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# Reliability analysis of series systems with multiple failure modes under epistemic and aleatory uncertainties

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**Abstract:** Uncertainty exists widely in engineering practice. An engineering system may have multiple failure criteria. In the current paper, system reliability analysis with multiple failure modes under both epistemic and aleatory uncertainties is presented. Epistemic uncertainty is modelled using p-boxes, while aleatory uncertainty is modelled using probability distributions. A first-order reliability method is developed and non-linear performance functions are linearized by the sampling method instead of the commonly used Taylor's expansion at the most probable point. Furthermore, multiple failure modes in a system are often correlated because they depend on the same uncertain variables. In order to consider these correlated failure modes, the methods proposed by Feng and Frank are extended in this paper in order to calculate the joint probability of failure for two arbitrary failure modes under both aleatory and epistemic uncertainties. The Pearson correlation coefficient of two arbitrary failure modes is determined by the sampling method. Since two types of uncertainty exist in the system, the probability of system failure is an interval rather than a point value. The probability of failure of the system can be obtained by the combination of the extension 'narrow' bound method and the interval arithmetic. A numerical example is presented to demonstrate the applicability of the proposed method.

**Keywords:** multiple failure modes, reliability analysis, epistemic uncertainty, aleatory uncertainty, series systems

## 1 INTRODUCTION

An engineering system often has multiple failure criteria. The occurrence of any failure mode in a series system will lead to failure of the system. The estimation of the probability of system failure may be difficult because these failure modes are often correlated as they depend on the same uncertain variables [**1**]. In traditional reliability analysis, there is an assumption that the probability distributions of all random variables are known or perfectly determinable. Many use this assumption anyway as a shortcut or mathematical convenience [**1**]. The probability of failure of a structural system theoretically involves multidimensional integration and is often difficult to calculate using direct integration methods [**2**]. Efforts searching for approximate methods have resulted in several approaches including bounding techniques, efficient Monte Carlo simulation (MCS) [**3**], and the important sampling method [**4**]. For a series system with multiple failure modes, Cornell [**5**] presented a 'wide' bound method for calculation of the probability of system failure. The main disadvantage of the method is that the bounds are too wide. Ditlevsen [**6**] presented a 'narrow' bound estimation method which can provide a narrow bound. However, the above methods

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cannot handle the situation when both epistemic and aleatory uncertainties exist in the system. Furthermore, the fast Fourier transform (FFT)-based reliability estimation technique and the multi-point approximation (MPA) method have been developed recently to estimate system reliability [**7–10**]. The algorithms reported solve the convolution integral in parts over several disjoint regions spanning the entire design space to estimate the system reliability accurately [**2**]. Generally, the combination of MPA and the FFT approach [**2**] can provide a more accurate result than the bound methods when multiple failure modes exist in the system. However, this approach is too complicated and thus has a low computational efficiency.

Many of the methods just discussed often assume that all probabilities or probability distributions are known or perfectly determinable. This assumption is not consistent with reality because two types of uncertainty may exist in a system [**11–13**]. Epistemic uncertainty comes from incomplete information or ignorance while aleatory uncertainty derives from inherent variations [**14–16**]. Reported mathematics to model epistemic uncertainty in structural reliability analysis include interval analysis [**17**, **18**], imprecise probability and Bayesian theory [**19**, **20**], evidence theory [**21**, **22**], fuzzy theory [**23**, **24**], and possibility theory and p-box models [**25**, **26**]. Meanwhile aleatory uncertainty is usually modelled using probability theory. Du [**15**] presented a unified reliability analysis method based on the first-order reliability method (FORM) and evidence theory. The method needs the most probable point (MPP) search which maybe computationally costly. Furthermore, this method can only model the system with a single failure mode rather than multiple correlated failure modes. Adduri and Penmetsa [**2**] presented a unified method for structural system reliability analysis under mixed variables. The computational efficiency of this approach is lower because it needs the MPP search, function approximations, and FFT of the system. In the present paper, the system variables are quantified based on the information available. Some variables are modelled using probability distributions because there is a large amount information about them. Other variables are modelled using p-boxes because of limited information about them. In order to capture the phenomenon where epistemic uncertainty and aleatory uncertainty both exist, the objective of the paper is to develop methodologies for structural reliability analysis that can efficiently handle multiple forms of uncertainties for multiple failure modes.

The paper is organized as follows. Section 2 provides a brief background about the interval arithmetic and section 3 provides a method for structural reliability analysis under epistemic uncertainty and aleatory uncertainty. Series systems reliability analysis under uncertainties is given in section 4. A numerical example is presented in section 5. A brief discussion and conclusion closes the paper in section 6.

## 2 INTERVAL ARITHMETIC

An interval is a closed set of the real line consisting of all values between an ordered pair of values known as the endpoints of the interval [**27**]. Mathematically, a closed bounded interval $[\underline{x}, \bar{x}] = \{\underline{x} \leqslant X \leqslant \bar{x}, X \in R\}$. is called an 'interval number', denoted as $x^{\mathrm{I}}$. When $\underline{x} = \bar{x} = x$, the interval number $x^{\mathrm{I}}$ degenerates into a real number $x$, which is generally called a 'point interval number'. The formulas of the four basic arithmetic operations are [**27**]

$$x^{\mathrm{I}} + y^{\mathrm{I}} = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \tag{1}$$

$$x^{\mathrm{I}} - y^{\mathrm{I}} = [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \tag{2}$$

$$x^{\mathrm{I}} \cdot y^{\mathrm{I}} = [\min(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}),\ \max(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y})] \tag{3}$$

$$x^{\mathrm{I}}/y^{\mathrm{I}} = [\underline{x}, \bar{x}] \cdot [1/\bar{y}, 1/\underline{y}], \qquad 0 \notin y^{\mathrm{I}} \tag{4}$$

