

# Design Optimization With Discrete and Continuous Variables of Aleatory and Epistemic Uncertainties

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Reliability based design optimization has received increasing attention for satisfying high requirements on reliability and safety in structure design. However, in practical engineering design, there are both continuous and discrete design variables. Moreover, both aleatory uncertainty and epistemic uncertainty may associate with design variables. This paper proposes the formulation of random/fuzzy continuous/discrete variables design optimization (RFCDV-DO) and two different approaches for uncertainty analysis (probability/possibility analysis). A method named random/fuzzy sequential optimization and reliability assessment is proposed based on the idea of sequential optimization and reliability assessment to improve efficiency in solving RFCDV-DO problems. An engineering design problem is utilized to demonstrate the approaches and the efficiency of the proposed method. [DOI: 10.1115/1.3066712]

**Keywords:** aleatory uncertainty, epistemic uncertainty, continuous and discrete variables, random/fuzzy continuous/discrete variables design optimization, sequential optimization and reliability assessment

## 1 Introduction

In recent years, increasing attention has been focused on the effect of uncertainties on structure design. Uncertainties can be categorized into aleatory uncertainty (AU) and epistemic uncertainty (EU). The design variables with AU can be treated as random variables. EU—reducible uncertainty, subjective uncertainty—caused by lack of knowledge, can be modeled with possibility theory. Design variables with EU can be treated as fuzzy variables [1,2].

To deal with the case of design variables associated with AU, reliability-based design optimization (RBDO), which assumes that there are sufficient data to construct probability distributions of inputs, is popular in structure design optimization [3–14]. Performance measure approach (PMA), which can efficiently decrease computational cost in reliability analysis, was proposed in Ref. [12]. Sequential optimization and reliability assessment (SORA) developed in Ref. [9] decouples the reliability assessment from optimization.

To deal with the case of design variables associated with EU, possibility theory was utilized in Refs. [2,15–17]. The case of design variables associated with both AU and EU was dealt with in Refs. [2,17]. In Ref. [2], the uncertainty analysis was based on the concept of conditional possibility of failure, and a method named maximal failure search (MFS) based on PMA was proposed. In Ref. [17], possibility constraints are treated as functions of the corresponding reliability index. Evidence theory was applied in design to deal with the case when both types of uncertainties exist [18]. In Ref. [19], Bayesian statistics was also used; an objective function of the confidence of design's reliability was added into the original optimization.

In this paper the RFCDV-DO problem is dealt with. Two approaches for uncertainty analysis (probability/possibility analysis)

are developed based on conditional possibility of failure, and formulations for probability/possibility analysis are proposed. To efficiently deal with RFCDV-DO problems, a method named random/fuzzy SORA (RFSORA) is developed based on SORA.

This paper is organized as follows. In Sec. 2, the mathematical formulation of RFCDV-DO is given. In Sec. 3, two different approaches for uncertainty analysis and their mathematical formulations are developed. In Sec. 4, the method RFSORA is proposed. An engineering design problem is utilized to demonstrate the approaches and the efficiency of RFSORA in Sec. 5, followed by the conclusions in Sec. 6.

## 2 Random/Fuzzy Continuous/Discrete Variables Design Optimization

The mathematical formulation of RFCDV-DO is given as

$$\min_{(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M)} f(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M)$$

such that

$$\Pi(G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0) \leq \alpha_t$$

$$g^{(j)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M) \leq 0$$

$$\mathbf{d}_c^L \leq \mathbf{d}_c \leq \mathbf{d}_c^U, \quad \mathbf{d}_d^L \leq \mathbf{d}_d \leq \mathbf{d}_d^U \quad (1)$$

$$\mathbf{X}_c^{M,L} \leq \mathbf{X}_c^M \leq \mathbf{X}_c^{M,U}, \quad \mathbf{X}_d^{M,L} \leq \mathbf{X}_d^M \leq \mathbf{X}_d^{M,U}$$

$$i = 1, 2, \dots, n_G, \quad j = 1, 2, \dots, n_g$$

where  $\mathbf{X}_c = \{\mathbf{X}_{rc}, \mathbf{X}_{fc}\}$ ,  $\mathbf{X}_d = \{\mathbf{X}_{rd}, \mathbf{X}_{fd}\}$ ,  $\mathbf{P}_c = \{\mathbf{P}_{rc}, \mathbf{P}_{fc}\}$ , and  $\mathbf{P}_d = \{\mathbf{P}_{rd}, \mathbf{P}_{fd}\}$ . Subscripts  $c$ ,  $d$ ,  $rc$ ,  $rd$ ,  $fc$ , and  $fd$  denote that the type of variables and parameters is continuous, discrete, continuous random, discrete random, continuous fuzzy, and discrete fuzzy, respectively. The fuzzy variable is continuous if the possibility  $\Pi\{X=x\}$  is a continuous function of  $x$ . The fuzzy variable is discrete if a countable sequence  $\{x_1, x_2, \dots\}$  exists such that  $\Pi\{X \neq x_1, X \neq x_2, \dots\} = 0$  [20]. The superscript  $M$  denotes the mean

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value of a random variable or parameter, and maximal grade point of a fuzzy variable or parameter, respectively. The maximal grade point of a fuzzy variable  $X$  is defined as  $X^M = \{x | \max\{\Pi_X(x)\}$  where  $\Pi_X(x)$  is the membership function of  $X$ .  $\mathbf{d}$  is a vector of deterministic design variables.  $\mathbf{X}$  is a vector composed by random and fuzzy variables while  $\mathbf{P}$  is a vector of random and fuzzy parameters. The mean value or maximal grade point of parameter with uncertainties is known and fixed while that of a variable with uncertainties is design variable.  $f(\cdot)$  is the objective function. The probability/possibility constraint is  $\Pi(\cdot) \leq \alpha_f$  for the failure event of  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0$ .  $\alpha_f$  is the allowable possibility of failure.  $g(\cdot) \leq 0$  are deterministic constraints.  $n_G, n_g$  are numbers of probability/possibility constraints and deterministic constraints, respectively. Superscripts  $L$  and  $U$  denote lower and upper bounds, respectively.

During optimization process, the feasibilities of probability/possibility constraints should be checked out at an obtained design point. This analysis process is called uncertainty analysis (probability/possibility analysis). In the Sec. 3, two different approaches for uncertainty analysis are proposed.

