A coordination method for fuzzy multi-objective optimization of system reliability

Hong-Zhong Huang\textsuperscript{a,}\textsuperscript{*}, Wei-Dong Wu\textsuperscript{b} and Chun-Sheng Liu\textsuperscript{b}

\textsuperscript{a}School of Mechatronics Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, China
\textsuperscript{b}Department of Mechanical Engineering, Heilongjiang Institute of Science and Technology, Harbin, 150027, China

Abstract. A fuzzy directed graph is used to represent the dynamic relationships among the sub-objectives of a fuzzy multi-objective optimization problem. Using the comprehensive coordination function with exponential weights, we transform a fuzzy multi-objective optimization problem into a single-objective optimization problem. The optimal solution of the single-objective optimization problem can then be obtained with conventional single-objective optimization methods. The optimal solution represents the best trade-off among the possibly conflicting sub-objectives in the original optimization problem. An illustrative example is given to illustrate the effectiveness of the proposed approach.

Keywords: Fuzzy multi-objective optimization, system reliability, coordination method, coordination function, exponential weights

1. Introduction

Most engineering design problems involve optimization of several objectives subject to multiple inequality and equality constraints. These multiple design objectives often compete against one another and a simultaneous optimization of all these objectives is rarely achieved. Thus, in most cases, the design is accomplished by considering the best comprise among the competing objectives. Although the methods of finding a compromise solution to a multi-objective design problem are fairly well established in the literature of engineering optimization [1], the designer often encounters a problem in the development of a precise mathematical model of the system. Vagueness and imprecision often arise due to poorly defined data, unclear system boundaries, unsatisfactory formulation of design objectives, and inability to evaluate the relative importance of the objectives. As the complexity of the system to be designed increases, more assumptions are made about its behavior and hence the ability of the designer to exactly model the system in precise mathematical terms is severely affected. To model the vague and imprecise nature of the design problem, one often uses the fuzzy sets theory [2].

The first application of fuzzy sets theory to the decision-making processes was presented by Bellman and Zadeh [3]. Their paper described the basic concepts and definitions associated with a decision-making process in a fuzzy environment. After their work, there come out a great number of articles dealing with the fuzzy optimization problems. The collection of papers on fuzzy optimization edited by Slowinski [4] and Delgado et al. [5] gives the main stream of this topic.

Formulating a fuzzy optimization problem entails developing membership functions for each constraint and each objective. These membership functions resemble normalization schemes. A variety of such functions are available in the literature (Rao) [6–8].
In their approach the fuzzy mathematical programming problems with fuzzy objectives and constraints can be combined to provide an efficient algorithm for optimization. In terms of fuzzy optimization, both of the objective functions and constraint functions are treated as modified constraints. Consequently, fuzzy optimization lends itself to multi-objective optimization where additional objective functions are modeled as constraints [6, 10].

Zimmermann [10] initiated the application of fuzzy theory to optimization by solving theoretical, fuzzy, linear programming problems. Zimmermann [11] also proposed a max-min approach, which was used for solving fuzzy mathematical problems with fuzzy objectives and fuzzy constraints. Delgado et al. [12], Cadenas and Verdegay [13] and Verdegay [14] discussed fuzzy mathematical programming problems with fuzzy objective coefficients. In their approach the $k$-th objective $\lambda$-constraint approach was used for solving fuzzy multi-objective problems with fuzzy objective coefficients. Chanas [15] presented the possibility of the identification of a complete fuzzy decision in fuzzy linear programming by use of the parametric programming technique. This parametric approach can analytically describe the set of solutions incorporating the whole range of possible values of the fuzzy decision and provides some information on other alternatives close to the maximizing solution. Lai and Hwang [16] proposed an augmented max-min approach, which is essentially an extension of Zimmermann’s approach in essence.

