the fuzzy set is obtained too. Fuzzy statistics [20], transformation of probability distributions to membership function [21–23], heuristic methods [24], and expert opinions [21] are a few commonly used methods for generating membership functions.

With heuristic methods, we first select a predefined shape of the membership function to be developed. The specific parameters of the membership

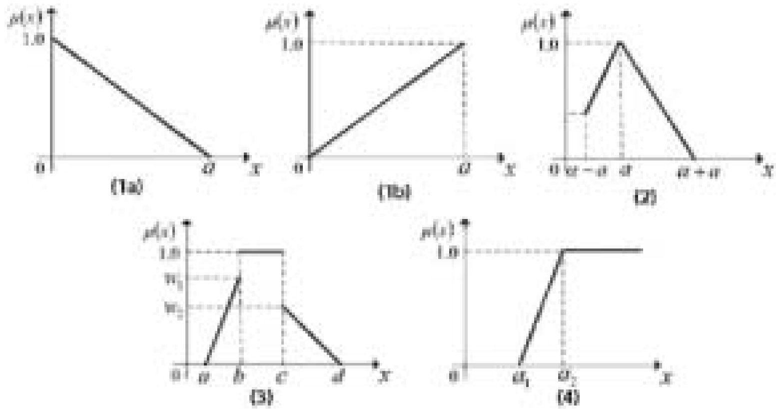

FIGURE 2
Piecewise linear membership functions.

function with the selected shape are determined from the data collected. In most real-life problems, the universe of discourse of the membership functions is the real number line R. The commonly used membership function shapes are the piecewise linear function and the piecewise monotonic function. Linear and piecewise linear membership functions have the advantages of reasonably smooth transitions and easy manipulation through fuzzy operations. However, the shapes of many heuristic membership functions are not flexible enough to model all kinds of data. Moreover, the parameters of the membership functions must be provided by experts. In many applications, the parameters need to be adjusted extensively to achieve a certain performance level.

A few commonly used piecewise linear functions are given below, and their corresponding figures are depicted in Figure 2.

(1) $\mu(x) = 1 - \frac{x}{a}$ or $\mu(x) = \frac{x}{a}$, where $x = [0, a]$;

(2) $$\mu(x) = \begin{cases} 1 - \frac{|a-x|}{\alpha}, & \alpha - a \le x \le \alpha + a \\ 0, & \text{otherwise} \end{cases};$$

(3) $$\mu(x) = \begin{cases} 0, & x \le a \\ w_1 \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ w_2 \frac{d-x}{d-c}, & c \le x \le d \\ 0, & x > d \end{cases};$$

(4) $$\mu(x) = \begin{cases} 0, & x < a_1 \\ \frac{x}{a_2-a_1} + \frac{a_1}{a_1-a_2}, & a_1 \le x \le a_2 \\ 1, & x > a_2 \end{cases}.$$

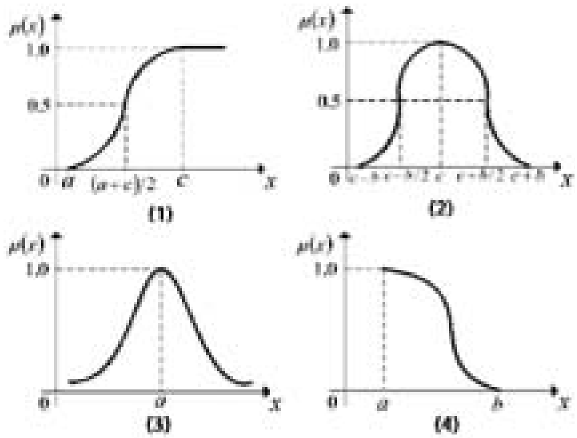

FIGURE 3
Piecewise monotonic membership functions.

Some commonly used piecewise monotonic functions are as follows with the figures shown in Figure 3.