### 3 Approaches for Uncertainty Analysis

The conditional possibility of failure developed in Ref. [2] is introduced. Suppose two continuous fuzzy variables  $X_1, X_2$  are mutually noninteractive with membership functions  $\Pi_{X_1}(x_1)$  and  $\Pi_{X_2}(x_2)$ , respectively, and the failure event is  $G(x_1, x_2) > 0$ . The possibility of failure  $\Pi_f$  can be computed by

$$\begin{aligned} \Pi_f &= \sup_{G(x_1, x_2) > 0} [\min\{\Pi_{X_1}(x_1), \Pi_{X_2}(x_2)\}] \\ &= \sup_{x_2} [\sup_{x_1: G(x_1, x_2) > 0} [\min\{\Pi_{X_1}(x_1), \Pi_{X_2}(x_2)\}]] \\ &= \sup_{x_2} [\min\{\sup_{x_1: G(x_1, x_2) > 0} \Pi_{X_1}(x_1), \sup_{x_1: G(x_1, x_2) > 0} \Pi_{X_2}(x_2)\}] \\ &= \sup_{x_2} [\min\{\Pi_f | \{X_2 = x_2\}, \Pi_{X_2}(x_2)\}] \end{aligned}$$

where  $\Pi_f | \{X_2 = x_2\} = \sup_{x_1: G(x_1, x_2) > 0} \Pi_{X_1}(x_1)$  is defined as the conditional possibility of failure when  $X_2 = x_2$  [2].

Based on the concept of the conditional possibility of failure, two different approaches for probability/possibility analysis are developed when continuous and discrete variables and parameters contain both AU and EU. First of all, the meanings of some symbols are explained as follows.  $F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd})$  is the joint cumulative distribution function (CDF) of discrete random variables  $\mathbf{X}_{rd}$ ;  $f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc})$  is the joint probability density function (PDF) of continuous random variables  $\mathbf{X}_{rc}$ ;  $F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd})$  is the joint CDF of discrete random parameters  $\mathbf{P}_{rd}$ ; and  $f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc})$  is the joint PDF of continuous random parameters  $\mathbf{P}_{rc}$ ;  $\Pi_{\mathbf{X}_f, \mathbf{P}_f}(\mathbf{x}_f, \mathbf{p}_f)$  is the membership function of fuzzy variables  $\mathbf{X}_f = \{\mathbf{X}_{fc}, \mathbf{X}_{fd}\}$  and fuzzy parameters  $\mathbf{P}_f = \{\mathbf{P}_{fc}, \mathbf{P}_{fd}\}$ , where  $\mathbf{X}_{fc}, \mathbf{X}_{fd}$  are continuous and discrete fuzzy variables, respectively, and  $\mathbf{P}_{fc}, \mathbf{P}_{fd}$  are continuous and discrete fuzzy parameters, respectively.

The constraint function  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d)$  is a continuous function when all variables and parameters are continuous. All random variables and parameters are assumed to be independent. All fuzzy variables and parameters are assumed noninteractive. Each continuous fuzzy variable and parameter is assumed with its membership function satisfying properties of unity, strong convexity, and boundedness (detailed definitions of these three properties can be found in Ref. [15]). In this paper, this type of discrete fuzzy variable and parameter is dealt with: assume  $\mu(x)$  is the membership function on  $\{x_1, x_2, \dots, x_n\}$ , only one value  $x_i \in \{x_1, x_2, \dots, x_n\}$  satisfies  $\mu(x_i) = \max_{h=1 \sim n} \mu(x_h) = 1$ ;  $x_1 < x_2 < \dots < x_i < x_{i+1} < \dots < x_n$ ,  $\mu(x_1) < \mu(x_2) < \dots < \mu(x_i), \mu(x_i) > \mu(x_{i+1}) > \dots > \mu(x_n)$ .

**3.1 Transformation.** This transformation for fuzzy variables and parameters in  $X$ -space into standard noninteractive ones in  $V$ -space is on: the membership is the same before and after transformation [15]. The standard normalized fuzzy variable  $V$  has an isosceles triangular membership function as

$$\Pi_V(v) = \begin{cases} v+1 & -1 \leq v \leq 0 \\ 1-v & 0 \leq v \leq 1 \end{cases} = 1 - |v|, \quad |v| \leq 1$$

This transformation can be written as

$$v = \begin{cases} \Pi_X(x) - 1 & x \leq X^M \\ 1 - \Pi_X(x) & x > X^M \end{cases} \quad (2)$$

where  $\Pi_X(x)$  is the membership function of fuzzy variable  $X$ . The discrete fuzzy variables and parameters of the type discussed here can also be uniquely transformed into the standard normalized fuzzy ones using Eq. (2).

Suppose there are two mutually noninteractive fuzzy variables  $X_1, X_2$  with their membership functions  $\Pi_{X_1}(x_1), \Pi_{X_2}(x_2)$  belonging to the type discussed before. After transforming into the standard normalized fuzzy ones, the joint membership function is given as

$$\begin{aligned} \Pi_{X_1, X_2}(x_1, x_2) &= \min\{\Pi_{X_1}(x_1), \Pi_{X_2}(x_2)\} \\ &= \min\{\Pi_{V_1}(v_1), \Pi_{V_2}(v_2)\} \\ &= \min\{1 - |v_1|, 1 - |v_2|\} \\ &= 1 - \|(v_1, v_2)\|_\infty \end{aligned}$$

**3.2 Approach 1 for Uncertainty Analysis.** The possibility of failure  $\Pi_f = \Pi(G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0)$  can be calculated by the following steps.

First, temporarily fix the fuzzy variables and parameters at  $\mathbf{X}_f = \mathbf{x}_f = (\mathbf{x}_{fc}, \mathbf{x}_{fd})$ , and  $\mathbf{P}_f = \mathbf{p}_f = (\mathbf{p}_{rc}, \mathbf{p}_{rd})$ ; the conditional probability of failure can then be calculated by

$$\begin{aligned} P_f | \{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\} &= \sum_{t=1}^N \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right. \\ &\quad \left. \times F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t) \right] \end{aligned}$$

where  $G(\cdot) = G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rc}, \mathbf{x}_{rd}^t, \mathbf{x}_{fc}, \mathbf{x}_{fd}, \mathbf{P}_{rc}, \mathbf{p}_{rd}^t, \mathbf{p}_{fc}, \mathbf{p}_{fd})$ .  $N$  is the total number of all possible combinations of discrete random variables and parameters.  $\mathbf{x}_{rd}^t, \mathbf{p}_{rd}^t$  are values of  $\mathbf{X}_{rd}, \mathbf{P}_{rd}$  in the combination mode  $t$ .

Second, set the conditional possibility of failure to be the same as the conditional probability of failure. It is a reasonable assumption because possibility is an alternative and a vague measure when probability is difficult to compute or when information is limited, and also the possibility of an event can be assigned as the upper bound of the probability when the probability of that event is unknown. If there exists the probability, the possibility can be set the same as the probability [2].