G.-Y. Wang and W.-Q. Wang [17] used a level-cuts approach to solve non-linear, structural problems with fuzzy constraints (and crisp objectives). Rao [6–8] used explicit, continuous membership functions for fuzzy constraints and fuzzy objectives to optimize mechanical systems and structure design; membership functions for the objective function and for the constraints are aggregated into a single, standard optimization problem. Rao’s method of $\lambda$-formulation yields a unique compromise solution with maximum overall satisfaction for fuzzy optimum structural design. Furthermore, he introduced the $\alpha$-cut approach which provided the results in a parametric form for multi-objective problems. Werner [18, 19] proposed an interactive decision support system which aids in solving multiple objective programming problems subject to crisp and fuzzy constraints. One part of the system is an extension of a well-known fuzzy sets approach evaluating possible solutions by their degrees of membership to objectives and constraints.

Xu [20] also transforms problems with fuzzy constraints into standard optimization problems, though with a slightly different format; the final solution is then determined with a bound-constrained optimization approach. Despite the significant amount of work that has been completed with fuzzy optimization, there are few investigations on the use of fuzzy theory to determine feasible points of constrained problems.

Shih [21] presented a global criterion method by fuzzy logic to obtain solutions for multi-criteria crisp or fuzzy structural design, which is not only capable of acquiring the non-dominated solution, but also capable of achieving the highest degree of satisfactory design.

However, the conflicting degree among objectives and the designer’s preferences are neglected to some extent. Loetamonphong et al. [22] studied the optimization problems that have multiple objective functions subject to a set of fuzzy relation equations. Huang [23] presented a fuzzy multi-objective optimization decision-making method, which can be used for the optimization decision-making on two or more objectives of system reliability.

A variety of other new techniques and applications for multi-objective optimization have been developed in recent years for overcoming the drawbacks of traditional methods. Representative work includes multi-objective collaborative optimization [24], interactive multi-objective optimization [25], physical programming [26], game theoretic approach [27], Taguchi routine based method [28], modified Dempster-Shafer theory [29], satisfaction metrics based method [30], genetic algorithms [31–33] and so on. They attempted to resolve the conflicting objectives so that the search for a compromise design over a feasible design space can be facilitated. A comprehensive survey of multi-objective optimization methods (traditional, evolutionary and interactive) is given in [34, 35].

As discussed previously, conflicts in design objectives are unavoidable in any multi-objective optimization problem. How to model and resolve the conflicts is the ongoing and important topic. In this paper, we propose a coordination method for solving multi-objective optimization problems. In Section 2, we summarize the fuzzy multi-objective optimization problem. In Section 3, we develop the coordination method for solv-
2. Model of fuzzy multi-objective optimization [6, 23]

Let the crisp multi-objective optimization problem be stated as follows:

Find $X$ which minimizes $f(X)$

Subject to

$$g_m(X) \in G_m; \ m = 1, 2, \ldots, q$$

where $G_m$ denotes the allowable interval for the constraint function $g_m; G_m = [g_m^{(l)}; g_m^{(u)}]$. When the constraints contain fuzzy information, the problem in Eq. (2) becomes

Find $X$ which minimizes $f(X)$

Subject to

$$g_m(X) \in \tilde{G}_m; \ m = 1, 2, \ldots, q$$

where the tilde is used to denote that the operators or variables contain fuzzy information. Thus the constraint $g_m(X) \in \tilde{G}_m$ means that $g_m(X)$ is a member of the fuzzy set $\tilde{G}_m$ in the sense that the membership function value is greater than 0, i.e., $\mu_{\tilde{G}_m}(g_m) > 0$.

To solve the fuzzy optimization problem in Eq. (3), we propose a coordination method in the following section.

