# A Hierarchical Statistical Sensitivity Analysis Method for Multilevel Systems With Shared Variables

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*Statistical sensitivity analysis (SSA) is an effective methodology to examine the impact of variations in model inputs on the variations in model outputs at either a prior or posterior design stage. A hierarchical statistical sensitivity analysis (HSSA) method has been proposed in literature to incorporate SSA in designing complex engineering systems with a hierarchical structure. However, the original HSSA method only deals with hierarchical systems with independent subsystems. For engineering systems with dependent subsystem responses and shared variables, an extended HSSA method with shared variables (named HSSA-SV) is developed in this work. A top-down strategy, the same as in the original HSSA method, is employed to direct SSA from the top level to lower levels. To overcome the limitation of the original HSSA method, the concept of a subset SSA is utilized to group a set of dependent responses from the lower level submodels in the upper level SSA and the covariance of dependent responses is decomposed into the contributions from individual shared variables. An extended aggregation formulation is developed to integrate local submodel SSA results to estimate the global impact of lower level inputs on the top level response. The effectiveness of the proposed HSSA-SV method is illustrated via a mathematical example and a multiscale design problem.* [DOI: 10.1115/1.4001211]

## 1 Introduction

A complex design problem often involves a large number of design variables and multidisciplinary analyses with excessive cost. The "all-in-one" (AIO) method in which the whole system analysis is treated as a black box, is usually not practical or even prohibitive due to the system's complexity, limited communications between subsystems belonging to various disciplines, and the associated high computational expense. To relieve the computational burden and manage the complexity in design processes, a complex system is often decomposed into several subsystems in either a hierarchical or nonhierarchical manner [1,2]. Hierarchical modeling is widely used to decompose a complex system into multilevel submodels according to their functional attributes [3,4], physical structures [5,6], or scale magnitudes [7], etc. A typical hierarchical system with a bilevel structure is illustrated in Fig. 1. Each submodel has only one "parent" submodel at a higher level but multiple "children" submodels at a lower level [2]. The information flow in the hierarchical structure follows a one way direction from the bottom level to the top level. Following the same terminology in multidisciplinary design optimization, we denote local input variables of each submodel as $\mathbf{X_0}, \mathbf{X_1}, \ldots, \mathbf{X_N}$ and shared variables that exist as a common set of input variables to "sibling" submodels as $\mathbf{X_s}$. As shown in Fig. 1, the existence of shared variables creates the functional dependence of the responses from the sibling submodels (e.g., $Y_1$ and $Y_2$ in Fig. 1). In multidisciplinary design, shared variables are common design (decision) variables shared by multiple subsystem designs. As design variables are frequently independent, it is assumed in this work that all input variables are independent. For designing hierarchical engineering systems, deterministic design methods have been developed in literature [1,3,8–10] with extensions to multilevel optimization formulations considering uncertainties [2,4,11].

Statistical sensitivity analysis (SSA) is the study of how the variation in the output of a model can be apportioned, qualitatively or quantitatively, to different sources of variation from input variables through statistical means [12]. By applying SSA, the importance of input variables can be identified and the engineering system can be simplified by fixing those unimportant variables [13–15]. A hierarchical statistical sensitivity analysis (HSSA) method was developed in our earlier research [11] to facilitate the application of SSA in complex multilevel engineering systems. The original HSSA method contains three features: (1) SSA is first applied to the top level model and a top-down analysis is executed level-by-level. (2) Instead of performing SSA in the all-in-one manner, SSA is separately applied to the critical submodels at each level. (3) The global statistical sensitivity index (GSSI) of any input variables with respect to the global system performance is derived from aggregating the local statistical sensitivity index (LSSI) of relevant submodels. The effectiveness and efficiency of the original HSSA method has been demonstrated by examples in Ref. [11].

However, the original HSSA method has a critical deficiency in that it can only be used for designing multilevel systems with independent submodel responses, i.e., no shared variables as inputs to multiple sibling submodels. Although previous literatures address SSA with correlated input variables [16–18], these methods do not concern the cases with multilevel submodels and therefore cannot be directly performed on a complex system with a hierarchical structure. To overcome the aforementioned limitations, an extended hierarchical statistical sensitivity analysis method, named hierarchical statistical sensitivity analysis with shared variables (HSSA-SV), is proposed in this work. The proposed method examines the importance of a local subset that contains dependent responses from lower levels. A top-down strategy, same as in Ref. [11], is invoked to direct SSA of submodels at different levels. An extended GSSI aggregation formulation is proposed to evaluate the GSSI of input variables in the multilevel

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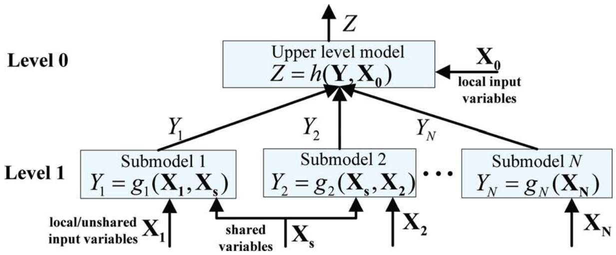


**Fig. 1 A bilevel hierarchical structure with shared input variables**

system by integrating the local subset SSIs and the decomposed covariance across multiple levels. The proposed HSSA-SV method has a broader application of SSA to complex engineering design problems compared with the original HSSA method.

