Yu PANG
Hong-Zhong HUANG
Liping HE
Zhonglai WANG
Ning-Cong XIAO

CONVEX SUBLATTICE BASED RELIABILITY THEORY

TEORIA NIEZAWODNOŚCI OPARTA NA POJĘCIU PODKRATY WYPUKŁEJ

Classical probability theory has been widely used in reliability analysis; however, it is hard to handle when the system is lack of adequate and sufficient data. Nowadays, alternative approaches such as possibility theory and fuzzy set theory have also been proposed to analyze vagueness and epistemic uncertainty regarding reliability aspects of complex and large systems. The model presented in this paper is based upon possibility theory and multistate assumption. Convex sublattice is addressed on congruence relation regarding the complete lattice of structure functions. The relations between the equivalence classes on the congruence relation and the set of all structure functions are established. Furthermore, important reliability bounds can be derived under the notion of convex sublattice. Finally, a numerical example is given to illustrate the results.

Keywords: *congruence relation, convex sublattice, lattice theory, multistate structure function, possibility theory, upper bound set.*

Klasyczna teoria prawdopodobieństwa ma szerokie zastosowanie w analizie niezawodności, jednak trudno jest się nią posługiwać, kiedy brak jest wystarczających i odpowiednich danych na temat systemu. Obecnie, proponuje się alternatywne podejścia, takie jak teoria możliwości czy teoria zbiorów rozmytych, za pomocą których można analizować niepewność epistemiczną oraz nieostrość w odniesieniu do aspektów niezawodności złożonych i dużych systemów. Model przedstawiony w niniejszym artykule oparto na teorii możliwości oraz na założeniu wielostanowości. Podkratę wklęsłą opisano na relacji kongruencji, odnoszącej się do całej kraty funkcji struktury. Ustalono relacje pomiędzy klasami równoważności na relacji kongruencji a zbiorem wszystkich funkcji struktury. Ponadto posługując się pojęciem podkraty wypukłej można wyprowadzać istotne kresy niezawodności. Wyniki zilustrowano przykładem numerycznym.

Słowa kluczowe: *relacja kongruencji, podkrata wypukła, teoria krat, wielostanowa funkcja struktury, teoria możliwości, górny kres zbioru.*

1. Introduction

The classical reliability theory is based upon binary structure functions and probability theory [19, 21]. In the binary probabilistic approach, the component state and system state may be assumed to be either perfectly functioning or completely failed, which is an oversimplification of reality [6]. The increasing complexity of real systems has brought the emergent need of intermediate states. With this background, the theory of multistate structure functions was proposed to overcome the problem [14, 15]. Moreover, in many real life cases, adequate statistical data is unavailable to obtain due to the limitation of experimental conditions [13]. Probability theory is shown not the only possible way of representing imprecision and uncertainty [7]. In fact, possibility theory has played a vital role in analyzing system uncertainty [8, 12, 17]. The models for reliability estimation studied from a non-probabilistic point of view are proposed to overcome the problems of approach in past literatures [1, 9, 10, 18, 20].

In order to better represent the system or component state space, lattice theory is essential in mathematical modelling using non-classical reliability theory [16]. By considering the complete lattice of a structure function, a general framework has given us a better foundation of reliability analysis [2, 4]. Cappelle [3] presented a theory of multistate structure functions on partially ordered sets (in casu complete lattices), which is able to solve several problems arising from the dichotomous model. Based on a combination of multistate structure functions and possibility theory, Cappelle and Kerre [7] derived a congruence relation on the complete lattice of structure functions which links several concepts and provides powerful tools to model physical systems. Based upon the congruence relation proposed by Cappelle and Kerre, the concept of convex sublattice is presented in reliability analysis in this paper. According to the convex sublattice properties, the upper (lower) bound set of structure functions on equivalence relations regarding the congruence relation is addressed to go along with the practical engineering. Given an equivalence class on structure functions, it can be verified that the upper (lower) bound set of the equivalence class is a convex lattice. Thus, several important boundaries of the structure function set are employed. Furthermore, the significance of the definitions and properties are explained, both from theoretical and practical point of view.