3. A coordination method for fuzzy multi-objective optimization

3.1. Fuzzy directed graph for multi-objective optimization

In the literature of fuzzy systems several approaches to the idea of fuzzy graph have been introduced [36–40]. Reference [41] overviewed most of the literature and gave a general way of treating fuzzy graph problems. The concept of fuzzy directed graph was summarized in [36]. Because of the characteristics of a fuzzy directed graph, we find that it can also be used to represent the dynamic relationship of the entities in a fuzzy multi-objective optimization problem. As shown in Fig. 1, $\{f_1(X), f_2(X), \ldots, f_p(X), F(X)\}$ is a set consisting of $p$ sub-objectives and one objective set function. The objective function measures the overall satisfaction of the $p$ sub-objectives. We use $\mu_j(f_j)$, or $\mu_j$ for simplicity, to denote the degree of satisfaction of the $j$th sub-objective and call it the coordination function of the $j$th sub-objective ($j = 1, 2, \ldots, p$). This coordination function is actually the membership function of $f_j(X)$. A parameter denoted by $\beta_j$ is used in the coordination function $\mu_j$ ($j = 1, 2, \ldots, p$). Correspondingly, we use $\mu(F)$ to denote the coordination function of the overall objective. In expressing the overall objective $F(X)$ as a function of the $p$ sub-objectives, we use $\alpha_j$ to represent the relative importance of the $j$th sub-objective with the requirement of $\sum_{j=1}^{p} \alpha_j = 1$.

3.2. The coordination function of a sub-objective

A membership function is used to fuzzify a sub-objective into a coordination functions $\mu_j$ with an exponential weight as a parameter. This exponential weight denoted by $\beta_j$ controls the pace at which the $j$th sub-objective changes between being satisfactory and nonsatisfactory. If the $j$th sub-objective should be maximized, we formalize its coordination function using a monotonically increasing function given below:

$$\mu_j = \begin{cases} 0, & f_j \leq f_j^l \\ \frac{(f_j^u - f_j)}{(f_j^u - f_j^l)} \beta_j, & f_j^l < f_j < f_j^u \\ 1, & f_j \geq f_j^u \end{cases}$$

(4)

If the $j$th sub-objective should be minimized, we formalize its coordination function using a monotonically decreasing function as follows:

$$\mu_j = \begin{cases} 1, & f_j \leq f_j^l \\ \frac{(f_j^u - f_j)}{(f_j^u - f_j^l)} \beta_j, & f_j^l < f_j < f_j^u \\ 0, & f_j \geq f_j^u \end{cases}$$

(5)

In both Eqs (4) and (5), $f_j^u$ and $f_j^l$ are the upper and lower bounds of the $j$th sub-objective, respectively, and $\beta_j$ is the exponential weight parameter. The upper and lower bounds are usually available from the description of the optimization problem while $\beta_j$ has to be selected based on the following discussions.
The parameter $\beta_j$ of the $j$th sub-objective used in Eqs (4) and (5) must be greater than 0. Its effects on the shape of the coordination function are illustrated in Fig. 2. When it is in the range between 0 and 1, we have the following observations.

When the value of the sub-objective function changes from being the least satisfactory to being the most satisfactory, the slope of the satisfaction decreases. Near the least satisfactory point, even if there is a little change in the sub-objective function value towards satisfaction, the coordination function will be improved greatly. Near the most satisfactory point, even if there is a great change in the sub-objective function value towards satisfaction, the value of the coordination function will change very little. Therefore, the smaller the $\beta_j$ value is, the more easily for the $j$th sub-objective to become satisfactory. However, when $\beta_j$ takes a value between 1 and $\infty$, the observations are just the opposite.

The exponential weights are usually determined by the designer’s knowledge on the relative importance of the sub-objectives. The knowledge is usually fuzzy and can’t lead to accurate $\beta_j$ values. They can also be determined with a systematic approach called fuzzy comprehensive evaluation documented in [42–44].

3.3. Development of the coordination function of the overall objective

Once we have obtained the coordination functions for all sub-objectives as described in Section 3.2, we need to develop a method to find the coordination function of the overall objective function.