The remainder of the paper is organized as follows: A technical background of SSA including the variance-based SSA and subset SSA is introduced in Sec. 2. The details of the HSSA-SV method are introduced in Sec. 3. Our proposed method is demonstrated and verified in Sec. 4 via a mathematical problem and SSA of a multiscale design system. The benefits of the proposed method are discussed in Sec. 5 followed by conclusions in Sec. 6.

## 2 Technical Background

**2.1 Variance-Based Statistical Sensitivity Analysis.** The variance-based SSA, a popular category among the global sensitivity analysis (GSA) methods, evaluates the statistical sensitivities based on the decomposition of the variance of the model outputs in accordance with the variation sources from the inputs [19,20]. In literature, a number of variance-based methods, including Sobol's methods [19,21], Fourier amplitude sensitivity test (FAST) [20], important measures [22], McKay's method [23], etc., were developed and applied in a variety of fields such as chemistry [22,24], environmental science [25–27], and mechanical engineering [13]. Among these existing methods, Sobol's method has been widely employed to rank the input variables based on their contributions to the total variance of the model output [19,21], and is adopted in this paper.

In the variance-based SSA methods, the total variance of an output $Y=f(\mathbf{X})$ is decomposed into the summation of $2^n-1$ variance terms, representing the various sources from input variables $\mathbf{X}=[X_1, X_2, \ldots, X_n]$ in a similar fashion as in an analysis of variance (ANOVA):

$$V^Y = \sum_i V^Y_{X_i} + \sum_{i<j} V^Y_{X_i X_j} + \ldots + V^Y_{X_1,\ldots,X_n} \tag{1}$$

where $V^Y$ is the total variance of the output and $V^Y_{X_i}$ is the first-order term that represents the partial variance in $V^Y$ due to the individual effect of a random variable $X_i$. The superscript $Y$ represents the model output of interest and the subscript denotes the index of an input variable. The higher-order terms such as $V^Y_{X_i X_j}$ and so on denote the effects from the interaction of two or more random variables. To measure the importance of an input variable with respect to an output variable, the SSI of $X_i$ is defined by the ratio of the partial variance contributed by $X_i$ to the total variance of the output $Y$

$$\mathrm{SSI}^Y_{X_i} = \frac{V^Y_{X_i}}{V^Y}, \quad 1 \le i \le n \tag{2}$$

Equation (2) calculates the main effect of $X_i$ on the variance of $Y$. A higher-order SSI is formulated as

$$\mathrm{SSI}^Y_{X_{i_1},\ldots,X_{i_j}} = \frac{V^Y_{X_{i_1},\ldots,X_{i_j}}}{V^Y}, \quad 1 \le i_1, \ldots, i_j \le n \tag{3}$$

which represents the interaction effect between random variables $X_{i_1}, \ldots, X_{i_j}$ to the variance of the model output $Y$. The total statistical sensitivity index (TSSI) of $X_i$, measures the contributions of $X_i$, including its main effect as well as its interaction effects with other input variables and it is defined as

$$\mathrm{SSI}^Y_{T_{X_i}} = \mathrm{SSI}^Y_{X_i} + \mathrm{SSI}^Y_{X_i \tilde{X}_i} \tag{4}$$

where $\mathrm{SSI}^Y_{X_i \tilde{X}_i}$ is the sum of all the higher-order SSIs involving the input variable $X_i$ and at least one other input variable from $\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}$.

To calculate the SSI under the condition that all the random variables are independent, Sobol's method [19,21] introduces a decomposition of the model function $Y=f(\mathbf{X})$ into an ANOVA formulation as

$$f(\mathbf{X}) = f_0 + \sum_i \phi_{X_i}(X_i) + \sum_{i_1<i_2} \phi_{X_{i_1}X_{i_2}}(X_{i_1}, X_{i_2}) + \ldots + \phi_{X_{i_1},\ldots,X_{i_n}}(X_{i_1}, \ldots, X_{i_n}) \tag{5}$$

where $f_0$ is the mean of $f(\mathbf{X})$ and other terms are formulated as [13,19,21]:

$$\phi_{X_i}(X_i) = \int f(\mathbf{X}) \prod_{k \neq i} [\rho_k(X_k) dX_k] - f_0 \tag{6}$$

$$\phi_{X_{i_1},\ldots,X_{i_j}}(X_{i_1}, \ldots, X_{i_j}) = \int f(\mathbf{X}) \prod_{k \neq i_1,\ldots,i_j} [\rho_k(X_k) dX_k] - \sum_{l=1}^{j-1} \sum_{k_1,\ldots,k_l \in (i_1,\ldots,i_j)} \phi_{X_{k_1},\ldots,X_{k_l}}(X_{k_1}, \ldots, X_{k_l}) - f_0 \tag{7}$$

In Eqs. (6) and (7), $\rho_k(X_k)$ is the probability density function (PDF) of variable $X_k$. As proposed in Sobol's methods [19,21], the total variance of $Y$ and the partial variance terms in Eq. (1) are calculated through integration of each term in Eq. (5)