This paper is organized as follows. In the next section, preliminary definitions such as structure functions and congruence relations are briefly reviewed. In Section 3, the notion of a convex sublattice is applied to reliability theory, along with the explanation of how the theorems and properties can be used in practical engineering. Afterwards, a numerical example is addressed in Section 4 to exemplify the usefulness of the introduced concepts. As a result, some conclusions are employed in Section 5.

2. Preliminary definitions

In this section, three useful notions regarding the theory of multistate structure functions on complete lattices are introduced. Considering that systems are with a finite number of components, we first give the concept of structure function, which can reflect the functional relationship between components states and system state.

Definition 1 [3] Let $(L_i,\leq)$, $1\leq i\leq n$, and $(L,\leq)$ be $n+1$ complete lattices. An $L_1\times\ldots\times L_n - L$-mapping ϕ, satisfying

(i) $\phi(0,\ldots,0) = 0$ and $\phi(1,\ldots,1) = 1$ (1)

(ii) ϕ is isotone, that is

$$\left(\forall(\mathbf{x},\mathbf{y})\in(L_1\times\cdots\times L_n)^2\right)\left(\mathbf{x}\leq\mathbf{y}\Rightarrow\phi(\mathbf{x})\leq\phi(\mathbf{y})\right) \quad (2)$$

is called to be a *structure function* from $(L_1\times\ldots\times L_n,\leq)$ to $(L,\leq)$.

$\mathcal{M}(L_1\times\ldots\times L_n, L)$ denotes the set of all the structure functions from complete lattice $(L_1\times\ldots\times L_n,\leq)$ to complete lattice $(L,\leq)$. The order relationship $\preceq$ is defined as follows: for any two $L_1\times\ldots\times L_n - L$ structure functions ϕ_1 and ϕ_2,

$$\phi_1\preceq\phi_2\Leftrightarrow(\forall\mathbf{x}\in L_1\times\cdots\times L_n)(\phi_1(\mathbf{x})\leq\phi_2(\mathbf{x})) \quad (3)$$

More properties of the complete lattice of structure functions will not be introduced here. For more details, the readers are referred to [3]. In the sequel, a core notion of congruence relation is addressed. All the equivalence classes employed in this paper are based upon the congruence relation.

Definition 2 [11] Let $(L,\leq)$ be a lattice and θ a binary relation on L; θ is a *congruence relation* if and only if

(i) θ is an equivalence relation on L,

(ii) for any elements x_1, x_2, y_1 and y_2 of L

$$x_1\in[y_1]_\theta \text{ and } x_2\in[y_2]_\theta\Rightarrow$$

$$\Rightarrow x_1\wedge x_2\in[y_1\wedge y_2]_\theta \text{ and } x_1\vee x_2\in[y_1\vee y_2]_\theta \quad (4)$$

In this definition, $[x]_\theta$ is the equivalence class of θ which is generated by x. The infimum (supremum) operator is denoted by $\wedge(\vee)$ on the lattice $(L,\leq)$, meanwhile denoted by $\cap(\cup)$ on the set of structure functions, that is, for any two structure functions ϕ_1 and ϕ_2,

$$\phi_1\cap\phi_2 : L_1\times\cdots\times L_n\to L:\mathbf{x}\mapsto\phi_1(\mathbf{x})\wedge\phi_2(\mathbf{x}) \quad (5)$$

The operation $\cup$ can be defined analogously. The subset S of the lattice L is called convex iff $a,b\in S$, $c\in L$, and $a\leq c\leq b$ imply that $c\in S$. Since the intersection of any number of convex sublattice is a convex sublattice unless void, the definition of convex sublattice is generated by a subset [11].