Mathematically, any function defined on real values can be extended to intervals in a straightforward way. The extension to intervals of a function $f$ defined on the real number is [**28**]

$$f\left(x_1^{\mathrm{I}}, x_2^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}\right) = \{f(X_1, X_2, \cdots, X_n), \\ X_1 \in x_1^{\mathrm{I}}, X_2 \in x_2^{\mathrm{I}}, \cdots, X_n \in x_n^{\mathrm{I}}\} \tag{5}$$

## 3 STRUCTURAL RELIABILITY ANALYSIS UNDER BOTH EPISTEMIC UNCERTAINTY AND ALEATORY UNCERTAINTY

### 3.1 Traditional structural reliability analysis

Consider two statistically independent, normally distributed random variables $R$ and $S$, where $R$ denotes resistance and $S$ denotes load effect. The performance function is $Z = R - S$. The system failure occurs when $R < S$, i.e. $Z < 0$. The probability of failure is given by [**29**]

$$\begin{aligned} P_{\mathrm{f}} = P(Z<0) &= \int_{-\infty}^{0} \frac{1}{\sigma_Z\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-\mu_Z}{\sigma_Z}\right)^2\right] \mathrm{d}z \\ &= \int_{-\infty}^{-\beta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u = \Phi(-\beta) \end{aligned} \tag{6}$$

where

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{7}$$

and $\mu_Z$ and $\sigma_Z$ are the mean and the standard deviation of the random variable $Z$, respectively, $\Phi$ is the standard normal cumulative distribution function (CDF), and $\beta$ is called the 'reliability index' or 'safety index'.

Generally, a linear performance function $G(\boldsymbol{X})$ with all normally distributed random variables can be expressed as

$$G(\boldsymbol{X}) = a_0 + a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_nX_n \tag{8}$$

From equations (6) and (7), the probability of failure becomes

$$P_f = P[G(\boldsymbol{X})<0] = \Phi(-\beta_G) \tag{9}$$

where

$$\beta_G = \frac{\mu_G}{\sigma_G} = \frac{a_0 + a_1\mu_{X_1} + a_2\mu_{X_2} + a_3\mu_{X_3} + \cdots + a_n\mu_{X_n}}{\sqrt{(a_1\sigma_{X_1})^2 + (a_2\sigma_{X_2})^2 + (a_3\sigma_{X_3})^2 + \cdots + (a_n\sigma_{X_n})^2}} \tag{10}$$

and $\mu_G$ and $\sigma_G$ are the mean value and the standard deviation of the function $G(\boldsymbol{X})$, respectively.

### 3.2 Reliability analysis under both aleatory and epistemic uncertainties

Because two types of uncertainty may exist simultaneously, one needs a unified method to handle these uncertainties. Generally, the uncertainties can be quantified based on the information available about the system variables. If a large amount of data or information about a variable is available, then its variation can be modelled using a probability distribution such as $X_i \sim N(\mu_i, \sigma_i)$. If information or data about a variable is sparse, then its variation can be modelled using a p-box such as $X_i \sim N\left(\left[\underline{\mu}_i, \bar{\mu}_i\right], [\underline{\sigma}_i, \bar{\sigma}_i]\right)$. Generally, the p-box is a closed-form function; for example, $X \sim N([4, 6], [1, 2])$, the CDF of parameter $X$, is shown in Fig. 1. When both epistemic and aleatory uncertainties are present in a system, the system performance function can be denoted by $G\left(X_{\mathrm{sys}}^{\mathrm{I}}\right)$. In this condition, the probability of system failure $P_f^{\mathrm{I}}$ in the interval becomes

$$P_f^{\mathrm{I}} = P\left[G(\boldsymbol{X}_{\mathrm{sys}}^{\mathrm{I}})<0\right] = \int \cdots \int_{G(\boldsymbol{X}_{\mathrm{sys}}^{\mathrm{I}})<0} f_X(\boldsymbol{X}_{\mathrm{sys}}^{\mathrm{I}})\mathrm{d}\boldsymbol{x}_{\mathrm{sys}} \tag{11}$$

where $f_X(X_{\mathrm{sys}}^{\mathrm{I}})$ is the joint probability density function (PDF) of the $n$-dimensional vector $X_{\mathrm{sys}}^{\mathrm{I}}$ of basic random variables.

From equation (11), the probability of system failure in interval form can be given by

$$P_f^{\mathrm{I}} = [\underline{P}_f, \bar{P}_f] = \left[\int \cdots \int_{G(\boldsymbol{X}_{\mathrm{sys}}^{\mathrm{I}})<0} \underline{f}_X(\boldsymbol{x}_{\mathrm{sys}})\mathrm{d}\boldsymbol{x}_{\mathrm{sys}}, \int \cdots \int_{G(\boldsymbol{X}_{\mathrm{sys}}^{\mathrm{I}})<0} \bar{f}_X(\boldsymbol{x}_{\mathrm{sys}})\mathrm{d}\boldsymbol{x}_{\mathrm{sys}}\right] \tag{12}$$

In order to maintain consistency and avoid conflict, the following constraint must be satisfied

$$0 \leqslant \underline{P}_f \leqslant \bar{P}_f \leqslant 1 \tag{13}$$

In engineering practice, the variables of the system may not all be normally distributed. In the present paper, all parameters are assumed to be mutually independent. A random vector $X = (X_1, X_2, X_3, \ldots, X_n)^{\mathrm{T}}$ can be transformed into the standardized normal random vector $Y = (Y_1, Y_2, \ldots, Y_n)$ through the Nataf transformation by $y_i = \Phi^{-1}[F_{X_i}(X_i)]$, where $F_{X_i}$ is the CDF of $X_i$ and $\Phi^{-1}$ is the inverse CDF of a standard normal distribution [**30**]. In this paper, for simplicity and illustration purposes, random variables are modelled using probability distributions and p-boxes are all normally distributed. A linear performance function can be expressed as

$$G(\boldsymbol{X}) = a_0 + a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_nX_n \tag{14}$$