(1) $s(x;a,b,c) = \begin{cases} 0, & x \le a \\ 2\left(\frac{x-a}{c-a}\right)^2, & a < x \le b \\ 1-2\left(\frac{x-a}{c-a}\right)^2, & b < x \le c \\ 1, & x > c \end{cases}$, where $b = \frac{a+c}{2}$;

(2) $\mu(x;a,b,c) = \begin{cases} s\left(x; c-b, c-\frac{b}{2}, c\right), & x \le c \\ 1 - s\left(x; c, c+\frac{b}{2}, c+b\right), & x > c \end{cases}$, where $b = \frac{a+c}{2}$;

(3) $\mu(x) = e^{-b(x-a)^2}, -\infty < x < +\infty$;

(4) $\mu(x) = \frac{1}{2} - \frac{1}{2}\sin\left(\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right), x \in [a,b]$.

In practical applications, we often combine fuzzy statistics with heuristic methods. First, the shape of the membership function is suggested by statistical data. Then, the suggested shape is compared with the predefined shape and the more appropriate ones are selected. Finally, the most suitable membership function is determined through practical tests.

In addition to the methods mentioned above, trichotomy [20], multiphase fuzzy statistics [20], and neural network based methods [24, 25] have been used in construction of membership functions. It should be pointed out that developing new methods for constructing membership functions is still a hot research topic. The reported methods for constructing membership functions are not as mature as those for constructing probability distribution functions.

Constructing membership functions still depends on experience and feedback from actual use and continuous revisions have to be made to achieve satisfactory results. This situation results in the immaturity of the methods for constructing possibility distributions.

4.2 Transformation of probability distributions to possibility distributions

The methods for transforming probability distributions to possibility distributions are based on the possibility/probability consistency principle. The possibility/probability consistency principle states:

If a variable X can take the value $u_1, \ldots, u_n$ with respective possibilities $\pi = (\pi(u_1), \ldots, \pi(u_n))$ and probabilities $p = (p(u_1), \ldots, p(u_n))$, then the degree of consistency of the probability distribution p with possibility distribution π is expressed by

$$C_z(\pi, p) = \sum_{i=1}^{n} \pi(u_i) p(u_i). \tag{27}$$

For more details on this principle, readers are referred to Ref. [18].

Directly transforming probability distribution into possibility distribution or fuzzy sets, such as the bijective transformation method [26], and the conservation of uncertainty method [27] are being argued [28].

4.3 Subjective manipulations of fatigue data

For products requiring high reliability, manufacturers often conduct laboratory tests to obtain a certain quantity of lifetime data. Due to budget limitations, it is usually difficult or impossible to obtain sufficient statistical data. Although the number of data points available may be too small for us to perform a statistical analysis, it may be sufficient for subjective estimation of the possibility distribution. If we have constructed a model of the possibilistic reliability of the device under study and derived the needed possibility distribution, we can perform a quantitative analysis of the possibilistic reliability of the device.

Assume that we have obtained fatigue life data of a device denoted by $(n_i^j)_{1 \le j \le N}$, $1 \le i \le M$, where M is the number of stress levels, N is the number of data points at each stress level. Then the mean fatigue life at stress level i can be expressed as

$$m_{n_i} = \frac{1}{N} \sum_{j=1}^{N} n_i^j, \; 1 \le i \le M. \tag{28}$$

The lifetime data at each stress level can be divided into two groups, that is,

$$G_1 = \left\{ n_i^j, \; j = 1, 2, \ldots, N | n_i^j < m_{n_i} \right\}, \tag{29}$$

$$G_2 = \left\{ n_i^j, \; j = 1, 2, \ldots, N | n_i^j > m_{n_i} \right\}. \tag{30}$$

The mean value m_{n_i} is assigned a possibility degree of 1 and the possibility degree of 0.5 is assigned to the means of the lifetime data in the two groups G_1 and G_2, that is,

$$m_{l_{n_i}} = \frac{1}{\#(G_1)} \sum_{n_i^j \in G_1} n_i^j, \quad \pi_{n_i}(m_{l_{n_i}}) = 0.5, \quad i = 1, 2, \ldots, M \tag{31}$$

$$m_{Y_{n_i}} = \frac{1}{\#(G_2)} \sum_{n_i^j \in G_2} n_i^j, \quad \pi_{n_i}(m_{Y_{n_i}}) = 0.5, \quad i = 1, 2, \ldots, M \tag{32}$$

where $\#(\cdot)$ denotes the number of data points in a set.