Definition 3[11] Let $(L,\leq)$ be a lattice and S a subset of L, S is a convex sublattice of L if and only if

$$(\forall a,b\in S)([a\wedge b,a\vee b]\subseteq S) \quad (6)$$

For $a,b\in L$, $a\leq b$, the interval $[a,b]=\{x|a\leq x\leq b\}$ is an important example of a convex sublattice. For a chain C, $a,b\in C$, $a\leq b$, the half-open intervals: $(a,b]=\{x|a<x\leq b\}$ and $[a,b)=\{x|a\leq x<b\}$, and the open interval: $(a,b)=\{x|a<x<b\}$, whenever nonvoid, are examples of convex sublattices.

3. Convex sublattice concept applied in reliability theory

Regarding the definition of convex sublattice, some interesting results are proposed to show how the convex sublattice concept is related to reliability theory in this section. First, two preliminary results, which are proposed by Cappelle and Kerre [7], are employed as lemmas. Then, three main theorems and one property are addressed with detailed proof. As a result, the significance of theoretic concepts applied in practical reliability engineering is addressed.

3.1. Preliminary results

The lemmas presented in this part are as a foundation of the main theoretical results. A typical equivalence class of structure functions is addressed in Lemma 1. On the basis of this equivalence class, different subsets result in different observations.

Lemma 1 [7] Let A be a subset of $L_1\times\ldots\times L_n$, ϕ and φ two arbitrary structure functions from $(L_1\times\ldots\times L_n,\leq)$ to $(L,\leq)$, then

$$\varphi\in[\phi]_{\theta_A}\Leftrightarrow(\forall\mathbf{x}\in A)(\varphi(\mathbf{x})=\phi(\mathbf{x})) \quad (7)$$

Lemma 2 [7] Let A and B be two subsets of $L_1\times\ldots\times L_n$ and ϕ a structure function from $(L_1\times\ldots\times L_n,\leq)$ to $(L,\leq)$, then

$$A\subseteq B\Rightarrow[\phi]_{\theta_B}\subseteq[\phi]_{\theta_A} \quad (8)$$

Lemma 2 can be intuitively understood from fig.1. That is, more observation will result in a smaller number of appropriate structure functions that meet with given information.

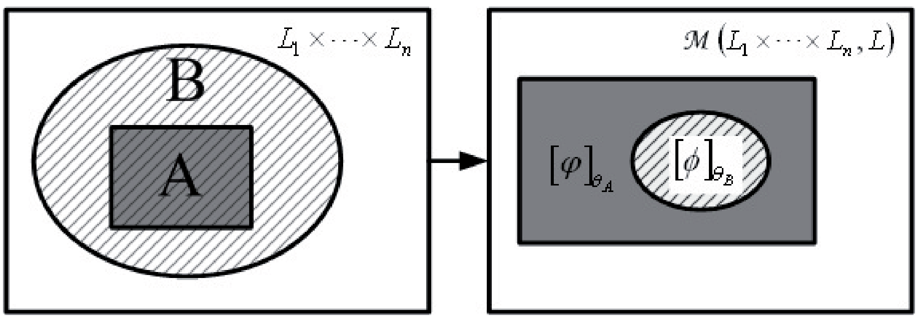

Fig.1. Relations between subsets and the relating equivalence class

3.2. Main theoretical results

All the theorems addressed in this part will provide us with several typical convex sublattices of ($\mathcal{M}(L_1\times...\times L_n,), \preceq$), which are with significant meaning in engineering application.

Theorem 1 Let A be a subset of $L_1\times...\times L_n$ and ϕ a structure function from $(L_1\times...\times L_n,\le)$ to $(L,\le)$, then $\left([\phi]_{\theta_A},\prec\right)$ is a convex sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$.