**2.2 Subset Decomposition and Statistical Sensitivity Analysis.** Based on the principle of variance-based SSA presented in Sec. 2.1, Sobol's method further introduces an SSI definition for studying the impact of a grouped, subset of input variables [21], [28]. It considers an arbitrary set of $m$ random variables as a group and evaluates the partial variance contributed by the variation in the grouped variables over the total variance of the output $Y$ [13]. With the subset decomposition, $n$ input random variables in $Y=f(\mathbf{X})$ are divided into $N$ mutually disjoint subsets denoted as $U_1, \ldots, U_N$, where $U_i=\{X_{i_1}, \ldots, X_{i_k}\}$, $k \ge 1$, and $1 \le i_1, \ldots, i_k \le n$. Under the condition that subsets are statistically independent, the ANOVA decomposition of $Y=f(\mathbf{X})$ can be expressed as

$$f(\mathbf{X}) = f_0 + \sum_{i=1} \phi_{U_i}(U_i) + \sum_{i_1<i_2} \phi_{U_{i_1}U_{i_2}}(U_{i_1}, U_{i_2}) + \ldots + \phi_{U_{i_1},\ldots,U_{i_N}}(U_{i_1}, \ldots, U_{i_N}) \tag{8}$$

where $X_{i_1}, \ldots, X_{i_j}$ in the original univariant ANOVA decomposition, see Eq. (5), are replaced by subsets $U_{i_1}, \ldots, U_{i_j}$. The variance of $Y$ can be decomposed into a summation of partial variances from the subsets

$$V^Y = \sum_i V^Y_{U_i} + \sum_{i<j} V^Y_{U_i U_j} + \ldots + V^Y_{U_1 U_2,\ldots,U_N} \tag{9}$$

Therefore, the main effect of each subset is defined by

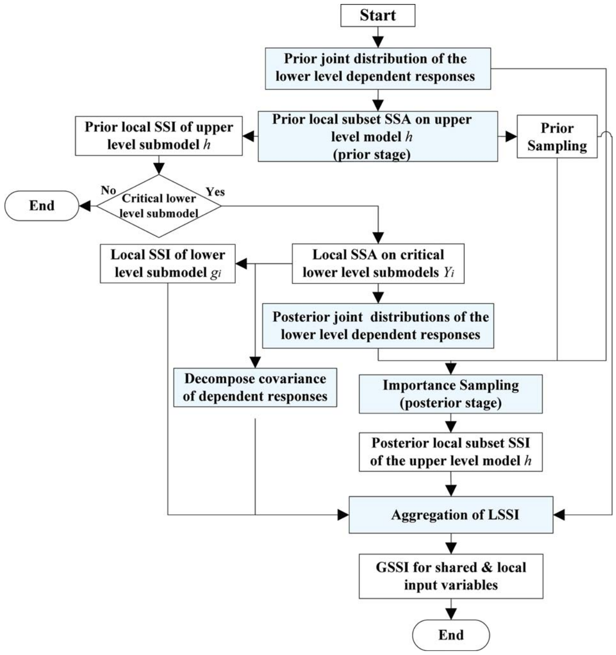


**Fig. 2 Flowchart of the HSSA-SV method with dependent lower level responses**

$$\mathrm{SSI}_{U_i}^{Y} = \frac{V_{U_i}^{Y}}{V^{Y}}, \quad 1 \le i \le N \tag{10}$$

and the higher-order SSI of the subsets is formulated as

$$\mathrm{SSI}_{U_{i_1},\ldots,U_{i_j}}^{Y} = \frac{V_{U_{i_1},\ldots,U_{i_j}}^{Y}}{V^{Y}}, \quad 1 \le i_1, \ldots, i_j \le N \tag{11}$$

The partial variance $V_{U_{i_1}\ldots U_{i_j}}^{Y}$ is calculated as:

$$V_{U_{i_1},\ldots,U_{i_j}}^{Y} = \int \phi_{U_{i_1},\ldots,U_{i_j}}^{2}(U_{i_1}, \ldots, U_{i_j}) \prod_{k=1}^{j} [\rho_{U_{i_k}}(U_{i_k}) dU_{i_k}] \tag{12}$$

where $\rho_{U_{i_k}}(U_{i_k})$ represents the joint probability density function of the subset variables $U_{i_k}$.

## 3 HSSA-SV

The HSSA-SV method developed in this paper follows a similar framework of the original HSSA method introduced in Sec. 1 [11] but with revised formulations to account for dependent submodel responses. The flowchart of the HSSA-SV method for a two-level model is shown in Fig. 2. In the first step of the proposed top-down strategy, SSA is applied to the upper level model. Because the exact information of the dependent responses from the lower level is not available at this stage, a prior joint distribution of the dependent responses is assigned for performing local subset SSA on the upper level system model. According to the rank of the TSSIs at the upper level, critical responses from the lower level can be identified and SSA is further applied to the lower level submodels with critical responses. Once the real information of the lower level responses is available and after applying SSA to the lower level submodels, a posterior local subset SSI of the upper level model can be computed. In this process, to account for the impact of shared variables, the covariance of the dependent responses is decomposed into the contributions from individual variation sources (see details in Sec. 3.1). The local subset SSA is used to evaluate the local subset SSI of the dependent responses and the importance sampling technique is employed for obtaining the posterior SSIs by reusing the existing sampling data from the prior stage of the upper level (see details in Sec. 3.2). The aggregation formulation used in the original HSSA method is then extended and the main effects of local independent input variables and shared variables can be assessed via integrating the posterior local subset SSIs of the upper level model, the LSSIs of lower level submodels, as well as the decomposed covariance of dependent responses (see details in Sec. 3.3).