By use of the above-mentioned analysis, we can express the $L - R$ type possibility distribution of fatigue lifetime as follows:

$$\pi_{n_i}\left(n_i^j\right) = \begin{cases} L\left(\frac{m_{n_i} - n_i^j}{\alpha_{n_i}}\right), & n_i^j \le m_{n_i} \\ R\left(\frac{n_i^j - m_{n_i}}{\beta_{n_i}}\right), & n_i^j > m_{n_i} \end{cases}, \tag{33}$$

where $\alpha_{n_i} = \frac{m_{n_i} - m_{l_{n_i}}}{L^{-1}(0.5)}$ and $\beta_{n_i} = \frac{m_{r_{n_i}} - m_{n_i}}{L^{-1}(0.5)}$.

Considering the various types of $L - R$ type possibility distributions mentioned earlier in this paper, we can use Eq. (33) to get specific possibility distributions to represent fatigue lifetime data. For example, the following triangular possibility distribution may be used to represent fatigue lifetime data:

$$\pi_{n_i}\left(n_i^j\right) = \begin{cases} 0, & n_i^j \le m_{n_i} - \alpha_{n_i} \\ 1 - \frac{m_{n_i} - n_i^j}{\alpha_{n_i}}, & m_{n_i} - \alpha_{n_i} \le n_i^j \le m_{n_i} \\ 1 - \frac{n_i^j - m_{n_i}}{\beta_{n_i}}, & m_{n_i} \le n_i^j \le m_{n_i} + \beta_{n_i} \\ 0, & m_{n_i} + \beta_{n_i} \le n_i^j \end{cases}, \tag{34}$$

where $\alpha_{n_i} = 2(m_{n_i} - m_{l_{n_i}})$ and $\beta_{n_i} = 2(-m_{n_i} + m_{r_{n_i}})$.

Similarly, we may use the following Gaussian possibility distribution to represent fatigue lifetime data:

$$\pi_{n_i}\left(n_i^j\right) = \begin{cases} \exp\left[-\left(\frac{m_{n_i} - n_i^j}{\alpha_{n_i}}\right)^2\right], & n_i^j \le m_{n_i} \\ \exp\left[-\left(\frac{n_i^j - m_{n_i}}{\beta_{n_i}}\right)^2\right], & n_i^j > m_{n_i} \end{cases}, \tag{35}$$

where $\alpha_{n_i} = \frac{m_{n_i} - m_{l_{n_i}}}{\sqrt{\ln 0.5}}$ and $\beta_{n_i} = \frac{m_{r_{n_i}} - m_{n_i}}{\sqrt{\ln 0.5}}$.

5 EXAMPLES

5.1 Example 1

To illustrate the methodology proposed in the previous section, we present an example of calculating the posbist reliability of a power generating system with its generator units connected in 2-out-of-3:G. We also assume that the lifetime of each component is a Gaussian fuzzy variable as follows,

$$\mu_{X_1}(t) = \begin{cases} \exp\left(-\left(\frac{5.0-t}{2.0}\right)^2\right), & t \leq 5.0 \\ \exp\left(-\left(\frac{t-5.0}{2.0}\right)^2\right), & t > 5.0 \end{cases}$$

$$\mu_{X_2}(t) = \begin{cases} \exp\left(-\left(\frac{5.8-t}{1.8}\right)^2\right), & t \leq 5.8 \\ \exp\left(-\left(\frac{t-5.8}{1.8}\right)^2\right), & t > 5.8 \end{cases}$$

$$\mu_{X_3}(t) = \begin{cases} \exp\left(-\left(\frac{6.5-t}{1.5}\right)^2\right), & t \leq 6.5 \\ \exp\left(-\left(\frac{t-6.5}{1.5}\right)^2\right), & t > 6.5 \end{cases}$$

and the metric of time unit is year.