Proof: Let ϕ_i and ϕ_j be two arbitrary elements of $[\phi]_{\theta_A}$, it can be addressed from Lemma 1 that

$$\left(\forall \mathbf{x}\in A\right)\left(\phi_i(\mathbf{x})=\phi(\mathbf{x})=\phi_j(\mathbf{x})\right) \tag{9}$$

Thus,

$$\left(\forall \mathbf{x}\in A\right)\left((\phi_i\cap\phi_j)(\mathbf{x})=\phi(\mathbf{x})=(\phi_i\cup\phi_j)(\mathbf{x})\right) \tag{10}$$

Let $\phi^{'}$ be any element belongs to $\left[\phi_i\cap\phi_j,\phi_i\cup\phi_j\right]$, then

$$\phi_i\cap\phi_j\preceq\phi^{'}\preceq\phi_i\cup\phi_j \tag{11}$$

Hence,

$$\left(\forall \mathbf{x}\in L_1\times\cdots\times L_n\right)\left((\phi_i\cap\phi_j)(\mathbf{x})\le\phi^{'}(\mathbf{x})\le(\phi_i\cup\phi_j)(\mathbf{x})\right) \tag{12}$$

which leads to

$$\left(\forall \mathbf{x}\in A\right)\left(\phi^{'}(\mathbf{x})=\phi(\mathbf{x})\right) \tag{13}$$

or equivalently

$$\left(\forall\phi^{'}\in\left[\phi_i\cap\phi_j,\phi_i\cup\phi_j\right]\right)\left(\phi^{'}(\mathbf{x})\in[\phi]_{\theta_A}\right) \tag{14}$$

Then, we can get

$$\left(\forall\phi_i,\phi_j\in[\phi]_{\theta_A}\right)\left(\left[\phi_i\cap\phi_j,\phi_i\cup\phi_j\right]\subseteq[\phi]_{\theta_A}\right) \tag{15}$$

Taking Definition 3 into account, the theorem is deduced. $\left([\phi]_{\theta_A},\prec\right)$ is a convex sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$ based on the equivalence relation θ_A. The lattices which are presented in the following two theorems are on the basis of equivalence class $[\phi]_{\theta_A}$.

Theorem 2 Let A be a subset of $L_1\times...\times L_n$, ϕ a structure function from $(L_1\times...\times L_n,\le)$ to $(L,\le)$ and M_aA (M_iA) denote the upper (lower) bound set of $[\phi]_{\theta_A}$ within $\mathcal{M}$, then $(M_aA,\le)$ $((M_iA,\le))$ is a complete sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$.

Proof: Only the proof of upper bound set M_aA a complete sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$ is given here. The results about the lower bound set M_iA can be proved analogously.

Let $(\phi_i|i\in I)$ be a non-empty family in M_aA, then

$$\left(\forall i\in I\right)\left(\forall \mathbf{x}\in A\right)\left(\phi_i(\mathbf{x})\ge\phi(\mathbf{x})\right) \tag{16}$$

Thus,

$$\left(\forall \mathbf{x}\in A\right)\left(\inf_{i\in I}\phi_i(\mathbf{x})\ge\phi(\mathbf{x})\right)\text{ and }\left(\forall \mathbf{x}\in A\right)\left(\sup_{i\in I}\phi_i(\mathbf{x})\ge\phi(\mathbf{x})\right) \tag{17}$$

Since both $\inf_{i\in I}\phi_i(\mathbf{x})$ and $\sup_{i\in I}\phi_i(\mathbf{x})$ are belonged to M_aA, $(M_aA,\le)$ is a complete sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$.

Theorem 3 Let A be a subset of $L_1\times...\times L_n$, ϕ a structure function from $(L_1\times...\times L_n,\le)$ to $(L,\le)$ and M_aA (M_iA) denote the upper (lower) bound set of $[\phi]_{\theta_A}$ within $\mathcal{M}$, then $(M_aA,\le)$ $((M_iA,\le))$ is a convex sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$.

Proof: As is proved in theorem 2, only the proof of upper bound set M_aA a convex sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$ is addressed here.