Finally the possibility of failure  $\Pi_f$  can be calculated by

$$\begin{aligned} \Pi_f &= \sup_{\mathbf{x}_f, \mathbf{p}_f} [\min\{\Pi_f | \{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\}, \Pi_{\mathbf{X}_f, \mathbf{P}_f}(\mathbf{x}_f, \mathbf{p}_f)\}] \\ &= \sup_{\mathbf{x}_f, \mathbf{p}_f} [\min\{P_f | \{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\}, \Pi_{\mathbf{X}_f, \mathbf{P}_f}(\mathbf{x}_f, \mathbf{p}_f)\}] \\ &= \sup_{\mathbf{x}_f, \mathbf{p}_f} \left[ \min \left\{ \sum_{t=1}^N \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right. \right. \right. \\ &\quad \left. \left. \left. \times F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t) \right], \Pi_{\mathbf{X}_f, \mathbf{P}_f}(\mathbf{x}_f, \mathbf{p}_f) \right\} \right] \quad (3) \end{aligned}$$

In approach 1 for uncertainty analysis, to avoid the huge compu-

tation in direct calculation of the conditional probability of failure, discrete random variables and parameters are initially transformed into discrete fuzzy ones based on possibility-probability consistency. In this paper the following possibility-probability transformation is used. Assume that  $p(y)$  is a probability distribution on  $Y = \{y_1, y_2, \dots, y_n\}$  whose elements have been indexed in descending order of their probabilities  $p_1 \geq p_2 \geq \dots \geq p_n$ . Then the possibility distribution on  $Y$  can be calculated as

$$\begin{aligned}\mu_n &= n \times p_n \\ \mu_j &= j(p_j - p_{j+1}) + \mu_{j+1} \\ j &= n - 1, \dots, 1\end{aligned}\quad (4)$$

If  $p_j = p_{j+1}$ , then  $\mu_j = \mu_{j+1}$ ; if  $p_j = 0$ , then  $\mu_j = 0$  [21].

In approach 1 for uncertainty analysis, this type of discrete random variables and parameters is dealt with:  $\mathbf{p}$  is a probability distribution on  $\{x_1, x_2, \dots, x_n\}$ ,  $(x_1 < x_2 < \dots < x_i < \dots < x_n)$ , only one value  $x_i \in \{x_1, x_2, \dots, x_n\}$  satisfies  $p(x_i) = \max_{m=1 \sim n} p(x_m)$  and  $p(x_1) < p(x_2) < \dots < p(x_i), p(x_i) > \dots > p(x_{n-1}) > p(x_n)$ . The discrete fuzzy variables and parameters transformed from discrete random ones are assumed to be noninteractive with  $\mathbf{X}_f, \mathbf{P}_f$ .

After the discrete random variables and parameters have been initially transformed into discrete fuzzy variables and parameters, there are no longer discrete random variables and parameters. Hence, the possibility of failure can be calculated as

$$\begin{aligned}\Pi_f &= \sup_{\substack{\mathbf{x}_f, \mathbf{x}_{\text{frd}}, \\ \mathbf{p}_{\text{frd}}, \mathbf{p}_f}} \left[ \min \left\{ \int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}}, \right. \right. \\ &\quad \left. \left. \Pi(\mathbf{x}_f, \mathbf{x}_{\text{frd}}, \mathbf{p}_{\text{frd}}, \mathbf{p}_f) \right\} \right]\end{aligned}\quad (5)$$

where  $\Pi(\cdot)$  is the membership function of  $\mathbf{X}_f, \mathbf{x}_{\text{frd}}, \mathbf{p}_{\text{frd}}, \mathbf{P}_f$ . The subscript frd denotes a discrete fuzzy variable or parameter transformed from the discrete random one.

All fuzzy variables and parameters  $\mathbf{X}_f, \mathbf{x}_{\text{frd}}, \mathbf{p}_{\text{frd}}, \mathbf{P}_f$ , including the ones transformed from discrete random variables and parameters, are transformed into standard normalized fuzzy variables and parameters  $\mathbf{V}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f$  in  $V$ -space. Equation (5) can be written as

$$\begin{aligned}\Pi_f &= \sup_{\substack{\mathbf{x}_f, \mathbf{x}_{\text{frd}}, \\ \mathbf{p}_{\text{frd}}, \mathbf{p}_f}} \left[ \min \left\{ \int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}}, \right. \right. \\ &\quad \left. \left. \Pi(\mathbf{x}_f, \mathbf{x}_{\text{frd}}, \mathbf{p}_{\text{frd}}, \mathbf{p}_f) \right\} \right] \\ &= \sup_{\substack{\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \\ \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f}} \left[ \min \left\{ \int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}}, \right. \right. \\ &\quad \left. \left. 1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} \right\} \right]\end{aligned}\quad (6)$$

When design point  $\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M$  is given, two ways can be utilized to check the feasibility of a probability/possibility constraint, which is given by  $\Pi(G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0) \leq \alpha_t$ .

The first way, whose computation is huge, directly computes the possibility of failure using Eq. (6) and then compares the result with  $\alpha_t$ . If the result is not larger than  $\alpha_t$ , the design point is feasible, otherwise infeasible.

The second way is when the design point  $(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M)$  is given, the probability distributions of random variables and parameters and the membership functions of fuzzy variables and

parameters are obtained. Among all points which satisfy  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0$ , there are three cases:

- (1)  $1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} < \alpha_t$
- (2)  $1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} = \alpha_t$
- (3)  $1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} > \alpha_t$

When the fuzzy parts satisfy cases (1) and (2), the value of  $\min\{\int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}}, 1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty}\}$  will be not larger than  $\alpha_t$ . These two cases do not affect the final result of  $\Pi_f \leq \alpha_t$ .

If the value of integration  $\int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}}$  is not larger than  $\alpha_t$  whenever the fuzzy part satisfies case (3), then the possibility of failure satisfies  $\Pi_f \leq \alpha_t$ . Transform continuous random variables and parameters  $\mathbf{X}_{\text{rc}}, \mathbf{P}_{\text{rc}}$  into standard normal ones  $\mathbf{U}_{\text{rc}}, \mathbf{UP}_{\text{rc}}$  in  $U$ -space using the Rosenblatt transformation [22]. Given the fuzzy part that satisfies  $1 - \|\mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} > \alpha_t$ , whether or not  $\int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}} \leq \alpha_t$  can be checked by following optimization based on the first order reliability method (FORM):

$$\max G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{U}_{\text{rc}}, \mathbf{UP}_{\text{rc}}, \mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f)$$

such that

$$\|\mathbf{U}_{\text{rc}}, \mathbf{UP}_{\text{rc}}\|_2 \leq -\Phi^{-1}(\alpha_t) \quad (7)$$

where  $\Phi(\cdot)$  is the CDF of the standard normal random variable. If the maximal value  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{U}_{\text{rc}}^*, \mathbf{UP}_{\text{rc}}^*, \mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f) \leq 0$  at the solution  $\mathbf{U}_{\text{rc}}^*, \mathbf{UP}_{\text{rc}}^*$ , the value of integration  $\int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}} \leq \alpha_t$  and vice versa. Hence, whether or not  $\Pi_f \leq \alpha_t$  at the current design point  $(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M)$ , optimization formulated as follows can be used to judge:

$$\max G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{U}_{\text{rc}}, \mathbf{UP}_{\text{rc}}, \mathbf{v}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f)$$

such that

$$\|\mathbf{U}_{\text{rc}}, \mathbf{UP}_{\text{rc}}\|_2 \leq -\Phi^{-1}(\alpha_t)$$

$$\|\mathbf{V}_f, \mathbf{v}_{\text{frd}}, \mathbf{VP}_{\text{frd}}, \mathbf{VP}_f\|_{\infty} < 1 - \alpha_t \quad (8)$$