Using Eq.(14), we can express the posbist reliability of the 2-out-of-3:G system as

$$\begin{aligned} &R_{k-outof-n}(t) \\ &\quad = \sup_{u>t} \mu_X(u) \\ &\quad = \begin{cases} 1, & t \leq 5.8 \\ \max\left(\exp\left(-\left(\frac{t-5.0}{2.0}\right)^2\right), \exp\left(-\left(\frac{t-5.8}{1.8}\right)^2\right)\right), & 5.8 < t \leq 6.5 \\ \max\left(\min\left(\exp\left(-\left(\frac{t-5.0}{2.0}\right)^2\right), \exp\left(-\left(\frac{t-5.8}{1.8}\right)^2\right)\right),\right. & \\ \quad \min\left(\exp\left(-\left(\frac{t-5.8}{1.8}\right)^2\right), \exp\left(-\left(\frac{t-6.5}{1.5}\right)^2\right)\right), & \\ \quad \left.\min\left(\exp\left(-\left(\frac{t-5.0}{2.0}\right)^2\right), \exp\left(-\left(\frac{t-6.5}{1.5}\right)^2\right)\right)\right), & t > 6.5 \end{cases} \end{aligned}$$

When $t = 6.0$ *years*, we have:

$$R_{k-outof-n}(6.0) = \exp\left(-\left(\frac{t-5.8}{1.8}\right)^2\right) = 0.9877.$$

5.2 Example 2

We illustrate the method presented in Section 4.3 for constructing the possibility distribution of the fatigue lifetime data. A fatigue test was conducted

	Loading torque levels S (N · m)		
Data points	$S_1 = 3800$	$S_2 = 4000$	$S_3 = 4200$
1	1.4321	0.4395	0.2932
2	1.5381	0.4572	0.3108
3	1.6362	0.4863	0.3159
4	1.6860	0.5021	0.3237
5	1.7205	0.5158	0.3412

TABLE 1
The fatigue lifetime data of crankshaft (in units of 10^6 loading cycles)

to investigate the reliability of crankshaft in a diesel engine. Three loading torque levels were used. The collected data of fatigue lifetime are shown in Table 1. Five data points at each torque level are given in Table 1 and they are sufficient for subjective estimation of the possibility distribution of fatigue lifetime.

First, we calculate the average fatigue lifetime at each torque level m_{n_i}. We will illustrate the procedure for constructing the possibility distribution of the lifetime using the data collected at the second torque level. The same procedure should be followed for analysis of data at other torque levels. Using Eq. (28) and Table 1, we have

$$\begin{aligned}
m_{n_1} &= \frac{1}{N}\sum_{j=1}^{N} n_1^j = \frac{1}{5}\sum_{j=1}^{5} n_1^j \\
&= \frac{1}{5}(0.4395 + 0.4572 + 0.4863 + 0.5021 + 0.5158) \\
&= 0.4802, \\
\pi_{n_1}(m_{n_1} &= 0.4802) = 1.
\end{aligned}$$

The data points at the second torque level are divided into two groups separated by the calculated mean value m_{n_1}, i.e.,

$$\begin{aligned}
G_1 &= \left\{0.4395, 0.4572 | n_1^j < m_{n_1}\right\}, \\
G_2 &= \left\{0.4863, 0.5021, 0.5158 | n_1^j > m_{n_1}\right\}.
\end{aligned}$$

Further, from Eqs. (70) and (71), we have

$$\begin{aligned}
m_{l_{n_1}} &= \frac{1}{\#(G_1)}\sum_{n_1^j \in G_1} n_1^j = \frac{1}{2}(0.4395 + 0.4572) = 0.4484, \\
\pi_{n_1}(m_{l_{n_1}} &= 0.4484) = 0.5;
\end{aligned}$$

$$m_{r_{n_1}} = \frac{1}{\#(G_2)} \sum_{n_1^j \in G_2} n_1^j = \frac{1}{3}(0.4863 + 0.5021 + 0.5158) = 0.5014,$$

$$\pi_{n_1}(m_{r_{n_1}} = 0.5014) = 0.5.$$

Finally, with these calculated results and Eq. (34), we can construct the triangular possibility distribution of the fatigue lifetime of crankshaft under the torque level of $4000\mathrm{N} \cdot \mathrm{m}$ as follows:

$$\alpha_{n_1} = 2\left(m_{n_1} - m_{l_{n_1}}\right) = 2(0.4802 - 0.4484) = 0.0636,$$