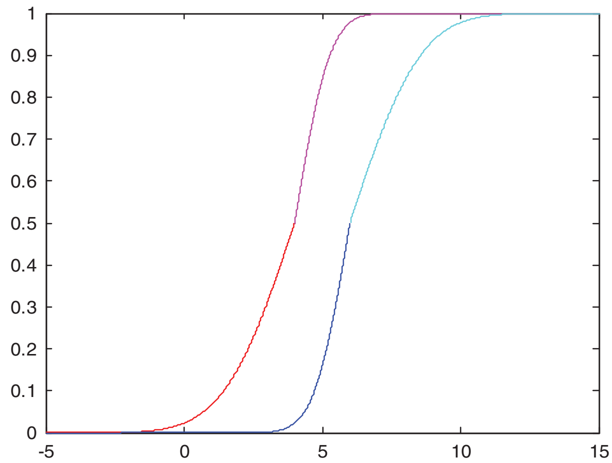


**Fig. 1** Cumulative distribution function of p-box $N([4, 6], [1, 2])$

where $a_i(i=0,1,2,\ldots,n)$ are constant, and one assumes $a_i(i=0,1,2,\ldots,n)>0$.

In order to consider both epistemic and aleatory uncertainties, a linear performance function under both epistemic and aleatory uncertainties can be expressed as

$$G\left(\boldsymbol{X}_{\text{sys}}^{\text{I}}\right)=a_0+a_1X_1+\cdots+a_iX_i+a_{i+1}X_{i+1}^{\text{I}}+\cdots+a_nX_n^{\text{I}} \tag{15}$$

$$\boldsymbol{X}_{\text{sys}}^{\text{I}}=\left(\boldsymbol{X};\boldsymbol{X}^{\text{I}}\right)=\left(X_1,X_2,\cdots,X_i;X_{i+1}^{\text{I}},X_{i+2}^{\text{I}},\cdots,X_n^{\text{I}}\right) \tag{16}$$

where $X_k\sim N\left(\mu_k,\sigma_k^2\right)$ $(k=0,1,\ldots,i)$ and $X_k^{\text{I}}\sim N\left(\left[\underline{\mu}_k,\bar{\mu}_k\right],\left[\underline{\sigma}_k^2,\bar{\sigma}_k^2\right]\right)$ $(k=i+1,i+2,\ldots,n)$.

From equations (9), (10), (15), and (16), the lower and the upper probability of failure and reliability index become [**11**]

$$P_{\text{f}}^{\text{I}}=\left[\underline{P}_{\text{f}},\bar{P}_{\text{f}}\right]=\Phi\left(-\beta^{\text{I}}\right)=\Phi\left[-\bar{\beta},-\underline{\beta}\right] \tag{17}$$

$$\beta^{\text{I}}=\frac{a_1\mu_{X_1}+a_2\mu_{X_2}+\cdots+a_i\mu_{X_i}+a_{i+1}\mu_{i+1}^{\text{I}}+\cdots+a_nb_n^{\text{I}}}{\sqrt{\left(a_1\sigma_{X_1}\right)^2+\left(a_2\sigma_{X_2}\right)^2+\cdots+\left(a_i\sigma_i\right)^2+\left(a_{i+1}\sigma_{i+1}^{\text{I}}\right)^2+\cdots+\left(a_n\sigma_n^{\text{I}}\right)^2}} \tag{18}$$

and

$$\underline{\beta}=\frac{a_1\mu_{X_1}+a_2\mu_{X_2}+\cdots+a_i\mu_{X_i}+a_{i+1}\underline{\mu}_{i+1}+\cdots+a_n\underline{\mu}_n}{\sqrt{\left(a_1\sigma_{X_1}\right)^2+\left(a_2\sigma_{X_2}\right)^2+\cdots+\left(a_i\sigma_i\right)^2+\left(a_{i+1}\bar{\sigma}_{i+1}\right)^2+\cdots+\left(a_n\bar{\sigma}_n\right)^2}} \tag{19}$$

and

$$\bar{\beta}=\frac{a_1\mu_{X_1}+a_2\mu_{X_2}+\cdots+a_i\mu_{X_i}+a_{i+1}\bar{\mu}_{i+1}+\cdots+a_n\bar{\mu}_n}{\sqrt{\left(a_1\sigma_{X_1}\right)^2+\left(a_2\sigma_{X_2}\right)^2+\cdots+\left(a_i\sigma_i\right)^2+\left(a_{i+1}\underline{\sigma}_{i+1}\right)^2+\cdots+\left(a_n\underline{\sigma}_n\right)^2}} \tag{20}$$

respectively.

In practice, the performance functions are usually of a more general form than the linear form. Traditionally, in order to use the FORM, the feasible method is to linearize $G(X)=0$ to $G_{\text{L}}(X)=0$. This can be done by expanding $G(X)=0$ as a first-order Taylor's expansion at the MPP $\boldsymbol{X}^*$. However, expanding a non-linear function at the MPP is very difficult because the MPP search is an optimization process, sometimes there is more than one MPP or the MPP search process does not convergence [**11**]. When both interval variables and random variables are present in the system, the MPP search is a double optimization process [**31**]. Therefore, in this paper, the performance functions are linearized by MCS or the importance sampling methods instead of the Taylor's expansion method. Assume that $X_j(j=1,2,\ldots,k)$ are $k$ samples generated by the sampling method. Those samples which bound the limit-state function contribute the greatest probability density or maximum likelihood for the limit-state function [**30**, **32**]. Therefore, those samples bounding the limit-state function are chosen to carry out the regression analysis. After carrying out the regression analysis, a linearized function $G_{\text{L}}(X)=0$ can be obtained which is used to replace the original performance function. The form of $G_{\text{L}}(X)=0$ can be expressed the same as in equation (15). The method to linearize $G(X)=0$ into $G_{\text{L}}(X)=0$ involves the following four steps.