According to Definition 3, we must prove that

$$\left(\forall\phi_i,\phi_j\in M_aA\right)\left(\left[\phi_i\cap\phi_j,\phi_i\cup\phi_j\right]\subseteq M_aA\right) \tag{18}$$

Since for any structure function $\phi_m\in[\phi]_{\theta_A}$,

$$\left(\forall \mathbf{x}\in A\right)\left(\phi_m(\mathbf{x})=\phi(\mathbf{x})\right) \tag{19}$$

Furthermore, for any $\phi_i,\phi_j\in M_aA$ and any $x\in L_1\times...\times L_n$,

$$\phi_i(\mathbf{x})\ge\phi_m(\mathbf{x}),\quad\phi_j(\mathbf{x})\ge\phi_m(\mathbf{x}) \tag{20}$$

Hence,

$$\left(\forall \mathbf{x}\in L_1\times\cdots\times L_n\right)\left((\phi_i\cap\phi_j)(\mathbf{x})\ge\phi_m(\mathbf{x}),\ (\phi_i\cup\phi_j)(\mathbf{x})\ge\phi_m(\mathbf{x})\right) \tag{21}$$

which leads to that for any $\phi^{'}\in\left[\phi_i\cap\phi_j,\phi_i\cup\phi_j\right]$,

$$\left(\forall \mathbf{x}\in L_1\times\cdots\times L_n\right)\left(\phi^{'}(\mathbf{x})\ge\phi_m(\mathbf{x})\right) \tag{22}$$

It is obvious that $\phi^{'}$ is an upper bound of $[\phi]_{\theta_A}$, that is $\phi^{'}\in M_aA$.

Thus, it can be obtained that (1) holds from the selection of $\phi^{'}$.

It turns out that $(M_aA,\le)$ $((M_iA,\le))$ is a convex sublattice of $(\mathcal{M}(L_1\times...\times L_n,L),\preceq)$, and complete sublattice at the meantime. That is to say, the upper (lower) bound set of $[\phi]_{\theta_A}$ within $\mathcal{M}$ exists and can be figured out. Adding subset B of $L_1\times...\times L_n$, more interesting results can be figured out in the following.

Corollary 1 Let A and B be two subsets of $L_1\times...\times L_n$ ($A\subseteq B$) and ϕ a structure function from $(L_1\times...\times L_n,\le)$ to $(L,\le)$, then the upper (lower) bound set of $[\phi]_{\theta_B}$ within $[\phi]_{\theta_A}$ is a convex sublattice of $\left([\phi]_{\theta_A},\preceq\right)$.

Proof: Immediate from Theorem 3 and Lemma 2.

Property 1 Let A be a subset of $L_1\times...\times L_n$, ϕ a structure function from $(L_1\times...\times L_n,\le)$ to $(L,\le)$ and $M_aA(M_iA)$ denote the upper (lower) bound set of $[\phi]_{\theta_A}$ within $\mathcal{M}$, then (i) the maximum and minimum of $(M_aA,\le)$ is the supremum of $(\mathcal{M},\le)$ and $\left([\phi]_{\theta_A},\le\right)$, respectively; (ii) the maximum and minimum of $(M_iA,\le)$ is the infimum of $(\mathcal{M},\le)$ and $\left([\phi]_{\theta_A},\le\right)$, respectively.

Proof: There are two parts in statement (i):

1) The maximum of $(M_aA,\le)$ is the supremum of $(\mathcal{M},\le)$;
2) The minimum of $(M_aA,\le)$ is the supremum of $\left([\phi]_{\theta_A},\le\right)$

Let φ denote the supremum of $(\mathcal{M},\le)$. According to $[\phi]_{\theta_A}\subseteq\mathcal{M}$, it can be immediately obtained that φ is an upper bound of $\left([\phi]_{\theta_A},\le\right)$, that is $\varphi\in M_aA$. For $\forall\eta\in M_aA$, η is an upper bound of $\left([\phi]_{\theta_A},\le\right)$ within $(\mathcal{M},\le)$, then $\eta\in(\mathcal{M},\le)$. Based on the denotation of φ, $\varphi(\mathbf{x})\ge\eta(\mathbf{x})$ holds for $\forall\ \mathbf{x}\in L_1\times\cdots\times L_n$. Thus, φ is an upper bound of $(M_aA,\le)$. It can be deduced that φ is the maximum of $(M_aA,\le)$.

The other statements can be addressed in a similar way.