$$\beta_{n_1} = 2\left(m_{r_{n_1}} - m_{n_1}\right) = 2(0.5014 - 0.4802) = 0.0212,$$

$$\pi_{n_1}\left(n_1^j\right) = \begin{cases} 0, & n_1^j \le m_{n_1} - \alpha_{n_1} \\ 1 - \frac{m_{n_1} - n_1^j}{\alpha_{n_1}}, & m_{n_1} - \alpha_{n_1} \le n_1^j \le m_{n_1} \\ 1 - \frac{n_1^j - m_{n_1}}{\beta_{n_1}}, & m_{n_1} \le n_1^j \le m_{n_1} + \beta_{n_1} \\ 0, & m_{n_1} + \beta_{n_1} \le n_1^j \end{cases}$$

$$= \begin{cases} 0, & n_1^j \le 0.4166 \\ 1 - \frac{0.4802 - n_1^j}{0.0636}, & 0.4166 < n_1^j \le 0.4802 \\ 1 - \frac{n_1^j - 0.4802}{0.0212}, & 0.4802 < n_1^j \le 0.5014 \\ 0, & 0.5014 < n_1^j \end{cases}$$

The shape of this obtained possibility distribution is shown in Figure 4.

Note that the procedure for constructing the possibility distributions at other torque levels is the same. If we are interested in constructing other $L - R$ types of possibility distributions such as the Gaussian possibility distribution for the fatigue lifetime data, a similar procedure can be followed.

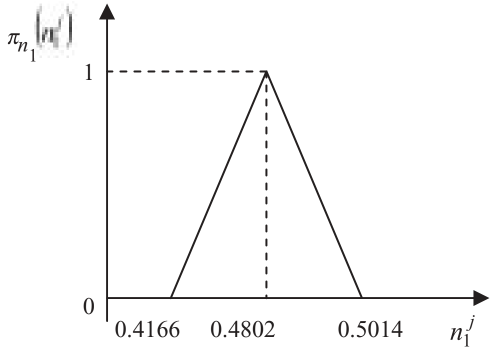

FIGURE 4
The possibility distribution of the fatigue lifetime at the second torque level.

After obtaining the possibility distribution of fatigue lifetime of the crankshaft, we can derive the possibilistic reliability of crankshaft at any time according to posbist reliability theory [8], e.g., under the torque level of 4000N · m, we can figure out the possibilistic reliability as follows:

$$R(t) = P_{oss}(X > t) = \sup_{x>t} P_{oss}(X = x) = \sup_{n_1^j > t} \pi_{n_1}(n_1^j)$$

$$= \begin{cases} 1, & t \leq 0.4802 \\ 1 - \frac{t-0.4802}{0.0212}, & 0.4802 < t \leq 0.5014 \\ 0, & t > 0.5014 \end{cases}.$$

6 CONCLUSIONS

Although the conventional reliability theory has been the dominant tool for evaluating system safety and analyzing failure uncertainty, the uncertainty within a system and its components cannot be always defined in the framework of probability. To analyze highly complex systems and deal with the vast variations of system characteristics, researchers have realized that the probability theory is not a panacea. Based on the posbist reliability theory, the lifetime of a system is considered to be a Gaussian fuzzy variable. The posbist reliability of k-out-of-n:G systems is derived. The universe of discourse on system lifetime defined in Ref.[8, 9] is expanded from $(0, +\infty)$ to $(-\infty, +\infty)$. We have illustrated in Section 3 that this expansion does not affect the nature of the problems to be solved. On the contrary, it makes the proofs in [8, 9] much more straightforward and the complexity of calculation is greatly reduced. In this paper, we addressed the critical problem in the posbist reliability theory which is the construction of the possibility distribution and pointed out that all methods for generating membership functions can be used to construct the corresponding possibility distributions. We also presented a new method for constructing the possibility distribution with the possibilistic reliability analysis of fatigue lifetime of mechanical parts. The methods for constructing possibility distributions are not as mature as those for constructing probability distributions. The present paper has provided a concise overview of the methods for constructing the possibility distributions in posbist reliability analysis. Further research is needed to develop a more general method for constructing possibility distributions, and how to transform data within Likert's and Osgood's scales into possibility context.

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