1. Generate $k$ sample points $x_l(l=1,2,\ldots,k)$ by MCS, for a p-box $X_j^{\text{I}}\sim N\left(\left[\underline{\mu}_j,\bar{\mu}_j\right],\left[\underline{\sigma}_j^2,\bar{\sigma}_j^2\right]\right)$; in order to use equations (15) to (20), the sample points are generated by $N\left(\tilde{\mu}_j,\tilde{\sigma}_j^2\right)$, where $\tilde{\mu}_j=\left(\underline{\mu}_j+\bar{\mu}_j\right)/2$ and $\tilde{\sigma}_j=\left(\underline{\sigma}_j+\bar{\sigma}_j\right)/2$.
2. Calculate the function values of $G(\boldsymbol{x}_l)$.
3. Give a constraint such as $-\infty<G(\boldsymbol{x}_l)<0$.
4. Assume that there are $K$ samples which satisfy the constraint $-\infty<G(\boldsymbol{x}_j)<0(j=1,2,\ldots,K)$. Carry out the regression analysis on function values $G(\boldsymbol{x}_j)$. One can obtain a linear tangent plane $G_{\text{L}}(X)=0$ of the original limit-state function $G(X)=0$. The bounds of the probability of system failure can be calculated using equation (17).

## 4 SERIES SYSTEM RELIABILITY ANALYSIS UNDER UNCERTAINTIES

### 4.1 Estimate the Pearson correlation coefficient between two arbitrary failure modes

In engineering practice, there may be many failure modes rather than a single one. These failure modes are often dependent because they depend on the same random variables. It is very difficult to quantify the correlation coefficients because the dependence between failure modes is infinite-dimensional and the single dimension of a correlation coefficient cannot capture the potential complexity of the dependence function [**1**]. In this paper, Pearson correlation between two arbitrary failure modes is the key correlation that is considered. The value of the Pearson correlation coefficient between two arbitrary failure modes can be estimated through a simulation method. The Pearson correlation coefficient $\rho_{XY}$ between two random variables $X$ and $Y$ is given by [**1**]

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)}\sqrt{V(Y)}} \tag{21}$$

where $E$ denotes the expectation and $V$ denotes the variance.

From equation (21), the Pearson correlation coefficient between two arbitrary failure modes becomes

$$\rho_{ij} = \frac{E\left[G_i(\boldsymbol{X})G_j(\boldsymbol{X})\right] - E[G_i(\boldsymbol{X})]E\left[G_j(\boldsymbol{X})\right]}{\sqrt{\sigma[G_i(\boldsymbol{X})]}\sqrt{\sigma\left[G_j(\boldsymbol{X})\right]}} \tag{22}$$

$$\sigma[G_i(\boldsymbol{X})] = E\left[G_i^2(\boldsymbol{X})\right] - E[G_i(\boldsymbol{X})]^2 \tag{23}$$

If there are $m$ failure modes in a system, the number of Pearson correlation coefficients between two arbitrary failure modes is $m^2$, and the Pearson correlation coefficients can be expressed as

$$\rho_{\text{sys}} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{m1} & \rho_{m2} & \cdots & \rho_{mm} \end{bmatrix} \tag{24}$$

where $\rho_{ii} = 1, \rho_{ij} = \rho_{ji}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, m)$.

Generally, when both p-boxes and random variables are present in the system, the Pearson correlation coefficients in equation (24) are in interval form, which can be expressed as

$$\rho_{\text{sys}} = \begin{bmatrix} \rho_{11}^{\text{I}} & \rho_{12}^{\text{I}} & \cdots & \rho_{1m}^{\text{I}} \\ \rho_{21}^{\text{I}} & \rho_{22}^{\text{I}} & \cdots & \rho_{2m}^{\text{I}} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{m1}^{\text{I}} & \rho_{m2}^{\text{I}} & \cdots & \rho_{mm}^{\text{I}} \end{bmatrix} \tag{25}$$

### 4.2 Joint probability of failure for two arbitrary failure modes

The joint probability of failure between the $i$th failure mode and $j$th failure mode can be expressed as

$$P_{\text{f}ij} = \int \underset{G_i(\boldsymbol{X})<0 \bigcup G_j(\boldsymbol{X})<0}{\cdots} \int f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{26}$$

where $f_X(X)$ is the joint PDF. However in reality, the integration in equation (26) is very difficult. Therefore, there are several approximation methods to estimate the joint probability of failure. Feng [**33**] proposed a model expressed as

$$P_{\text{f}ij} = (P_{\text{fA}} + P_{\text{fB}}) \cdot \left[1 - \arccos(\rho_{ij})/\pi\right] \tag{27}$$

$$P_{\text{fA}} = \Phi(-\beta_i) \cdot \Phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right) \tag{28}$$

$$P_{\text{fB}} = \Phi(-\beta_j) \cdot \Phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \tag{29}$$

where $\rho_{ij}$ is the Pearson correction between the $i$th and $j$th failure modes.