Corollary 2 Let A and B be two subsets of $L_1\times...\times L_n$ $(A\subseteq B)$, ϕ a structure function from$(L_1\times...\times L_n,\le)$ to $(L,\le)$ and $M_aB(M_lB)$ denote the upper (lower) bound set of $[\phi]_{\theta_B}$ within $[\phi]_{\theta_A}$, then (i) maximum and minimum of $(M_aB,\le)$ is the supremum of $([\phi]_{\theta_A},\le)$ and $([\phi]_{\theta_B},\le)$, respectively; (ii) maximum and minimum of $(M_lB,\le)$ is the infimum of $([\phi]_{\theta_B},\le)$ and $([\phi]_{\theta_A},\le)$, respectively.

Proof: Immediate from Property 1 and Lemma 2.

In the preceding paragraphs, main theoretical results have been addressed, together with the boundary of bound set. It will be shown how to apply these results of convex sublattices to actual problems.

3.3. Explanations and discussions

In real life situations, it is necessary to estimate structure functions. How can we narrow the scope of appropriate structure functions from a set of observation? Mathematically, considering a subset A of $L_1\times...\times L_n$, set $A_\phi=\{(\mathbf{x},y)|\mathbf{x}\in A\}$ is called an observation set of ϕ in which $\phi(\mathbf{x})=y$. Thus, an ordered couple $(\mathbf{x},y)$ is called an observation, which is an element of A_ϕ [5]. As a matter of fact, it is rarely possible to investigate all the observations. Suppose that system state space is presented as a limited amount of elements of $L_1\times...\times L_n$, denoted by A, thus the set of observation A_ϕ is determined. Additionally, given the observation A_ϕ, $[\phi]_{\theta_A}$ represents the equivalence class of structure functions which satisfy Equation (7). Hence, the bounds of set $[\phi]_{\theta_A}$ can be figured out. Based on the determined observations, engineers are always fond of the structure functions superior to any in $[\phi]_{\theta_A}$. In fact, for any $\mathbf{x}\in A$, $\mathbf{x}$ denotes the state vector of n subsystems (components) and different structure function corresponds to a different system structure. As for the same state vector of n subsystems (components), for instance, parallel and series system may lead to different results of system state. This system structure can be represented by the structure function. Undoubtedly, people are willing to find structure for system which can be under better state based on the same subsystem (component) state. This is why it is essential to study the upper bound set of $[\phi]_{\theta_A}$. According to the order relation within the set of all the structure functions, those are superior to any in $[\phi]_{\theta_A}$ should be superior to any element of the upper bound set of $[\phi]_{\theta_A}$.

It can be proven from Theorem 2 that both the supremum and the infimum of the upper bound set of $[\phi]_{\theta_A}$ exist. It is indicated in Theorem 3 that any structure function situated between the supremum and the infimum is an upper bound of $[\phi]_{\theta_A}$. Furthermore, the lower and upper bound of M_aA can be substituted and transformed through Property 1, which will result in useful bounds. Given a data of subsystem (component) state, good structure function is capable of leading to a good system state. Engineers are able to compare the characteristics between the examining structure function and those within $[\phi]_{\theta_A}$. Comparison of the examining structure function and those within $[\phi]_{\theta_A}$ is directly converted to the comparison of the examining structure function and the infimum of the upper bound of $[\phi]_{\theta_A}$ or the supremum of the lower bound of $[\phi]_{\theta_A}$. Consider a structure function ϕ from $([0,1]^2,\le)$ to $([0,1],\le)$, $\phi:[0,1]^2\to[0,1]:(x_1,x_2)\mapsto$ $\mapsto\dfrac{x_1+x_2}{2}$, Let A be the set of $\left\{(0,0),\left(\dfrac{1}{2},\dfrac{1}{2}\right),(1,1)\right\}$, it can be figured out that $\varphi_1(x_1,x_2)=\min(x_1,x_2)$ and $\varphi_2(x_1,x_2)=\max(x_1,x_2)$ is the supremum of the lower bound of $[\phi]_{\theta_A}$ and the infimum of the upper bound of $[\phi]_{\theta_A}$, respectively. Thus, given the examining structure function $\varphi(x_1,x_2)=x_1\cdot x_2$, it is easy to find out that both φ_1 and φ_2 are superior to φ. Therefore, comparison of the examining structure function and those within $[\phi]_{\theta_A}$ is given.