**3.1 Decomposition of Covariance.** The idea of the covariance decomposition is to decompose the total covariance of two dependent outputs due to the shared variables into separate items. In this subsection, two outputs, $Y_1$ and $Y_2$, from the models $Y_1 = f_1(\mathbf{X_s}, \mathbf{X_1})$ and $Y_2 = f_2(\mathbf{X_s}, \mathbf{X_2})$, respectively, are used to demonstrate the decomposition of covariance. $\mathbf{X_s}$ is a vector of $N_S$ independent shared input variables $\{X_{s1}, \ldots, X_{sN_S}\}$, $\mathbf{X_1} = \{X_{11}, \ldots, X_{1N_1}\}$, and $\mathbf{X_2} = \{X_{21}, \ldots, X_{2N_2}\}$ are two vectors with $N_1$ and $N_2$ independent local input variables for the two models, respectively. $Y_1$ and $Y_2$ are functionally and statistically dependent because of the existence of the shared variables $\mathbf{X_s}$.

Similar to the ANOVA method, the covariance of $Y_1$ and $Y_2$ are decomposed into $2^{N_S} - 1$ contribution items as

$$\mathrm{Cov}[Y_1, Y_2] = \sum_i \mathrm{Cov}_{X_{si}}^{Y_1Y_2} + \sum_{i<j} \mathrm{Cov}_{X_{si}X_{sj}}^{Y_1Y_2} + \ldots + \mathrm{Cov}_{X_{s1},\ldots,X_{sN_s}}^{Y_1Y_2} \tag{13}$$

where $\mathrm{Cov}_{X_{si}}^{Y_1Y_2}$ is the first-order covariance contribution from the shared variable $X_{si}$ and $\mathrm{Cov}_{X_{si}X_{sj}}^{Y_1Y_2}$ denotes the second-order covariance contribution due to the interaction of the shared variables $X_{si}$ and $X_{sj}$, and so on.

PROPOSITION. The partial covariance contribution in Eq. (13) is calculated as

$$\mathrm{Cov}_{X_{si}}^{Y_1Y_2} = \mathrm{Cov}[\phi_{X_{si}}(X_{si}), \varphi_{X_{si}}(X_{si})], \quad 1 \le i \le N_S \tag{14}$$

for $1 \le i_1, \ldots, i_j \le N_S$ and

$$\mathrm{Cov}_{X_{si_1},\ldots,X_{si_j}}^{Y_1Y_2} = \mathrm{Cov}[\phi_{X_{si_1},\ldots,X_{si_j}}(X_{si_1}, \ldots, X_{si_j}), \varphi_{X_{si_1},\ldots,X_{si_j}}(X_{si_1}, \ldots, X_{si_j})] \tag{15}$$

where $\phi_{X_{si}}(X_{si})$ and $\phi_{X_{si_1},\ldots,X_{si_j}}(X_{si_1}, \ldots, X_{si_j})$ are the ANOVA decomposed terms of the function $Y_1 = f_1(\mathbf{X_s}, \mathbf{X_1})$, as shown in Eq. (5), and $\varphi_{X_{si}}(X_{si})$ and $\varphi_{X_{si_1},\ldots,X_{si_j}}(X_{si_1}, \ldots, X_{si_j})$ are the ANOVA decomposed terms of function $Y_2 = f_2(\mathbf{X_s}, \mathbf{X_2})$. See Appendix A for the proof of Eqs. (14) and (15).

The first-order covariance term in the covariance decomposition is derived as (see details in Appendix A)

$$\mathrm{Cov}_{X_{si}}^{Y_1Y_2} = \mathrm{Cov}[E(f_1|X_{si}), E(f_2|X_{si})] \tag{16}$$

and the higher-order covariance decomposition term is derived as (see details in Appendix A)

$$\mathrm{Cov}_{X_{si_1},\ldots,X_{si_j}}^{Y_1Y_2} = \mathrm{Cov}[E(f_1|X_{si_1}, \ldots, X_{si_j}), E(f_2|X_{si_1}, \ldots, X_{si_j})] - \sum_{k=1}^{j-1} \sum_{j_1,\ldots,j_k \in (i_1,\ldots,i_j)} \mathrm{Cov}_{X_{sj_1},\ldots,X_{sj_k}}^{Y_1Y_2} \tag{17}$$

It can be concluded that, the covariance of $Y_1$ and $Y_2$ is equal to the covariance of their conditional expectations with respect to their shared variables, i.e.,

$$\mathrm{Cov}[Y_1, Y_2] = \mathrm{Cov}[E(f_1|\mathbf{X_s}), E(f_2|\mathbf{X_s})] \tag{18}$$

The proof is shown in Appendix A.

**3.2 Local Subset SSA.** Due to the dependency shown in Fig. 1, local statistical sensitivity analysis (LSSA) cannot be directly applied to the lower level responses $Y_1$ and $Y_2$ [17,28]. We propose to use the subset SSA approach to evaluate the local subset SSIs for dependent responses. These dependent responses are regarded as a subset denoted by $U_{Y_1Y_2}$ and the subset becomes independent of the other submodel responses $Y_i, 3 \le i \le N$. A prior joint distribution of the dependent responses, i.e., the subset $U_{Y_1Y_2}$, is assigned, based on the designer's knowledge, before applying the local subset SSA. If a designer does not have any knowledge to assign a prior joint distribution, a joint uniform distribution with a large range that encompasses as much of the real performance as possible should be assigned. Based on the empirical study in our earlier work [11], we found such treatment can improve the posterior estimation when applying the importance sampling method in the correction step. Based on the resulting local subset SSIs, the LSSA is next performed on critical submodels at the lower level.