When both epistemic and aleatory uncertainties exist in the system, the joint probability of failure between the $i$th failure mode and $j$th failure mode is an interval, which can be calculated by

$$P_{\text{f}ij}^{\text{I}} = \left(P_{\text{fA}}^{I} + P_{\text{fB}}^{I}\right) \cdot \left[1 - \arccos(\rho_{ij}^{\text{I}})/\pi\right] \tag{30}$$

$$P_{\text{fA}}^{I} = \Phi(-\beta_i^{\text{I}}) \cdot \Phi\left(-\frac{\beta_j^{\text{I}} - \rho_{ij}^{\text{I}}\beta_i^{\text{I}}}{\sqrt{1-\left(\rho_{ij}^{\text{I}}\right)^2}}\right) \tag{31}$$

$$P_{\text{fB}}^{\text{I}} = \Phi(-\beta_j^{\text{I}}) \cdot \Phi\left(-\frac{\beta_i^{\text{I}} - \rho_{ij}^{\text{I}}\beta_j^{\text{I}}}{\sqrt{1-\left(\rho_{ij}^{\text{I}}\right)^2}}\right) \tag{32}$$

From equations (30) to (32), the joint probability of failure $P_{\text{f}ij}^{\text{I}}$ in equation (30) can also be calculated by an optimization model as follows

$$\begin{cases} \min(\max)P_{\text{f}ij} = \min(\max)[P_{\text{fA}} + P_{\text{fB}}]\left[1 - \arccos(\rho_{ij})/\pi\right] \\ P_{\text{fA}} = \Phi(-\beta_i) \cdot \Phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right) \\ P_{\text{fB}} = \Phi(-\beta_j) \cdot \Phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \\ \beta_{i_{\min}} \leqslant \beta_i \leqslant \beta_{i_{\max}} \\ \beta_{j_{\min}} \leqslant \beta_j \leqslant \beta_{j_{\max}} \\ \rho_{ij_{\min}} \leqslant \rho_{ij} \leqslant \rho_{ij_{\max}} \end{cases} \tag{33}$$

$$P_{\text{f}ij}^{\text{I}} = \left[P_{\text{f}ij}^{\min}, P_{\text{f}ij}^{\max}\right] \tag{34}$$

Another method used to estimate the joint probability of failure is the Frank model. In the Frank model, the joint probability of events $X$ and $Y$ is given by [**1**]

$$P(X\&Y) = \begin{cases} \min(a, b) & \text{if } \rho_{XY} = 1 \\ ab & \text{if } \rho_{XY} = 0 \\ \max(a + b - 1, 0) & \text{if } \rho_{XY} = -1 \\ \log_s\left[1 + (s^a - 1)(s^b - 1)/(s - 1)\right] & \text{otherwise} \end{cases} \tag{35}$$

where $s = \tan[\pi(1-\rho_{XY})/4]$, $a = P(A)$, $b = P(B)$, and $\rho_{XY}$ is the Pearson correlation between events $X$ and $Y$.

From equation (35), the joint probability of failure between the $i$th and $j$th failure modes using the Frank model can be expressed as

$$P_{\mathrm{f}}\left[G_i(\boldsymbol{X})\&G_j(\boldsymbol{X})\right] = P_{\mathrm{f}ij} = \begin{cases} \min(P_{\mathrm{f}i}, P_{\mathrm{f}j}) & \text{if } \rho_{ij} = 1 \\ P_{\mathrm{f}i}P_{\mathrm{f}j} & \text{if } \rho_{ij} = 0 \\ \max(P_{\mathrm{f}i} + P_{\mathrm{f}j} - 1, 0) & \text{if } \rho_{ij} = -1 \\ \log_s\left[1 + \left(s^{P_{\mathrm{f}i}} - 1\right)\left(s^{P_{\mathrm{f}j}} - 1\right)/(s-1)\right] & \text{otherwise} \end{cases} \tag{36}$$

where $s = \tan\left[\pi(1-\rho_{ij})/4\right]$, $P_{\mathrm{f}i}$ and $P_{\mathrm{f}j}$ are the probability of failure for the $i$th and $j$th failure mode, respectively, and $\rho_{ij}$ is the Pearson correlation between the $i$th and $j$th failure modes.

The joint probability of failure of the $i$th failure mode and $j$th failure mode in interval form becomes

$$P_{\mathrm{f}}^{\mathrm{I}}[G_i(\mathbf{X})\&G_j(\mathbf{X})] = P_{\mathrm{f}ij}^{\mathrm{I}} = \begin{cases} \min(P_{\mathrm{f}i}^{\mathrm{I}}, P_{\mathrm{f}j}^{\mathrm{I}}) & \text{if } \rho_{ij}^{\mathrm{I}} = 1 \\ P_{\mathrm{f}i}^{\mathrm{I}}P_{\mathrm{f}j}^{\mathrm{I}} & \text{if } \rho_{ij}^{\mathrm{I}} = 0 \\ \max(P_{\mathrm{f}i}^{\mathrm{I}} + P_{\mathrm{f}j}^{\mathrm{I}} - 1, 0) & \text{if } \rho_{ij}^{\mathrm{I}} = -1 \\ \log_{s^{\mathrm{I}}}\left[1 + \left((s^{\mathrm{I}})^{P_i^{\mathrm{I}}} - 1\right)\left((s^{\mathrm{I}})^{P_j^{\mathrm{I}}} - 1\right)/\left((s^{\mathrm{I}}) - 1\right)\right] & \text{otherwise} \end{cases} \tag{37}$$

From equation (37), the probability of failure $P_{\mathrm{f}ij}^{\mathrm{I}}$ in interval form can also be calculated by an optimization model as follows

$$\begin{cases} \min(\max) P_{\mathrm{f}ij} = \min(\max)\log_s[1 + (s^{P_{\mathrm{f}i}} - 1) \\ (s^{P_{\mathrm{f}i}} - 1)/(s-1)] \text{ subject to } s_{\min} \leqslant s \leqslant s_{\max} \\ \Phi\left(-\beta_{i_{\max}}\right) \leqslant P_{\mathrm{f}i} \leqslant \Phi\left(-\beta_{i_{\min}}\right) \\ \Phi\left(-\beta_{j_{\max}}\right) \leqslant P_{\mathrm{f}j} \leqslant \Phi\left(-\beta_{j_{\min}}\right) \end{cases} \tag{38}$$

where $s_{\min} = \tan[\pi(1-\rho_{ij}^{\max})/4]$, $s_{\max} = \tan[\pi(1-\rho_{ij}^{\min})/4]$; $P_{\mathrm{f}i_{\min}} = \Phi(-\beta_{i\max})$, $P_{\mathrm{f}i_{\max}} = \Phi(-\beta_{i\min})$; $P_{\mathrm{f}j_{\min}} = \Phi(-\beta_{j\max})$, $P_{\mathrm{f}j_{\max}} = \Phi(-\beta_{j\min})$; and $P_{\mathrm{f}ij}^{\mathrm{I}} = [P_{\mathrm{f}ij}^{\min}, P_{\mathrm{f}ij}^{\max}]$.