Given a solution $X$, we can find the coordination function values of all sub-objectives, that is, $\mu_j$ for $j = 1, 2, \ldots, p$. The total degree of dissatisfaction of the $j$th sub-objective is measured by $1 - \mu_j$. The total degree of dissatisfaction of all $p$ sub-objectives is equal to $\sum_{j=1}^{p} (1 - \mu_j)$. The relative degree of dissatisfaction of the $j$th sub-objective becomes $\frac{(1 - \mu_j)}{\sum_{i=1}^{p} (1 - \mu_i)}$. As we are in the process of finding a solution that would maximize a comprised satisfaction of all sub-objectives, we can use this relative degree of dissatisfaction as the weight in calculating the degree of satisfaction of the overall objective. In other words, we can calculate $\alpha_j$ using the following equation:

$$\alpha_j = \frac{(1 - \mu_j)}{\sum_{j=1}^{p} (1 - \mu_j)}, \quad j = 1, 2, \ldots, p \quad (6)$$

With $\alpha_j$ calculated with Eq. (6), we find the following expression of the coordination function of the overall objective:

$$\mu(F) = \sum_{j=1}^{p} \alpha_j \mu_j = \frac{\sum_{j=1}^{p} (1 - \mu_j) \mu_j}{\sum_{j=1}^{p} (1 - \mu_j)} \quad (7)$$
At any solution, if \( \mu_j = 1 \), that is, the \( j \)th sub-objective is totally satisfied, then this sub-objective has no effect on the comprehensive decision-making process any more. This sub-objective can then be removed from this comprehensive decision making process.

When the \( \mu_j \) \((1 \leq j \leq p)\) values do not reach their maximum value 1 simultaneously, then there exists a point \( X^{**} \) and the \( k \)th sub-objective which satisfy the following equations:

\[
\frac{\partial \mu(F(X^{**}))}{\partial (\mu_k(f_k(X^{**})))} = \\
\left[ \sum_{j=1}^{p} (1 - \mu_j(f_j(X^{**}))) (1 - 2\mu_k(f_k(X^{**}))) \right] \\
+ \mu_j(f_j(X^{**})) \\
\left\{ \sum_{j=1}^{p} (1 - \mu_j(f_j(X^{**}))) \right. \\
\left. \right\}^2 = 0 \tag{8}
\]

\[
\frac{\partial^2 \mu(F(X^{**}))}{\partial (\mu_k(f_k(X^{**})))^2} = -4 \cdot \\
\sum_{j=1,j\neq k}^{p} (1 - \mu_j(f_j(X^{**}))) \sum_{l\geq j,l\neq k}^{p} (1 - \mu_l(f_l(X^{**}))) \\
\left[ \sum_{j=1}^{p} (1 - \mu_j(f_j(X^{**}))) \right]^{3} < 0 \tag{9}
\]

Based on Eqs (8) and (9), once the values of the coordination function of all other sub-objectives are specified, there exists an optimal value of the coordination function of the \( k \)th sub-objective at which the coordination function of the overall objective attains its maximum. We take a multi-objective optimization problem with two sub-objectives as an example. When the value of one sub-objective function is specified (correspondingly its coordination function value is specified), there is an optimal value of the other sub-objective at which the overall objective function has the highest satisfaction value, as shown in Fig. 3.

4. Hypothetical case example

A system with five series sub-systems is considered. The relationship between the cost \( C_i \) and reliability \( R_i \) of the \( i \)th sub-system can be represented as

\[
C_i(R_i) = a_i \ln(1/(1 - R_i)) + b_i \tag{10}
\]

where \( a_i \) and \( b_i \) are the reliability cost coefficient of the \( i \)th sub-system. For illustrative purposes, we use the following values of the parameters:

\[
a_1 = 24, \quad a_2 = 8, \quad a_3 = 8.75, \quad a_4 = 7.14, \quad a_5 = 3.33
\]

\[
b_1 = 120, \quad b_2 = 80, \quad b_3 = 70, \quad b_4 = 50, \quad b_5 = 30
\]

These parameters are in generic units. In addition, we are also given requirements that the cost of the system \( C_S \) should be between 500 and 600 and the system reliability \( R_S \) should be above 0.90. We need to determine the sub-systems’ reliabilities \( R = [R_1, R_2, \cdots, R_5]^T \) to maximize \( R_S \) and minimize \( C_S \).