If a subset of dependent responses has a high LSSI, the LSSA needs to be performed on the submodels related to all the dependent responses in this subset. By sampling simultaneously the outputs from dependent submodels with shared and local input variables, the real joint distribution of the dependent outputs $Y_1$ and $Y_2$, as well as the LSSIs of the local input variables and shared variables for each submodel, can be determined.

Using the importance sampling strategy as described in Ref. [11], posterior local subset SSIs of the upper level models and statistical indices (e.g., variance) can be recalculated without additional samples, reducing the computational cost significantly. For example, in the upper level model SSA, if the lower level responses $Y_1, \ldots, Y_n$, acting as inputs to the upper level model, are dependent, the integral of an arbitrary function $f(Y_1, \ldots, Y_n)$ with a posterior joint PDF of dependent responses $Y_1, \ldots, Y_n$ can be written in terms of a prior joint PDF as

$$\int f(Y_1, \ldots, Y_n)\rho^{pst}_{Y_1,\ldots,Y_n} dY_1, \ldots, dY_n = \int f(Y_1, \ldots, Y_n)\frac{\rho^{pst}_{Y_1,\ldots,Y_n}}{\rho^{prr}_{Y_1,\ldots,Y_n}}\rho^{prr}_{Y_1,\ldots,Y_n} dY_1, \ldots, dY_n \approx \frac{1}{M}\sum_{k=1}^{M} f(Y_1^k, \ldots, Y_n^k)\frac{\rho^{pst}_{Y_1,\ldots,Y_n}}{\rho^{prr}_{Y_1,\ldots,Y_n}} \quad (19)$$

where $\rho^{pst}_{Y_1,\ldots,Y_n}$ and $\rho^{prr}_{Y_1,\ldots,Y_n}$ are the posterior and prior joint PDFs of dependent submodel responses, respectively, $(Y_1^k, \ldots, Y_n^k)$ are pairs of samples subject to a prior joint distribution. $M$ is the number of sampling points. In SSA of the upper level model, the posterior $\rho^{pst}_{Y_1,\ldots,Y_n}$ is the real joint PDF of the dependent responses $Y_1, \ldots, Y_n$.

**3.3 Extended Aggregation Approach.** An extended aggregation approach is proposed to estimate the GSSIs of lower level input variables, including the shared input variables. Since the aggregations of interaction effects and total effects are mathematically difficult and the main effects are usually the most dominating effects in typical engineering systems [11], the proposed method focuses on the evaluation of the GSSIs of input variables for main effects.

For demonstration purpose, we start with a bilevel system as shown in Fig. 1 in which only two submodels ($g_i$ and $g_j$) have the shared variables $\mathbf{X_s}$. When the upper level model function is linear with respect to $Y_i$ and $Y_j$, $h(\mathbf{X_0}, \mathbf{Y})$ can be written as

$$h(\mathbf{X_0}, \mathbf{Y}) = S(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})Y_i + T_j(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})Y_j \quad (20)$$

where $\mathbf{Y}_{\widetilde{ij}}$, which excludes $Y_i$ and $Y_j$, is the vector of independent responses from the lower level submodels. $S(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})$, $T_i(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})$, and $T_j(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})$ are any integrable functions in terms of $\mathbf{X_0}$ and $\mathbf{Y}_{\widetilde{ij}}$. The GSSIs for the main effects of the local input variables $X_{ik}$ and $X_{jk}$ are expressed as

$$\mathrm{SSI}^Z_{X_{ik}} = \frac{V^Z_{X_{ik}}}{V^Z} = \mathrm{SSI}^Z_{U_{Y_iY_j}} \cdot \mathrm{SSI}^{Y_i}_{X_{ik}} \cdot \frac{\tilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_iY_j}}}, \quad X_{ik} \in \mathbf{X_i} \quad (21)$$

$$\mathrm{SSI}^Z_{X_{jk}} = \frac{V^Z_{X_{jk}}}{V^Z} = \mathrm{SSI}^Z_{U_{Y_iY_j}} \cdot \mathrm{SSI}^{Y_j}_{X_{jk}} \cdot \frac{\tilde{T}_j^2 \cdot V^{Y_j}}{V^Z_{U_{Y_iY_j}}}, \quad X_{jk} \in \mathbf{X_j} \quad (22)$$

The GSSIs for the main effects of the shared variables $X_{sk} \in \mathbf{X_s}$ can be calculated as

$$\mathrm{SSI}^Z_{X_{sk}} = \frac{V^Z_{X_{sk}}}{V^Z} = \mathrm{SSI}^Z_{U_{Y_iY_j}} \cdot \left(\mathrm{SSI}^{Y_i}_{X_{sk}} \cdot \frac{\tilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_iY_j}}} + \mathrm{SSI}^{Y_j}_{X_{sk}} \cdot \frac{\tilde{T}_j^2 \cdot V^{Y_j}}{V^Z_{U_{Y_iY_j}}}\right) + \frac{2 \cdot \tilde{T}_i \cdot \tilde{T}_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sk}}}{V^Z} \quad (23)$$

where $\tilde{T}_i = \int T_i(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\widetilde{ij}}} d\mathbf{X_0} d\mathbf{Y}_{\widetilde{ij}}$ and $\tilde{T}_j = \int T_j(\mathbf{X_0}, \mathbf{Y}_{\widetilde{ij}})\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\widetilde{ij}}} d\mathbf{X_0} d\mathbf{Y}_{\widetilde{ij}}$. $V^{Y_i}$ and $V^{Y_j}$ represent the variance of submodels outputs $Y_i$ and $Y_j$, respectively. $V^Z_{U_{Y_iY_j}}$ is the first-order term that denotes the partial variance contributed by the subset $U_{Y_iY_j}$ to the total variance of the upper level output $Z$. $V^Z$ is the total variance of $Z$. $\mathrm{Cov}^{Y_iY_j}_{X_{sk}}$ is the first-order term from the covariance decomposition. The proofs of Eqs. (21)–(23) are given in Appendix B.