### 4.3 Reliability analysis with multiple failure models under uncertainties

A series structural system has $m$ ($m \geqslant 2$) failure modes. The occurrence of any failure mode causes the system to fail. A series system which has $m$ failure modes is shown in Fig. 2.


**Fig. 2** A series system

Assume that the performance function of the $i$th failure mode and $j$th failure mode are expressed as $G_{Gi}(X_{Gi}^{\mathrm{I}})$ and $G_{Gj}(X_{Gj}^{\mathrm{I}})$. After using the simulation method, their linearized functions are denoted by $G_{\mathrm{L}Gi}(X_{Gi}^{\mathrm{I}})$ and $G_{\mathrm{L}Gj}(X_{Gj}^{\mathrm{I}})$. From equation (15), the linearized functions $G_{\mathrm{L}Gi}(X_{Gi}^{\mathrm{I}})$ and $G_{\mathrm{L}Gj}(X_{Gj}^{\mathrm{I}})$ can be expressed as

$$G_{\mathrm{L}Gi}(\boldsymbol{X}_{Gi}^{\mathrm{I}}) = a_0 + a_1X_1 + \cdots + a_{i1}X_{i1} + a_{i1+1}X_{i1+1}^{\mathrm{I}} + \cdots + a_{n1}X_{n1}^{\mathrm{I}} \tag{39}$$

$$G_{\mathrm{L}Gj}\left(\boldsymbol{X}_{Gj}^{\mathrm{I}}\right) = c_0 + c_1X_1 + \cdots + c_{i2}X_{i2} + c_{i2+1}X_{i2+1}^{\mathrm{I}} + \cdots + c_{n2}X_{n2}^{\mathrm{I}} \tag{40}$$

From equations (17) to (20), the approximate probabilities of failure for the $i$th and $j$th failure modes become

$$P_{\mathrm{f}i}^{\mathrm{I}} = \left[\underline{P}_{\mathrm{f}i}, \bar{P}_{\mathrm{f}i}\right] = \Phi(-\beta_i^I) = \left[\Phi(-\bar{\beta}_i), \Phi\left(-\underline{\beta}_i\right)\right] \tag{41}$$

and

$$P_{\mathrm{f}j}^{\mathrm{I}} = \left[\underline{P}_{\mathrm{f}j}, \bar{P}_{\mathrm{f}j}\right] = \Phi(-\beta_j^I) = \left[\Phi\left(-\bar{\beta}_j\right), \Phi\left(-\underline{\beta}_j\right)\right] \tag{42}$$

where

$$\underline{\beta}_i = \frac{a_1\mu_{X_1} + a_2\mu_{X_2} + \cdots + a_{i1}\mu_{X_{i1}} + a_{i1+1}\underline{\mu}_{i1+1} + \cdots + a_{n1}\underline{\mu}_{n1}}{\sqrt{(a_1\sigma_{X_1})^2 + (a_2\sigma_{X_2})^2 + \cdots + (a_{i1}\sigma_{i1})^2 + (a_{i1+1}\bar{\sigma}_{i1+1})^2 + \cdots + (a_{n1}\bar{\sigma}_{n1})^2}} \tag{43}$$

$$\bar{\beta}_i = \frac{a_1\mu_{X_1} + a_2\mu_{X_2} + \cdots + a_{i1}\mu_{X_{i1}} + a_{i1+1}\bar{\mu}_{i1+1} + \cdots + a_{n1}\bar{\mu}_{n1}}{\sqrt{(a_1\sigma_{X_1})^2 + (a_2\sigma_{X_2})^2 + \cdots + (a_{i1}\sigma_{i1})^2 + (a_{i1+1}\underline{\sigma}_{i1+1})^2 + \cdots + (a_{j1}\underline{\sigma}_{n1})^2}} \tag{44}$$

and

$$\underline{\beta}_j = \frac{c_1\mu_{X_1} + c_2\mu_{X_2} + \cdots + c_{i1}\mu_{X_{i2}} + c_{i2+1}\underline{\mu}_{i2+1} + \cdots + c_{n2}\underline{\mu}_{n2}}{\sqrt{(c_1\sigma_{X_1})^2 + (c_2\sigma_{X_2})^2 + \cdots + (c_{i2}\sigma_{i2})^2 + (c_{i2+1}\bar{\sigma}_{i2+1})^2 + \cdots + (c_{n2}\bar{\sigma}_{n2})^2}} \tag{45}$$

$$\bar{\beta}_j = \frac{c_1\mu_{X_1} + c_2\mu_{X_2} + \cdots + c_{i1}\mu_{X_{i2}} + c_{i2+1}\bar{\mu}_{i2+1} + \cdots + c_{n2}\bar{\mu}_{n2}}{\sqrt{(c_1\sigma_{X_1})^2 + (c_2\sigma_{X_2})^2 + \cdots + (c_{i2}\sigma_{i2})^2 + (c_{i2+1}\underline{\sigma}_{i2+1})^2 + \cdots + (c_{n2}\underline{\sigma}_{n2})^2}} \tag{46}$$

respectively.