The solutions are the most probable/possible point (MPPP)  $(\mathbf{U}_{\text{rc}}^*, \mathbf{UP}_{\text{rc}}^*, \mathbf{V}_f^*, \mathbf{v}_{\text{frd}}^*, \mathbf{VP}_{\text{frd}}^*, \mathbf{VP}_f^*)$  and  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{U}_{\text{rc}}^*, \mathbf{UP}_{\text{rc}}^*, \mathbf{V}_f^*, \mathbf{v}_{\text{frd}}^*, \mathbf{VP}_{\text{frd}}^*, \mathbf{VP}_f^*)$ , which is the value of the performance measure at the MPPP. If  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{U}_{\text{rc}}^*, \mathbf{UP}_{\text{rc}}^*, \mathbf{V}_f^*, \mathbf{v}_{\text{frd}}^*, \mathbf{VP}_{\text{frd}}^*, \mathbf{VP}_f^*) > 0$ , the current design point  $\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M$  is not feasible and vice versa.

This way is called PMA. Although there are different kinds of transformations from discrete random variables and parameters into fuzzy ones, the differences in transformations do not affect the final results whenever the following requirements are satisfied: (1) Discrete random variables and parameters belong to the type discussed before, and (2) the minimum value of membership function of the discrete fuzzy variable or parameter transformed from the discrete random one is larger than  $\alpha_t$  and the maximum value is equal to 1.

**3.3 Approach 2 for Uncertainty Analysis.** In approach 2 for uncertainty analysis, there is no requirement about each discrete random variable and parameter. The steps of calculating the possibility of failure are the same as those in Sec. 3.2.

Equation (3) in Section 3.2 can be further written as

$$\begin{aligned}\Pi_f &\leq \sup_{\mathbf{x}_f, \mathbf{p}_f} \left[ \min \left\{ \max_{\mathbf{x}_{\text{rd}}, \mathbf{p}_{\text{rd}}} \left[ \int_{\mathbf{x}_{\text{rc}}, \mathbf{p}_{\text{rc}}: G(\cdot) > 0} f_{\mathbf{X}_{\text{rc}}}(\mathbf{x}_{\text{rc}}) f_{\mathbf{P}_{\text{rc}}}(\mathbf{p}_{\text{rc}}) d\mathbf{x}_{\text{rc}} d\mathbf{p}_{\text{rc}} \right] \right. \right. \\ &\quad \left. \left. \times \sum_{i=1}^N [F_{\mathbf{X}_{\text{rd}}}(\mathbf{x}_{\text{rd}}^i) \times F_{\mathbf{P}_{\text{rd}}}(\mathbf{p}_{\text{rd}}^i)], \Pi_{\mathbf{X}_f, \mathbf{P}_f}(\mathbf{x}_f, \mathbf{p}_f) \right\} \right]\end{aligned}\quad (9)$$

where  $\sum_{i=1}^N [F_{\mathbf{X}_{\text{rd}}}(\mathbf{x}_{\text{rd}}^i) \times F_{\mathbf{P}_{\text{rd}}}(\mathbf{p}_{\text{rd}}^i)]$  can be obtained initially in design and is the sum of probabilities of all combinations of discrete

random variables and parameters. First, all fuzzy variables and parameters (continuous and discrete)  $(\mathbf{X}_f, \mathbf{P}_f)$  are transformed into the standard fuzzy ones  $(\mathbf{V}_f, \mathbf{VP}_f)$  ( $\mathbf{V}_f = (\mathbf{V}_{fc}, \mathbf{V}_{fd})$ , and  $\mathbf{VP}_f = (\mathbf{VP}_{fc}, \mathbf{VP}_{fd})$ ) in the  $V$ -space using Eq. (2). Equation (9) can be written as

$$\Pi_f \leq \sup_{\mathbf{v}_f, \mathbf{VP}_f} \left[ \min \left\{ \max_{\mathbf{x}_{rc}, \mathbf{p}_{rc}} \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right] \times \sum_{t=1}^N [F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)], 1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty \right\} \right] \quad (10)$$

If the right part of Eq. (10) is not larger than  $\alpha_t$ , then  $\Pi_f \leq \alpha_t$ . Based on Eq. (10), two ways can be utilized to check the feasibility of  $\Pi_f \leq \alpha_t$  at a design point.

The first way, whose computational price is too expensive to afford, directly calculates the value of the right part of Eq. (10) and then compares the result with  $\alpha_t$ . If the result is not larger than  $\alpha_t$ , the current design point is feasible; otherwise infeasible.

The second one is PMA, which is based on this: given a design point  $\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M$ , the probability distributions of random variables and parameters, and the membership functions of fuzzy variables and parameters are obtained. Among all points which satisfy  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d) > 0$ , there are still three cases similar to those in approach 1 for uncertainty analysis.

- (1)  $1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty < \alpha_t$
- (2)  $1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty = \alpha_t$
- (3)  $1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty > \alpha_t$

When the fuzzy parts satisfy cases (1) and (2), the value of

$$\min \left\{ \max_{\mathbf{x}_{rc}, \mathbf{p}_{rc}} \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right] \times \sum_{t=1}^N [F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)], 1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty \right\}$$

will be not larger than  $\alpha_t$ . These two cases do not affect the final result of  $\Pi_f \leq \alpha_t$ . If the value

$$\max_{\mathbf{x}_{rc}, \mathbf{p}_{rc}} \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right] \times \sum_{t=1}^N [F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)]$$

is not larger than  $\alpha_t$  whenever the fuzzy part satisfies case (3), then  $\Pi_f \leq \alpha_t$ . Transform all continuous random variables and parameters  $(\mathbf{X}_{rc}, \mathbf{P}_{rc})$  into the standard normal ones  $(\mathbf{U}_{rc}, \mathbf{UP}_{rc})$  in  $U$ -space using Rosenblatt transformation. Given the fuzzy part that satisfies  $1 - \|\mathbf{v}_f, \mathbf{VP}_f\|_\infty > \alpha_t$ , whether or not

$$\max_{\mathbf{x}_{rc}, \mathbf{p}_{rc}} \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right] \times \sum_{t=1}^N [F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)] \leq \alpha_t$$

can be checked by following optimization:

$$\max G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}, \mathbf{U}_{rc}, \mathbf{V}_{fd}, \mathbf{V}_{fc}, \mathbf{P}_{rd}, \mathbf{UP}_{rc}, \mathbf{VP}_{fd}, \mathbf{VP}_{fc}) \quad (11)$$

such that

$$\|(\mathbf{U}_{rc}, \mathbf{UP}_{rc})\|_2 \leq -\Phi^{-1} \left[ \frac{\alpha_t}{\sum_{t=1}^N \{F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)\}} \right]$$