4. Numerical example

In this section, the numerical example in Ref. [4] is used to illustrate the results in Section 3.

Consider a structure function ϕ from $([0,1]^2,\le)$ to $([0,1],\le)$,

$$\phi:[0,1]^2\to[0,1]:(x_1,x_2)\mapsto\frac{x_1+x_2}{2} \tag{23}$$

It is easy to know the value of ϕ in some specific points, such as,

$$\phi\left(\frac{1}{5},\frac{4}{5}\right)=\frac{1}{2}\ ,\ \phi\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{2}\ \text{ and }\ \phi\left(\frac{1}{3},\frac{2}{3}\right)=\frac{1}{2} \tag{24}$$

For the sake of simplicity, set $\left\{(0,0),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},\frac{2}{3}\right),(1,1)\right\}$ is denoted by A and set $\left\{(0,0),\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{3},\frac{2}{3}\right),(1,1)\right\}$ is denoted by B. It is indicated that $[\phi]_{\theta_A}$and $[\phi]_{\theta_B}$ are presented as closed intervals denoted by $[l(A,\phi),u(A,\phi)]$ and $[l(B,\phi),u(B,\phi)]$, respectively [7]. The denotations in the intervals are expressed as follows,

$$l(A,\phi)(\mathbf{x})=\sup_{\mathbf{y}\in[0,\mathbf{x}]\cap A}\phi(\mathbf{y})\ ,\quad u(A,\phi)(\mathbf{x})=\inf_{\mathbf{y}\in[\mathbf{x},1]\cap A}\phi(\mathbf{y}) \tag{25}$$

$$l(B,\phi)(\mathbf{x})=\sup_{\mathbf{y}\in[0,\mathbf{x}]\cap B}\phi(\mathbf{y})\ ,\quad u(B,\phi)(\mathbf{x})=\inf_{\mathbf{y}\in[\mathbf{x},1]\cap B}\phi(\mathbf{y}) \tag{26}$$

Hence, the following expressions are obtained after some calculations:

$$l(A,\phi):[0,1]^2\to[0,1]:(x_1,x_2)\mapsto$$

$$\mapsto\begin{cases}1 & ;\quad x_1=x_2=1\\ \dfrac{1}{2} & ;\quad (x_1,x_2)\in\left[\dfrac{1}{4},1\right]\times\left[\dfrac{3}{4},1\right]\bigcup\left[\dfrac{1}{3},1\right]\times\left[\dfrac{2}{3},1\right]\ \backslash\{(1,1)\}\\ 0 & ;\quad \text{elsewhere}\end{cases} \tag{27}$$

$$u(A,\phi):[0,1]^2\to[0,1]:(x_1,x_2)\mapsto$$

$$\mapsto\begin{cases}0 & ;\quad x_1=x_2=0\\ \dfrac{1}{2} & ;\quad (x_1,x_2)\in\left[0,\dfrac{1}{4}\right]\times\left[0,\dfrac{3}{4}\right]\bigcup\left[0,\dfrac{1}{3}\right]\times\left[0,\dfrac{2}{3}\right]\ \backslash\{(0,0)\}\\ 1 & ;\quad \text{elsewhere}\end{cases} \tag{28}$$

$$l(B,\phi):[0,1]^2 \to [0,1]:(x_1,x_2) \mapsto$$

$$\mapsto \begin{cases} 1 \quad ; & x_1 = x_2 = 1 \\ \frac{1}{2} \quad ; & (x_1,x_2) \in \left[\frac{1}{5},1\right] \times \left[\frac{4}{5},1\right] \bigcup \left[\frac{1}{4},1\right] \times \left[\frac{3}{4},1\right] \\ & \bigcup \left[\frac{1}{3},1\right] \times \left[\frac{2}{3},1\right] \setminus \{(1,1)\} \\ 0 \quad ; & \text{elsewhere} \end{cases} \tag{29}$$