The joint probability of failure between the $i$th failure mode and the $j$th failure mode is denoted by $P_{\mathrm{f}ij}$. Generally, $P_{\mathrm{f}ij}$ can be calculated by the method proposed by Feng or Frank discussed in section 4.2. A system with $m$ failure modes may have $m \times m$ joint probabilities of failures, expressed as

$$[P_{fij}] = \begin{bmatrix} P_{f11}P_{f12}\cdots P_{f1m} \\ P_{f21}P_{f22}\cdots P_{f2m} \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ P_{fm1}P_{fm2}\cdots P_{fmm} \end{bmatrix} \tag{47}$$

where $P_{fii} = P_{fi}$, $P_{fij} = P_{fji}$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, m)$.

When p-boxes and random variables both exist in the system, the joint probabilities of failure are intervals given by

$$\left[P^I_{fij}\right] = \begin{bmatrix} P^I_{f11} & P^I_{f12} & \cdots & P^I_{f1m} \\ P^I_{f21} & P^I_{f22} & \cdots & P^I_{f2m} \\ \vdots & \vdots & \cdots & \\ P^I_{fm1} & P^I_{fm2} & \cdots & P^I_{fmm} \end{bmatrix} \tag{48}$$

where $P^I_{ii} = P^I_i$, $P^I_{fij} = P^I_{fji}$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, m)$.

Because the evaluation of the multidimensional integration is very difficult, approximate methods have been proposed and developed. The 'wide' bound approach for estimating the probability of failure of a series structural system is expressed as [**5**]

$$\max_{1\leqslant i\leqslant m}(P_{fi}) \leqslant P_f \leqslant 1 - \prod_{i=1}^{m}(1 - P_{fi}) \tag{49}$$

where $P_{fi}$ is the probability of failure for the $i$th failure mode, $P_f$ is the probability of failure of the system.

The 'wide' bound approach is very simple while the resulting bounds are usually too wide, especially for a complex system. The 'narrow' bound estimation method for series systems is expressed as [**6**]

$$\begin{aligned} &P_{f1} + \sum_{i=2}^{m}\max\left(P_{fi} - \sum_{j=1}^{i-1}P_{fij}, 0\right) \leqslant \\ &P_f \leqslant \sum_{i=1}^{m}P_{fi} - \sum_{i=2}^{m}\max_{j<i}(P_{fij}) \end{aligned} \tag{50}$$

where $P_{fij}$ is the joint failure probability of the $i$th and the $j$th failure modes. From equation (49), the 'wide' bound method in interval form can be expressed as

$$\max_{1\leqslant i\leqslant m}(P^I_{fi}) \leqslant P^I_f \leqslant 1 - \prod_{i=1}^{m}(1 - P^I_{fi}) \tag{51}$$

In this paper, the 'narrow' bound method is used for its accuracy. When both epistemic and aleatory uncertainties are present in the system, after the combination of the interval arithmetic, the Feng model, and the extended 'narrow' bound method, the probability of system failure in interval form becomes

$$\begin{aligned} P_f \in \Bigg[ &\Bigg(\min(P^I_{f1}) + \sum_{i=2}^{m}\max \\ &\left(\min(P^I_{fi}) - \sum_{j=1}^{i-1}\min\left(P^I_{fij}\right), 0\right)\Bigg), \\ &\left(\sum_{i=1}^{m}\max(P^I_{fi}) - \sum_{i=2}^{m}\max_{j<i}\left(P^I_{fij}\right)\right)\Bigg] \end{aligned} \tag{52}$$

## 5 NUMERICAL EXAMPLE

In this section, an example is provided to demonstrate the application of the proposed method as well as its effectiveness. In the example, three parameters are expressed by precise probability distributions while one is expressed by a p-box.

Consider a cantilevered beam with an end load as shown in Fig. 3. The performance functions of displacement, stress, and reaction moment are expressed as follows [**34**]

$$\begin{aligned} &G_{Disp}(L, P, E, B, H) = 4.0 - \frac{4PL^3}{EBH^3} \\ &G_{Stress}(L, P, B, H) = 4000.0 - \frac{6PL}{BH^2} \\ &G_{Moment}(L, P) = 25000.0 - PL \end{aligned} \tag{53}$$

The system is a series system which is controlled by the above three performance functions. The system fails if any performance function is smaller than zero. The values of the cross-sectional height, cross-sectional width, length of the beam, Young's modulus of the material, and the applied force are given in Table 1.

According to the proposed method, the first step is to linearize the three non-linear performance functions $G_{Disp}$, $G_{Stress}$, and $G_{Moment}$ to their linearized form $G_{LDisp}$, $G_{LStress}$, and $G_{LMoment}$ by the

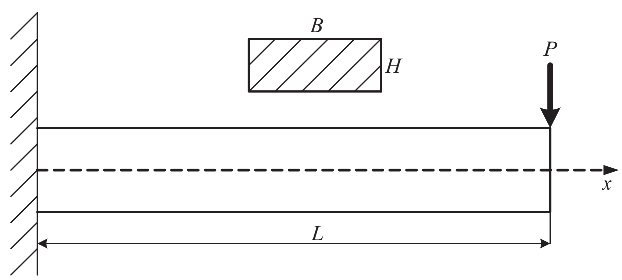


**Fig. 3** A cantilevered beam

**Table 1** System parameters and their distribution information

| Variable | Mean or interval | Standard deviation | Distribution |
|---|---|---|---|
| $L$ | 200 | 10 | Normal |
| $B$ | 4 | 0.2 | Normal |
| $H$ | 3 | 0.15 | Normal |
| $P$ | [95,105] | [4.5,5.5] | Normal |
| $E$ | $1.0\times10^7$ | – | – |

sampling simulation. Therefore, the probability of failure in interval form for each performance function is given by

$$\left[P_{\mathrm{f}}^{\mathrm{I}}\right]=\begin{bmatrix}P_{\mathrm{fDisp}}^{\mathrm{I}}\\P_{\mathrm{fStress}}^{\mathrm{I}}\\P_{\mathrm{fMoment}}^{\mathrm{I}}\end{bmatrix}=\begin{cases}[0.0554,0.1247]\\ [0.0406,0.1531]\\ [1.5740\times10^{-5},0.0050]\end{cases}\tag{54}$$