The approach can be extended to a general case with $n$ dependent lower level responses $Y_1, \ldots, Y_n$. When the upper level function $h(\mathbf{X_0}, \mathbf{Y})$ is linear with respect to all the dependent responses, the following relationship holds:

$$h(\mathbf{X_0}, \mathbf{Y}) = S(\mathbf{X_0}, \mathbf{Y}_{\widetilde{1,\ldots,n}}) + \sum_{i=1}^{n} T_i(\mathbf{X_0}, \mathbf{Y}_{\widetilde{1,\ldots,n}})Y_i \quad (24)$$

where $\mathbf{Y}_{\widetilde{1\ldots n}}$ is the vector of independent responses from the lower level submodels excluding $Y_1, \ldots, Y_n$.

The formulation of the GSSI for the main effects of the local input variable $X_{ik} \in \mathbf{X}_i$ is written as

$$\mathrm{SSI}^Z_{X_{ik}} = \frac{V^Z_{X_{ik}}}{V^Z} = \mathrm{SSI}^Z_{U_{Y_1,\ldots,Y_n}} \cdot \mathrm{SSI}^{Y_i}_{X_{ik}} \cdot \frac{\tilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_1,\ldots,Y_n}}} \quad (25)$$

The GSSIs for the main effects of the shared variable $X_{sk} \in \mathbf{X_s}$ are expressed as

$$\mathrm{SSI}^Z_{X_{sk}} = \frac{V^Z_{X_{sk}}}{V^Z} = \mathrm{SSI}^Z_{U_{Y_1,\ldots,Y_n}} \cdot \sum_{i=1}^{n}\left(\mathrm{SSI}^{Y_i}_{X_{sk}} \cdot \frac{\tilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_1,\ldots,Y_n}}}\right) + \frac{2}{V^Z}\sum_{i=1}^{n}\sum_{j<i}^{n-1} \tilde{T}_i \cdot \tilde{T}_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sk}} \quad (26)$$

where $\tilde{T}_i = \int T_i(\mathbf{X_0}, \mathbf{Y}_{\widetilde{1,\ldots,n}})\boldsymbol{\rho}_{\mathbf{X_0}}\boldsymbol{\rho}_{\mathbf{Y}_{\widetilde{1,\ldots,n}}} d\mathbf{X_0} d\mathbf{Y}_{\widetilde{1,\ldots,n}}$.

Oftentimes the upper level model function is not linear with respect to the dependent responses from the lower level, the GSSI formulations (Eqs. (25) and (26)) no longer hold for providing accurate SSA results. To overcome this limitation and estimate the GSSI in a nonlinear situation, a multivariable weighted linear regression [29] is employed to capture the global linear trend of the upper level function.

$$h_{U_{Y_1,\ldots,Y_n}}(U_{Y_1,\ldots,Y_n}) = \int h(\mathbf{X_0}, \mathbf{Y})\boldsymbol{\rho}_{\mathbf{X_0}} d\mathbf{X_0} \boldsymbol{\rho}_{\mathbf{Y}_{\widetilde{1,\ldots,n}}} d\mathbf{Y}_{\widetilde{1,\ldots,n}} \approx A + \sum_{i=1}^{n} B_i \cdot Y_i \quad (27)$$

where $A$ is constant and $B_i$ is the linear coefficient for each de-

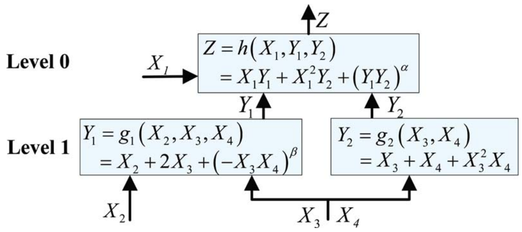


**Fig. 3 System structure for the mathematical example**

pendent response $Y_i$. The coefficient values can be obtained through the weighted linear regression based on the upper level SSA samples collected at the prior stage and the posterior joint distribution of the lower level responses. The multivariable weighted linear regression is written as

$$\boldsymbol{\beta} = (\boldsymbol{\xi}^T \mathbf{W} \boldsymbol{\xi})^{-1} \boldsymbol{\xi}^T \mathbf{W} \mathbf{Z} \tag{28}$$

where $\boldsymbol{\beta}=[A, B_1, \ldots, B_n]^T$; $\boldsymbol{\xi}=[\mathbf{1}, \mathbf{Y}_1, \ldots, \mathbf{Y_n}]$ is a matrix composed of a column vector of unity and all the samples of dependent responses from the lower level submodels with prior joint probability density function. Each component in the weight diagonal matrix $\mathbf{W}$ is set as the probabilities of the sampled dependent responses based on the posterior joint probability density function. The introduction of weights in the linear regression is to capture the linear trend of the upper level model based on the lower submodel responses. Hence, the GSSI for the main effects of $X_{ik}$ can be approximated as

$$\mathrm{SSI}_{X_{ik}}^{Z} = \mathrm{SSI}_{U_{Y_1,\ldots,Y_n}}^{Z} \cdot \mathrm{SSI}_{X_{ik}}^{Y_i} \cdot \frac{B_i^2 \cdot V^{Y_i}}{V_{U_{Y_1,\ldots,Y_n}}^{Z}}, \quad X_{ik} \in \mathbf{X_i} \tag{29}$$