$$u(B,\phi):[0,1]^2 \to [0,1]:(x_1,x_2) \mapsto$$

$$\mapsto \begin{cases} 0 \quad ; & x_1 = x_2 = 0 \\ \frac{1}{2} \quad ; & (x_1,x_2) \in \left[0,\frac{1}{5}\right] \times \left[0,\frac{4}{5}\right] \bigcup \left[0,\frac{1}{4}\right] \times \left[0,\frac{3}{4}\right] \\ & \bigcup \left[0,\frac{1}{3}\right] \times \left[0,\frac{2}{3}\right] \setminus \{(0,0)\} \\ 1 \quad ; & \text{elsewhere} \end{cases} \tag{30}$$

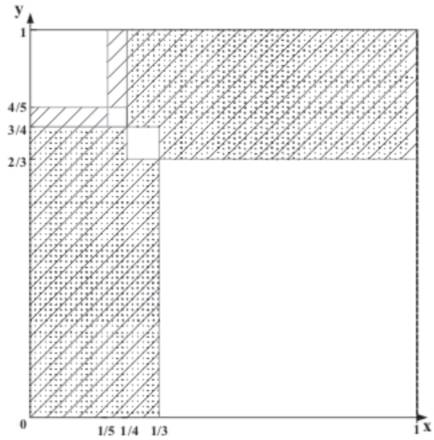

Fig.2 comparison of the domains regarding A and B

The virtual and hatched part in Fig.2 states the domain related to A and B, respectively, in the calculation of boundary structure functions. From this figure, it can be found out that $l(A,\phi) \prec l(B,\phi)$ and $u(A,\phi) \succ u(B,\phi)$. Thus, it is obvious that $[\phi]_{\theta_B} \subset [\phi]_{\theta_A}$, which can be obtained from $A \subset B$ and Lemma 2.

Furthermore, it is easily deduced that if $M_a B$ ($M_l B$) (the upper (lower) bound set of $[\phi]_{\theta_B}$ within $[\phi]_{\theta_A}$) is the closed interval $[u(B,\phi),u(A,\phi)]$ ($[l(A,\phi),l(B,\phi)]$), then the following statements can be seen in this numerical example:

1) The maximum of $(M_a B,\leq)$ is the supremum of $\left([\phi]_{\theta_A},\leq\right)$;
2) The minimum of $(M_a B,\leq)$ is the supremum of $\left([\phi]_{\theta_B},\leq\right)$;
3) The maximum of $(M_l B,\leq)$ is the infimum of $\left([\phi]_{\theta_B},\leq\right)$;
4) The minimum of $(M_l B,\leq)$ is the infimum of $\left([\phi]_{\theta_A},\leq\right)$.

These results meet with the theoretical results in the previous sections. It is stated in a practical point of view that lower and upper bound of the bound set can be substituted and transformed, which will lead to some useful reliability bounds.

4. Conclusion

Based on the notion of congruence relationship, a convex sublattice on the complete lattice of structure functions is presented in this paper. It is indicated that the relationship between lattices of equivalence classes and set of all the structure functions gives a better comprehension in system reliability, from both theoretical and practical point of view. The upper bound set of equivalence class regarding congruence relation presented in this paper has been shown to be a vital notion in engineering applications. Finally, theoretic properties are testified in the numerical example.

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This research is partially supported by the National Natural Science Foundation of China under the contract number 51075061, and the Research Fund for the Doctoral Program of Higher Education of China (New Faculty) under the contract number 20100185120029.

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Yu PANG, M. Sc.
Prof. Hong-Zhong HUANG, Ph.D.
Prof. Liping HE, Ph.D.
Zhonglai WANG, Ph.D.
Ning-Cong XIAO, M. Sc.
School of Mechanical, Electronic, and Industrial Engineering
University of Electronic Science and Technology of China
Chengdu, Sichuan, 611731, P. R. China
E-mail: hzhuang@uestc.edu.cn