The multi-objective optimization model of the problem is expressed as

\[
\text{find } R = [R_1, R_2, \cdots, R_5]^T \tag{11}
\]

\[
to \text{maximize } R_S(R) = \prod_{i=1}^{5} R_i \tag{12}
\]
and minimize
\[
C_S = \sum_{i=1}^{5} \{a_i \ln[1/(1 - R_i)] + b_i\} \tag{13}
\]

The coordination functions of the reliability and the cost of the system are expressed, respectively, as
\[
\mu_1(R_S) =
\begin{cases}
1, & R_S = 1 \\
((R_S - 0.9)/0.1)^{\beta_R}, & 0.9 < R_S < 1 \\
0, & R_S \leq 0.90
\end{cases} \tag{14}
\]
\[
\mu_2(C_S) =
\begin{cases}
1, & C_S \leq 500 \\
((600 - C_S)/100)^{\beta_C}, & 500 < C_S < 600 \\
0, & C_S \geq 600
\end{cases} \tag{15}
\]

The comprehensive coordination function considering both sub-objectives can be defined as
\[
\mu(F(R, C)) = \alpha_1 \mu_1(R_S) + \alpha_2 \mu_2(C_S), \tag{16}
\]

where
\[
\alpha_1 = \frac{1 - \mu_1(R_S)}{2 - \mu_1(R_S) - \mu_2(C_S)},
\alpha_2 = \frac{1 - \mu_2(R_S)}{2 - \mu_1(R_S) - \mu_2(C_S)}.
\tag{17}
\]

These expressions in Eq. (17) are obtained from Eq. (6).

The exponential weights \(\beta_R\) and \(\beta_C\) of the reliability and the cost of the system can be set at different values. The results obtained using an unconstrained optimization method [45] are shown in Table 1.

From the results given in Table 1, we can see the larger the exponential weight \(\beta_R\) of system reliability is, the larger the reliability \(R_S\) and the cost \(C_S\) will be, and the smaller the satisfaction degree in cost will be. When the exponential weight \(\beta_C\) of system cost increases, system reliability \(R_S\) decreases, and system cost \(C_S\) decreases. Then the satisfactory degree in reliability decreases, and the satisfactory degree in cost increases. When \(\beta_R\) and \(\beta_C\) are equal, the satisfactory degree in each sub-objective will be the lowest, the unsatisfactory degree in each sub-objective is the highest, and the comprehensive coordination ability of decision maker is the weakest. With the gradually increasing of the difference of the two exponential weights, the satisfactory degree of each sub-objective becomes larger, the total unsatisfactory degree becomes smaller, and the comprehensive coordination ability of decision maker becomes stronger, until the single-objective optimization is reached.

5. Conclusion

A coordination method for fuzzy multi-objective optimization is developed. It incorporates not only fuzzy factors in the decision-making process but also the designer’s experience and knowledge. The optimization results of the numerical example illustrate that when
the exponential weight of system reliability is equal to
that of system cost, comprehensive coordination abil-
ity of decision maker is weakest. When the difference
of the two exponential weights increases, the com-
prehensive coordination ability of decision maker is also
strengthened until the single-objective optimization is
reached.

The exponential weights of sub-objectives can be
positive number, which is convenient to use in compar-
ing and judging the complex relationship among sub-
objectives. The range of an exponential weight should
be determined based on specific problems. If the ex-
ponential weight is much larger than one, the slope of
the coordination function will be very large near the
most satisfactory point, and the ideal satisfactory solu-
tion will probably not be obtained. Therefore, the
exponential weight is suggested to be less than one.

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Table 1

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