For a shared variable $X_{sk}$, its GSSI for the main effect is approximated as (see the proof in Appendix C).

$$\mathrm{SSI}_{X_{sk}}^{Z} = \mathrm{SSI}_{U_{Y_1,\ldots,Y_n}}^{Z} \cdot \sum_{i=1}^{n} \left( \mathrm{SSI}_{X_{sk}}^{Y_i} \cdot \frac{B_i^2 \cdot V^{Y_i}}{V_{U_{Y_1,\ldots,Y_n}}^{Z}} \right) + \frac{2}{V^Z} \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \mathrm{Cov}_{X_{sk}}^{Y_iY_j} \tag{30}$$

## 4 Case Studies

Two example problems are used to demonstrate and verify our proposed HSSA-SV formulations. The first problem is a mathematical example with explicit functions defined. The purpose is to help readers understand how our method works and to illustrate the impact of the nonlinearity and dependency on the accuracy of the proposed method. The second example is associated with a multiscale bracket system, where the functions are implicit. The purpose is to illustrate the use of HSSA-SV in a highly correlated multiscale system to identify critical subscale inputs and responses with respect to their impact on performance at the coarse scale. The AIO SSA are presented in both case studies to verify the accuracy of the proposed HSSA-SV formulations in capturing the correlation of dependent responses but not as a competing method to HSSA.

**4.1 Mathematical Example.** A mathematical example of a bilevel hierarchical system is shown in Fig. 3. All the input variables $X_i$ are assumed to be independent with uniform distributions over the range of $[0,1]$. Submodels 1 and 2 have two shared variables $X_3$ and $X_4$. To study the impact of the nonlinearity and dependency on the accuracy of the proposed method, different values are assigned to $\alpha$ and $\beta$ that control the functional nonlinearity. Changing $\alpha$ from 0 to 4 increases the nonlinearity between the lower level responses and the upper level outputs while changing $\beta$ from 0 to 2 provides an increasing interaction effect of the local variables in submodel 1. Since the function is cheap to compute in this example, $1.0 \times 10^5$ Monte Carlo samples are generated for calculating the integrals in Eq. (12). Following the proposed

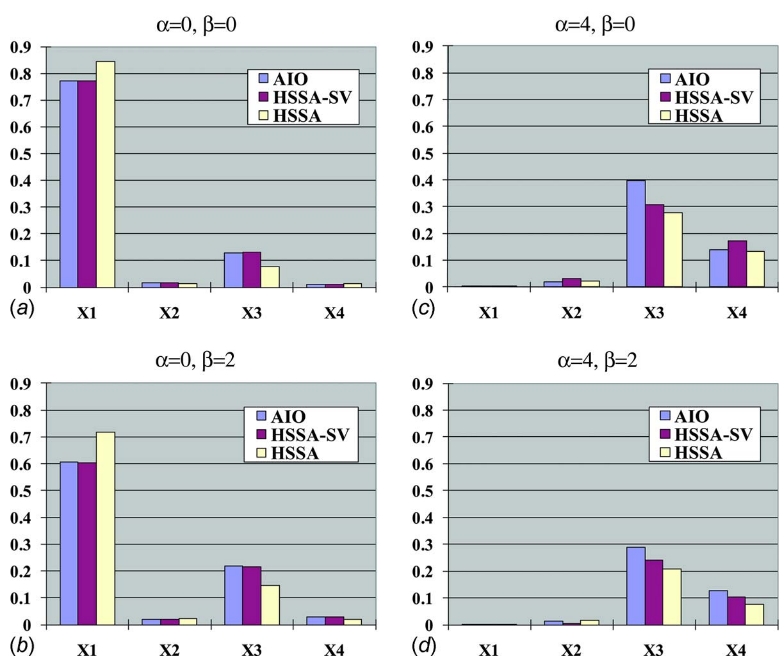


**Fig. 4 The GSSI for the main effects of the input variables in different scenarios**

**Table 1 Comparison of variance estimations of top level performance *Z* using HSSA, HSSA-SV, and AIO methods**

| $V_Z$ | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| AIO | 1.46 | 1.07 | 1.98 × 105 | 0.98 × 106 |
| HSSA | 1.33(8.90%) | 0.90(15.9%) | 6.29 × 105 (217.7%) | 1.07 × 105 (89.1%) |
| HSSA-SV | 1.45(0.7%) | 1.06(0.9%) | 2.05 × 105 (3.5%) | 1.02 × 106 (4.1%) |

method, local subset SSA is performed first at level 0. The two dependent responses $Y_1$ and $Y_2$ are regarded as a subset $U_{Y_1Y_2}$ with prior joint uniform distributions as

$$f_{Y_1,Y_2}(Y_1,Y_2) = \begin{cases} 1/14, & 0.5 \le Y_1 \le 4.5, \quad 0 \le Y_2 \le 3.5 \\ 0, & \text{otherwise} \end{cases} \tag{31}$$