If the maximal value  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}^*, \mathbf{U}_{rc}^*, \mathbf{V}_{fd}, \mathbf{V}_{fc}, \mathbf{P}_{rd}^*, \mathbf{UP}_{rc}^*, \mathbf{VP}_{fd}, \mathbf{VP}_{fc}) \leq 0$  at the solution  $\mathbf{X}_{rd}^*, \mathbf{U}_{rc}^*, \mathbf{P}_{rd}^*, \mathbf{UP}_{rc}^*$ , the value

$$\max_{\mathbf{x}_{rd}, \mathbf{p}_{rd}} \left[ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc}: G(\cdot) > 0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \right] \times \sum_{t=1}^N [F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)] \leq \alpha_t$$

Whether or not  $\Pi_f \leq \alpha_t$  at the current design point  $(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M)$ , an optimization can be used to check as

$$\max G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}, \mathbf{U}_{rc}, \mathbf{V}_{fd}, \mathbf{V}_{fc}, \mathbf{P}_{rd}, \mathbf{UP}_{rc}, \mathbf{VP}_{fd}, \mathbf{VP}_{fc}) \quad (12)$$

such that

$$\|(\mathbf{U}_{rc}, \mathbf{UP}_{rc})\|_2 \leq -\Phi^{-1} \left[ \frac{\alpha_t}{\sum_{t=1}^N \{F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^t) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^t)\}} \right]$$

$$\|(\mathbf{V}_{fd}, \mathbf{V}_{fc}, \mathbf{VP}_{fd}, \mathbf{VP}_{fc})\|_\infty \leq 1 - \alpha_t$$

where  $\mathbf{X}_{rd}, \mathbf{P}_{rd}$  vary in all possible combined modes. The solutions are the MPPP  $(\mathbf{X}_{rd}^*, \mathbf{U}_{rc}^*, \mathbf{V}_{fd}^*, \mathbf{V}_{fc}^*, \mathbf{P}_{rd}^*, \mathbf{UP}_{rc}^*, \mathbf{VP}_{fd}^*, \mathbf{VP}_{fc}^*)$  and  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}^*, \mathbf{U}_{rc}^*, \mathbf{V}_{fd}^*, \mathbf{V}_{fc}^*, \mathbf{P}_{rd}^*, \mathbf{UP}_{rc}^*, \mathbf{VP}_{fd}^*, \mathbf{VP}_{fc}^*)$ , which is the value of the performance measure at the MPPP. If  $G(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}^*, \mathbf{U}_{rc}^*, \mathbf{V}_{fd}^*, \mathbf{V}_{fc}^*, \mathbf{P}_{rd}^*, \mathbf{UP}_{rc}^*, \mathbf{VP}_{fd}^*, \mathbf{VP}_{fc}^*) > 0$ , the current design point  $\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M$  is not feasible and vice versa. The MPPP  $(\mathbf{X}_{rd}^*, \mathbf{X}_{rc}^*, \mathbf{X}_{fd}^*, \mathbf{X}_{fc}^*, \mathbf{P}_{rd}^*, \mathbf{P}_{rc}^*, \mathbf{P}_{fd}^*, \mathbf{P}_{fc}^*)$  in  $X$ -space can be obtained by the inverse Rosenblatt transformation and Eq. (2).

There are double loops in solving the RFCDV-DO problems. The outer loop is to minimize the value of objective function while executing uncertainty analysis in the inner loop. To efficiently solve RFCDV-DO problems, RFSORA is proposed in Sec. 4.

#### 4 Random/Fuzzy Sequential Optimization and Reliability Assessment

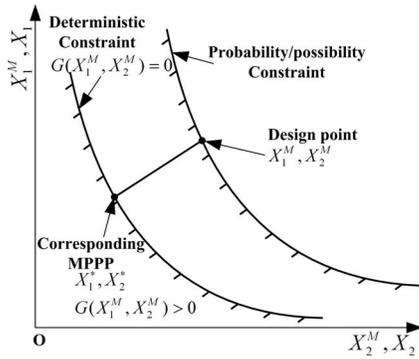
In this section, to efficiently deal with RFCDV-DO problems, RFSORA is developed based on the idea of SORA.

**4.1 Strategy of RFSORA.** To reduce the computational cost, two critical technologies are adopted:

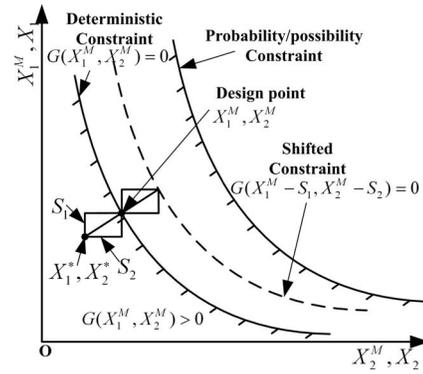
- (1) *Performance measure approach.* In RBDO, PMA is more efficient than directly calculating the probability of failure [9]; PMA is also found very efficient for PBDO [2,14–16]. Hence, Eqs. (8) and (12) are utilized in uncertainty analysis.
- (2) *Sequential optimization and reliability assessment.* Within the SORA, the original optimization is decoupled into sequential deterministic optimization and reliability analysis [9]. This idea is adopted in solving RFCDV-DO problems.

From Eqs. (8) and (12), to satisfy the probability/possibility constraint  $\Pi(G^{(i)}(\cdot) > 0) \leq \alpha_t$ , the value of performance measure at the MPPP must be not larger than zero  $G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_{rd}^*, \mathbf{X}_{rc}^*, \mathbf{X}_{fd}^*, \mathbf{X}_{fc}^*, \mathbf{P}_{rd}^*, \mathbf{P}_{rc}^*, \mathbf{P}_{fd}^*, \mathbf{P}_{fc}^*) \leq 0$ .

An example is utilized to demonstrate how a probability/possibility constraint is converted into a deterministic constraint in Fig. 1 similar to Ref. [9]. In this example, only two variables no matter random or fuzzy variables are considered. Two coordinate systems are plotted in Fig. 1. One is the design space composed by  $X_1^M, X_2^M$ , another is the uncertainty space composed by  $X_1, X_2$ . If no uncertainty is considered, probability/possibility constraint becomes deterministic constraint  $G(X_1^M, X_2^M) \leq 0$  with the constraint boundary  $G(X_1^M, X_2^M) = 0$  as plotted in Fig. 1. From the discussion before, to satisfy a probability/possibility constraint, the value of performance measure at the MPPP  $(X_1^*, X_2^*)$  must satisfy  $G(X_1^*, X_2^*) \leq 0$ . This indicates that the MPPP must be within the deterministic feasible area ( $G(X_1^M, X_2^M) \leq 0$ ) or at least on the de-