The second step is to estimate the Pearson correlation coefficients between two arbitrary performance functions. We can obtain a $3\times3$ Pearson correlation matrix expressed as

$$\rho_{\mathrm{system}}=\begin{bmatrix}[1.000,1.000] & [0.927,0.931] & [0.617,0.628]\\ [0.927,0.931] & [1.000,1.000] & [0.503,0.557]\\ [0.617,0.628] & [0.503,0.557] & [1.000,1.000]\end{bmatrix}\tag{55}$$

The next step is to calculate the joint probability of failure for two arbitrary performance functions by the extended Feng's model or Frank's model described in section 4.2. The joint probability of failure for two arbitrary performance functions obtained by the extension of Feng's model is denoted by $[P_{\mathrm{f}ij}^{\mathrm{I}}]_{\mathrm{Fe}_{3\times3}}$, and that by the extension of Frank's model is denoted by $[P_{\mathrm{f}ij}^{\mathrm{I}}]_{\mathrm{Fr}_{3\times3}}$. From equations (33), (38), (54), and (55), the joint probabilities of failure in interval form by extending the above two models become

$$\left[P_{\mathrm{f}ij}^{\mathrm{I}}\right]_{\mathrm{Fe}3\times3}=\begin{bmatrix}[0.0554,0.1247] & [0.0303,0.1003] & [1.1303\times10^{-5},0.0033]\\ [0.0303,0.1003] & [0.0406,0.1531] & [1.3049\times10^{-5},0.0032]\\ [1.1303\times10^{-5},0.0033] & [1.3049\times10^{-5},0.0032] & [1.5740\times10^{-5},0.0050]\end{bmatrix}\tag{56}$$

and

$$\left[P_{\mathrm{f}ij}^{I}\right]_{\mathrm{Fr}3\times3}=\begin{bmatrix}[0.0554,0.1247] & [0.0060,0.0423] & [2.4540\times10^{-6},9.8458\times10^{-4}]\\ [0.0060,0.0423] & [0.0406,0.1531] & [2.3015\times10^{-6},1.1118\times10^{-3}]\\ [2.4540\times10^{-6},9.8458\times10^{-4}] & [2.3015\times10^{-6},1.1118\times10^{-3}] & [1.5740\times10^{-5},0.0050]\end{bmatrix}\tag{57}$$

From equations (52) and (54) to (56), the probability of system failure is $[0.066,0.176]$. This method is denoted by PM1. From equations (52), (54), (55), and (57), the bounds of the probabilities of system failure are $[0.090,0.239]$, which is denoted by PM2. In order to demonstrate the applicability, efficiency, and accuracy of the proposed method, the MCS-based method is adopted to validate the proposed method. The bounds of the probability of system failure using the MCS are $[0.065,0.182]$. The probabilities of system failure calculated by the three methods are shown in Table 2. From Table 2 it can be known that the results calculated by the proposed method PM1 are more accurate than those calculated by PM2. The results obtained using proposed method PM1 are almost identical to those calculated using the MCS-based method. Furthermore, it should be noted that when both aleatory and epistemic uncertainties are present in a system, it is very difficult to estimate the probability of system failure using traditional methods. The widely used and efficient reliability analysis method is the MCS-based method. However, when both aleatory and epistemic uncertainties exist in a system, the computational burden of using MCS is much higher than having only probability distributions because it is a double-loop sampling process [**35**].

**Table 2** Calculation results of the probability of system failure $P_{\mathrm{f}}^{\mathrm{I}}$

| Results ($P_{\mathrm{f}}$) | PM1 | PM2 | MCS |
|---|---|---|---|
| Upper ($P_{\mathrm{f}}$) | 0.176 | 0.239 | 0.182 |
| Lower ($P_{\mathrm{f}}$) | 0.066 | 0.090 | 0.065 |

From Table 2 it is also known that all the results are intervals rather than point values due to the influence of epistemic uncertainty. Due to the influence of many uncertainties and vagueness in the available information, the system reliability analysis under epistemic uncertainty and aleatory uncertainty is more reasonable than under the traditional reliability analysis because the latter needs all probabilities or probability distributions to be known or perfectly determinable.

## 6 CONCLUSIONS

In structural systems, two types of uncertainty may exist. Epistemic uncertainty (subjective uncertainty) comes from incomplete information or ignorance while aleatory uncertainty (stochastic uncertainty) derives from inherent variations. The present paper proposes a novel and unified system reliability analysis method which can model both epistemic and aleatory uncertainties. In order to avoid the most probable point search, the performance functions are linearized by the sampling method, and the probability of failure in interval form of each linearized performance function can be obtained using the proposed method. The Pearson correlation coefficients between two arbitrary performance functions are defined by the simulation method. Finally, the probability of system failure is calculated based on the extended 'narrow' bound method. The results of the numerical example have shown that the proposed method is effective because it provides a means of system reliability analysis under either epistemic uncertainty, aleatory uncertainty, or both. Generally, it is more robust than traditional reliability methods such as those based on the first-order reliability method because it does not require the most probable point search. Furthermore, the proposed method is superior to the Monte Carlo simulation-based method in terms of computational time. From the numerical example discussed in the paper it is also known that the probability of system failure is an interval rather than a precise value when both epistemic and aleatory uncertainties are present.

It should be noted that there are limitations in the proposed method. Generally, the results calculated using the proposed method are approximate results because it uses linearized function surrogates instead of the original functions, which causes a problem of losing information. Furthermore, if there are many highly non-linear functions in the system, the proposed method may cause large errors. Future work involves accuracy improvement, especially when the probability of system failure is very small and the computational time is huge.

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