**Fig. 1 Conversion of a probability/possibility constraint into deterministic constraint**



**Fig. 2 Shifted constraint**

terministic constraint boundary  $G(X_1^M, X_2^M) = 0$  as plotted in Fig. 1. Therefore, Eq. (1) can be written as

$$\min_{(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M)} f(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M)$$

such that

$$\begin{aligned} G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^{*(i)}, \mathbf{X}_d^{*(i)}, \mathbf{P}_c^{*(i)}, \mathbf{P}_d^{*(i)}) &\leq 0 \\ g^{(j)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^M, \mathbf{X}_d^M, \mathbf{P}_c^M, \mathbf{P}_d^M) &\leq 0 \\ \mathbf{d}_c^L &\leq \mathbf{d}_c \leq \mathbf{d}_c^U, \quad \mathbf{d}_d^L &\leq \mathbf{d}_d \leq \mathbf{d}_d^U \\ \mathbf{X}_c^{M,L} &\leq \mathbf{X}_c^M \leq \mathbf{X}_c^{M,U}, \quad \mathbf{X}_d^{M,L} &\leq \mathbf{X}_d^M \leq \mathbf{X}_d^{M,U} \end{aligned} \quad (13)$$

$$i = 1, 2, \dots, n_G, \quad j = 1, 2, \dots, n_g$$

where  $\mathbf{X}_c^{*(i)} = \{\mathbf{X}_{rc}^{*(i)}, \mathbf{X}_{fc}^{*(i)}\}$ ,  $\mathbf{X}_d^{*(i)} = \{\mathbf{X}_{rd}^{*(i)}, \mathbf{X}_{fd}^{*(i)}\}$ ,  $\mathbf{P}_c^{*(i)} = \{\mathbf{P}_{rc}^{*(i)}, \mathbf{P}_{fc}^{*(i)}\}$ ,  $\mathbf{P}_d^{*(i)} = \{\mathbf{P}_{rd}^{*(i)}, \mathbf{P}_{fd}^{*(i)}\}$  is the MPPP of  $\mathbf{X}_c, \mathbf{X}_d, \mathbf{P}_c, \mathbf{P}_d$  corresponding to the  $i$ th probability/possibility constraint.  $G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^{*(i)}, \mathbf{X}_d^{*(i)}, \mathbf{P}_c^{*(i)}, \mathbf{P}_d^{*(i)})$  is the constraint value at the MPPP and  $G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^{*(i)}, \mathbf{X}_d^{*(i)}, \mathbf{P}_c^{*(i)}, \mathbf{P}_d^{*(i)}) \leq 0$  is equivalent to the probability/possibility constraint in Eq. (1) based on Eqs. (8) and (12).

**4.2 Procedure of RFSORA.** In this section, the procedure of RFSORA is provided step by step as follows:

*Step 1.* Set initial values for  $\mathbf{d}_c^{(0)}, \mathbf{d}_d^{(0)}, \mathbf{X}_c^{M,(0)}, \mathbf{X}_d^{M,(0)}$ ;  $k=1$

*Step 2.* Solve the deterministic optimization. This step is to obtain values of  $\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k}, \mathbf{X}_d^{M,k}$ . Since there is no information about the MPPPs in the first cycle, the MPPPs are set to be equal to  $\mathbf{X}_c^{M,(0)}, \mathbf{X}_d^{M,(0)}, \mathbf{P}_c^M, \mathbf{P}_d^M$ . From the second cycle, the MPPPs obtained from the previous cycle are used to reconstruct deterministic constraints until the value of objective function converges and requirements of probability/possibility constraints are all satisfied.

• *Step 3.* Perform probability/possibility analysis. The probability/possibility analysis is carried out to check the feasibility of each probability/possibility constraint at the current design point. The results are MPPP and value of performance measure at the MPPP corresponding to each probability/possibility constraint.

*Step 4.* Check convergence. If requirements of probability/possibility constraints are all satisfied and the value of the objective function is stable ( $G^{(i)} \leq 0, i=1 \sim n_G; |f(k) - f(k-1)| \leq \varepsilon$ ) where  $\varepsilon$  is an arbitrary small positive constant, stop the solving process; otherwise set  $k=k+1$  and go to Step 2 with the MPPPs obtained in Step 3.

If the requirement of probability/possibility constraint ( $\Pi(G^{(i)}(\cdot) > 0) \leq \alpha_i$ ) is not satisfied in the  $(k-1)$ th cycle, the MPPP  $\mathbf{X}_c^{*(i),(k-1)}, \mathbf{X}_d^{*(i),(k-1)}, \mathbf{P}_c^{*(i),(k-1)}, \mathbf{P}_d^{*(i),(k-1)}$  obtained from cycle  $k-1$  will be used to modify the constraint in the  $k$ th deterministic optimization formulation. To ensure the feasibility of probability/possibility constraint, the MPPP of the  $k$ th cycle should fall into the deterministic feasible region.

Let  $\mathbf{S}$  be a shift vector. The shift is based on the idea of SORA as that used in Ref. [9].

$$\mathbf{S}^{(i),k} = \{\mathbf{S}_c^{(i),k}, \mathbf{S}_d^{(i),k}\}$$

$$\mathbf{S}_c^{(i),k} = \mathbf{X}_c^{M,(k-1)} - \mathbf{X}_c^{*(i),(k-1)}$$

$$\mathbf{S}_d^{(i),k} = \mathbf{X}_d^{M,(k-1)} - \mathbf{X}_d^{*(i),(k-1)}$$

where  $\mathbf{S}^{(i),k}$  is the shift vector for the  $i$ th probability/possibility constraint in the  $k$ th cycle.  $\mathbf{S}_c^{(i),k}$  indicates the shifts of continuous random and fuzzy variables, while  $\mathbf{S}_d^{(i),k}$  stands for the shifts of discrete random and fuzzy variables.  $\mathbf{X}_c^{M,(k-1)}, \mathbf{X}_d^{M,(k-1)}$  are the mean values or the maximal grade points of variables with uncertainties obtained in the  $(k-1)$ th cycle.

Because there is no means to control the random and fuzzy parameters, the same shift strategy is not used. But from Eq. (13), the deterministic constraint function must satisfy  $G^{(i)}(\mathbf{d}_c, \mathbf{d}_d, \mathbf{X}_c^*, \mathbf{X}_d^*, \mathbf{P}_c^*, \mathbf{P}_d^*) \leq 0$  to achieve the possibility requirement. So the MPPP  $\mathbf{P}_c^{*(i),(k-1)}, \mathbf{P}_d^{*(i),(k-1)}$  obtained in the previous cycle is used in constructing a deterministic model.

The deterministic constraint in the  $k$ th cycle is modified as

$$G^{(i)}(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k} - \mathbf{S}_c^{(i),k}, \mathbf{X}_d^{M,k} - \mathbf{S}_d^{(i),k}, \mathbf{P}_c^{*(i),(k-1)}, \mathbf{P}_d^{*(i),(k-1)}) \leq 0$$

The example in Sec. 4.1 is used to illustrate the above shift strategy in Fig. 2. In the first cycle, there is no information about the MPPP because probability/possibility analysis has not been performed. In the deterministic optimization, the constraint is  $G(X_1^M, X_2^M) \leq 0$ . The worst case is that the design point is on the boundary of the constraint. As one can expect, the MPPP must fall into the area of  $G(X_1^M, X_2^M) > 0$  after uncertainty analysis. To satisfy the probability/possibility constraint, the MPPP should be within the deterministic feasible region. When constructing the equivalent deterministic formulation in cycle 2, the constraint should be modified to shift the MPPP at least onto the deterministic boundary to make sure the feasibility of probability/possibility constraint. The shifted constraint is plotted in Fig. 2 as dashed line. The feasibility of the violated probability/possibility constraint will be improved remarkably using this shift strategy.

### 4.3 Formulations of Deterministic Optimization and Probability/Possibility Analysis of the $k$ th Cycle

**4.3.1 Deterministic Optimization of the  $k$ th Cycle.** The deterministic optimization of the  $k$ th cycle is given as

$$\min_{(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k}, \mathbf{X}_d^{M,k}, \mathbf{P}_c^M, \mathbf{P}_d^M)} f(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k}, \mathbf{X}_d^{M,k}, \mathbf{P}_c^M, \mathbf{P}_d^M)$$

$$i = 1, 2, \dots, n_G; \quad j = 1, 2, \dots, n_g \quad (14)$$

such that

$$G^{(i)}(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k} - \mathbf{S}_c^{(i),k}, \mathbf{X}_d^{M,k} - \mathbf{S}_d^{(i),k}, \mathbf{P}_c^{*(i),(k-1)}, \mathbf{P}_d^{*(i),(k-1)}) \leq 0$$

$$g^{(j)}(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k}, \mathbf{X}_d^{M,k}, \mathbf{P}_c^M, \mathbf{P}_d^M) \leq 0$$

$$\mathbf{d}_c^L \leq \mathbf{d}_c^k \leq \mathbf{d}_c^U, \quad \mathbf{d}_d^L \leq \mathbf{d}_d^k \leq \mathbf{d}_d^U$$

$$\mathbf{X}_c^{M,L} \leq \mathbf{X}_c^{M,k} \leq \mathbf{X}_c^{M,U}, \quad \mathbf{X}_d^{M,L} \leq \mathbf{X}_d^{M,k} \leq \mathbf{X}_d^{M,U}$$

where  $\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{X}_c^{M,k}, \mathbf{X}_d^{M,k}$  are design variables in the  $k$ th cycle. Superscript  $k$  indicates cycle  $k$ .  $\mathbf{P}^{*(i),(k-1)}$  denote the MPPPs of  $\mathbf{P}$  corresponding to the  $i$ th probability/possibility constraint obtained in the  $(k-1)$ th cycle.

**4.3.2 Probability/Possibility Analysis in the  $k$ th Cycle.** First of all, transformations should be done according to each approach in Sec. 3. Formulation of approach 1 for uncertainty analysis in the  $k$ th cycle is given as

$$\max_{\left( \begin{array}{l} \mathbf{U}_{rc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \\ \mathbf{UP}_{rc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k} \end{array} \right)} G^{(i)}(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{U}_{rc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{UP}_{rc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k})$$

such that

$$\|(\mathbf{U}_{rc}^{(i),k}, \mathbf{UP}_{rc}^{(i),k})\|_2 \leq \beta_t$$

$$\|(\mathbf{V}_{fd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k})\|_\infty < 1 - \alpha_t \quad (15)$$

$$i = 1, 2, \dots, n_G$$

where  $\mathbf{d}_c^k, \mathbf{d}_d^k$  are optimal values of deterministic design variables obtained from the  $k$ th deterministic optimization.  $\beta_t$  is equal to  $-\Phi^{-1}(\alpha_t)$ . The symbols  $\mathbf{U}$  and  $\mathbf{UP}$  indicate the standard normal variable and parameter in  $U$ -space, respectively.  $\mathbf{V}$  and  $\mathbf{VP}$  denote standard fuzzy variable and parameter in  $V$ -space, respectively. The superscript  $(i)$  indicates that the variables and parameters cor-

respond to the  $i$ th probability/possibility constraint because each probability/possibility constraint has its own MPPP.

The solutions are MPPPs  $(\mathbf{U}_{rc}^{*(i),k}, \mathbf{V}_{fd}^{*(i),k}, \mathbf{V}_{fc}^{*(i),k}, \mathbf{V}_{fd}^{*(i),k}, \mathbf{UP}_{rc}^{*(i),k}, \mathbf{VP}_{fd}^{*(i),k}, \mathbf{VP}_{fc}^{*(i),k}, \mathbf{VP}_{fd}^{*(i),k})$  ( $i=1 \sim n_G$ ) and values of performance measure at MPPPs.

Formulation of approach 2 for uncertainty analysis is given as

$$\max_{\left( \begin{array}{l} \mathbf{U}_{rc}^{(i),k}, \mathbf{X}_{rd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \\ \mathbf{UP}_{rc}^{(i),k}, \mathbf{P}_{rd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k} \end{array} \right)} G^{(i)}(\mathbf{d}_c^k, \mathbf{d}_d^k, \mathbf{U}_{rc}^{(i),k}, \mathbf{X}_{rd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{UP}_{rc}^{(i),k}, \mathbf{P}_{rd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k})$$

such that

$$\|(\mathbf{U}_{rc}^{(i),k}, \mathbf{UP}_{rc}^{(i),k})\|_2 \leq \beta'_t$$

$$\|(\mathbf{V}_{fd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k})\|_\infty < 1 - \alpha_t \quad (16)$$

$$i = 1, 2, \dots, n_G$$

where  $\beta'_t$  is equal to  $-\Phi^{-1}[\alpha_t / \sum_{i=1}^N \{F_{\mathbf{X}_{rd}}(\mathbf{x}_{rd}^i) \times F_{\mathbf{P}_{rd}}(\mathbf{p}_{rd}^i)\}]$ .  $\mathbf{X}_{rd}^{(i)}, \mathbf{P}_{rd}^{(i)}$  are discrete random variables and parameters of  $\mathbf{X}, \mathbf{P}$ , respectively.

The solutions are MPPPs  $(\mathbf{U}_{rc}^{*(i),k}, \mathbf{X}_{rd}^{*(i),k}, \mathbf{V}_{fc}^{*(i),k}, \mathbf{V}_{fd}^{*(i),k}, \mathbf{UP}_{rc}^{*(i),k}, \mathbf{P}_{rd}^{*(i),k}, \mathbf{VP}_{fc}^{*(i),k}, \mathbf{VP}_{fd}^{*(i),k})$  ( $i=1 \sim n_G$ ) and values of performance measure at the MPPPs.

Then the MPPPs in  $X$ -space  $\mathbf{X}_{rc}^{*(i),k}, \mathbf{X}_{rd}^{*(i),k}, \mathbf{X}_{fc}^{*(i),k}, \mathbf{X}_{fd}^{*(i),k}, \mathbf{P}_{rc}^{*(i),k}, \mathbf{P}_{rd}^{*(i),k}, \mathbf{P}_{fc}^{*(i),k}, \mathbf{P}_{fd}^{*(i),k}$  can be obtained using the inverse Rosenblatt transformation and Eq. (2). The constraints in the deterministic optimization will be modified using the MPPPs when requirements of probability/possibility constraints are not all satisfied. Figure 3 shows the flowchart of RFSORA. The details of RFSORA have been given above.

To deal with the discrete-continuous optimization in Eqs.

(14)–(16), the algorithm MDOP in Ref. [23] is utilized. But different from the methods used to find a feasible discrete point in MDOP such as genetic algorithm and random test method and so on, the method of “TRANS” in MDOD [23] is utilized, where the unit vector of the feature vector is used directly for rounding. During one-dimension searching and adjacent point-checking in the discrete unit area, when a new point is obtained, the point is first compared with those saved. If there exists a same point, set the values of objective function and constraints the same as those saved; otherwise save the point, calculate, and save its values of objective function and constraints. Finally, starting from the optimal solution obtained by MDOP, the original optimization problem is solved using algorithms for continuous optimization while fixing the discrete part at relevant values of that optimal solution.

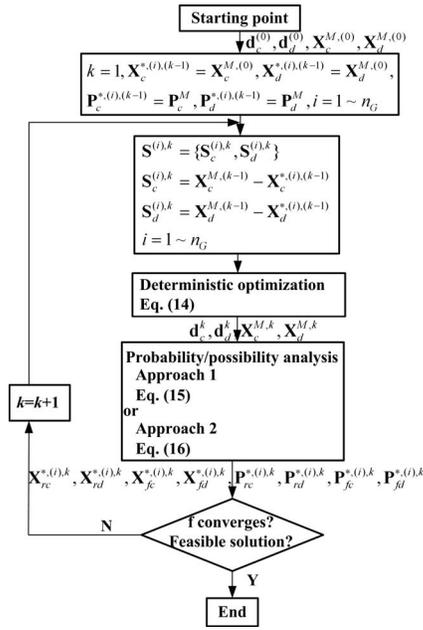


Fig. 3 Flowchart of RFSORA

## 5 Example

The example of pressure vessel design is derived from Ref. [24]. Design variables are radius ( $R$ ), length ( $L$ ), and thickness ( $T$ ). There are two design parameters: internal pressure ( $P$ ) and allowable tensile strength of the material ( $S_t$ ). The objective is to maximize the internal volume while minimizing weight. In this paper, this problem is modified to be a RFCDV-DO problem.

In this paper,  $T$  and  $R$  are continuous random variables while  $L$  is a discrete random variable. Table 1 shows the uncertainty descriptions of design variables and parameters.

Due to the manufacturing practice, mean values of  $T$ ,  $R$  are integer multiples of 0.01. When the mean value of  $T$  is obtained as  $T^M$ , the practical dimension is subjected to  $N(T^M, 0.01)$ . The case of  $R$  is similar as  $T$ . The length  $L$  is discretely distributed according to the following probability:

Table 1 Uncertainty descriptions of design variables and parameters

Variables or parameters	Mean value	Standard deviation	Distribution	Lower bound of mean value	Upper bound of mean value
$R$		0.01	Normal	0.1	36
$T$		0.01	Normal	0.5	6.0
$L$				0.1	140
$S_t$	40	4	Normal		
	Maximal grade point	Deviation	Membership function		
$P$	3.89	1.167	Triangular: (3.89–1.167, 3.89, 3.89+1.167)		

Table 2 Results of pressure vessel design

	$T^M$	$R^M$	$L^M$	$v$	$v_1$	$v_2$	NFE	$k$
Approach 1	6.0000	33.2300	71.3000	$-2.0447 \times 10^5$	$1.9658 \times 10^5$	$4.0105 \times 10^5$	612	3
Approach 2	6.0000	33.2300	71.3000	$-2.0447 \times 10^5$	$1.9658 \times 10^5$	$4.0105 \times 10^5$	617	3

Table 3 Value of performance measure at MPPP

	$G_1$	$G_2$	$G_3$	$G_4$
Approach 1	-3.0770	-0.7276	$-2.0724 \times 10^{-4}$	-0.0551
Approach 2	-3.0770	-0.7276	$-2.0724 \times 10^{-4}$	-0.0551

$$\Pr\{L = \tau\} = \begin{cases} 0.1 & \tau = L^M + 0.1 \\ 0.8 & \tau = L^M \\ 0.1 & \tau = L^M - 0.1 \end{cases}$$

where  $L^M$  is integer multiple of 0.1.

The probability/possibility constraints are as follows:

$$\Pi\{G_1 = 5T - R > 0\} \leq \alpha_t$$

$$\Pi\{G_2 = T + R - 40 > 0\} \leq \alpha_t$$

$$\Pi\left\{G_3 = \frac{PR}{T} - S_t > 0\right\} \leq \alpha_t$$

$$\Pi\{G_4 = L + 2R + 2T - 150 > 0\} \leq \alpha_t$$

The objective function is defined as follows:

$$v = v_1 - v_2$$

$$v_1 = \frac{4}{3}\pi(T^M + R^M)^3 + \pi(T^M + R^M)^2L^M - \left[\frac{4}{3}\pi(R^M)^3 + \pi(R^M)^2L^M\right]$$

$$v_2 = \frac{4}{3}\pi(R^M)^3 + \pi(R^M)^2L^M$$

The target possibility of failure is  $\alpha_t = 1 - 0.9987 = 0.0013$ . In the optimization process, the optimum points obtained from the previous cycle are used as the starting points for the current cycle. The optimum results are showed in Table 2. Although the optimal designs obtained with different approaches for uncertainty analysis are the same, the numbers of function evaluations are different. RFSORA solves this problem with three cycles listed in column nine in Table 2. Table 3 lists the values of performance measure of each probability/possibility constraint at the relevant MPPP. From Table 3, all values of performance measure at MPPPs are less than zeros, which indicates that probability/possibility constraints are all satisfied at each optimum point.

## 6 Conclusions

This paper proposes a formulation of RFCDV-DO, two different approaches for uncertainty analysis, and a method (RFSORA) based on the idea of SORA. Due to the presence of both types of uncertainties (AU and EU) as well as the existence of both continuous and discrete design variables and parameters, the cost of direct calculation of possibility of failure is huge and very expensive. Based on the concept of conditional possibility of failure, two approaches for uncertainty analysis are developed to reduce the computational cost.

In the method of RFSORA, based on the ideas of SORA the solving process of a RFCDV-DO problem is decoupled into deterministic optimization and probability/possibility analysis, which are carried out sequentially instead of nested. Constraints in the deterministic optimization model are shifted to make sure

MPPPs falling into the deterministic feasible region. In the first cycle, the values of MPPPs are set to be equal to the mean values or maximal grade points of variables and parameters with uncertainties. From the second cycle, the MPPPs obtained from the previous cycle are used to reconstruct constraints in the deterministic optimization model to improve the feasibility of design. As demonstrated by example, the RFSORA can efficiently solve RFCDV-DO problem in a few cycles.

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