Figure 4 illustrates the GSSIs for the main effect of the four input variables using the HSSA-SV method and the AIO method. When $\alpha=0$ (see cases (a) and (b)), both the AIO and the proposed method provide identical GSSI values. This is because the dependent responses passed from level 1 are linear with respect to $Z$ and the proposed method is able to provide accurate GSSI as discussed in Sec. 3.3. When $\alpha$ is increased from 0 to 4, the nonlinearity of the upper level model becomes higher. The formulation in Eq. (27) is applied to capture the linear trend of the level 0 model in terms of the dependent responses $Y_1$ and $Y_2$. The ranking of GSSIs for the main effects changes for different values of $\alpha$ (see cases (c) and (d)). The GSSIs obtained from the proposed method differs slightly from the one using the AIO methods. The difference can be attributed to the approximation of the global nonlinearity trend at the upper level. Another observation from (a) versus (b) is that the magnitudes of the GSSIs for main effects change when $\beta$ changes. The same observation can be found in (c) versus (d). It is because $\beta$ controls the nonlinearity of the lower level model functions. A large $\beta$ magnifies the interactions between local input variables and reduces the main effects. However, it shows that in all the cases, the proposed method provides the same ranking compared with the one using the AIO method.

To verify the improvement, the GSSI results obtained from the HSSA-SV are also compared with those from the original HSSA in Fig. 4. It is observed that, in most instances, the GSSI estimations using HSSA-SV are more accurate than the ones using the original HSSA in which the impact of covariance is not taken into account. Ignoring the dependency of shared variables with the original HSSA method has introduced a large estimation error in the variance of an upper level output. Table 1 lists the top level performance variance estimations from AIO, the original HSSA and the HSSA-SV method. Here the results of the AIO method are as references to calculate the relative errors (presented in parentheses) from the original HSSA and the HSSA-SV method. As observed from Table 1 with increasing nonlinearity from cases (a) to (d), the error in the variance estimations of an upper level output $Z$ when using HSSA become larger compared with the reference solution. Additionally from Table 1, using the HSSA-SV method provides a good estimation of the variance.

**4.2 Multiscale Bracket System Problem.** Designing multiscale systems based on the advancements of multiscale modeling theories and techniques is an emerging research topic in engineering design [30–32]. To simplify complex multiscale systems, SSA can be applied in a multiscale design process to identify critical variables and subscale analyses with respect to their impact on performance at the coarse scale. Figure 5 illustrates the framework of the multiscale system, which contains two material models at scale 2 and one product model at scale 1. At scale 2, representative volume element (RVE) material models are employed to construct the microstructure-constitutive property relation of an aluminum alloy material. Silicon particle volume fraction ($PVF$) and particle density ($PD$) quantitatively characterize the material microstructure, and are considered as shared material design variables at scale 2. The two submodel responses, strength index ($k$) and strain hardening index ($n$) are two interrelated responses. Together, they represent the material constitutive properties, which are passed to scale 1 as inputs. As shown in Fig. 5, there are three product design variables ($C_x, C_y, R$ location and radius of the hole) as local input variables of the top level bracket model. Due to the high computational cost in RVE simulations, the Kriging metamodels of material property parameters ($k$ and $n$) are constructed in terms of $PVF$ and $PD$ while the maximum stress is expressed in terms of $C_x$, $C_y$, $R$, $k$, and $n$ [33].

The HSSA-SV is applied to rank the importance of local model inputs at two different scales, including the three product design variables ($C_x, C_y, R$) and the two material design variables ($PVF$ and $PD$). Without knowing the optimal solutions of the five design variables at a prior design stage, uniform distributions are assigned to each design variable to show that the optimal solution could be anywhere in the design domain with equal possibility. In this example, $C_x \in [40,80]$, $C_y \in [-100,-50]$, $R \in [20,35]$, $PVF \in [0.03,0.11]$ and $PD \in [3,7]$. Due to the statistical dependency, $k$ and $n$ are regarded as a subset $U_{kn}=\{k,n\}$, and a prior joint uniform distribution is assigned to the subset $U_{kn}$.

The HSSA-SV results shown in Table 2, at scale 1, indicate that both the local input variable $R$ and the subset $U_{kn}$ have a large impact (the normalized TSSIs are 43.75% and 55.44%, respectively) on the total variance of the maximum stress $S_{\max}$. Hence, SSA is further applied to the material model at scale 2 and the SSA results are listed in Table 2. Once the true joint distribution of $k$ and $n$ are available, the LSSIs and TSSIs at the product scale can be updated at the posterior stage. As shown in Table 2, the posterior LSSI and TSSI are slightly different from the SSA results at the prior stage but the ranking of $R$ and subset $U_{kn}$ remains the same as at the prior stage.

To apply the proposed aggregation approach, the approximate linear regression function of $S_{\max}$ with respect to $k$ and $n$ is calculated as

$$h_{U_{kn}}(U_{kn}) \approx B_1k + B_2n + A = 170{,}760.8k - 205{,}857.9n + 273{,}091{,}883 \tag{32}$$

which is illustrated in Fig. 6. The decomposition of the covariance of $k$ and $n$ is written as

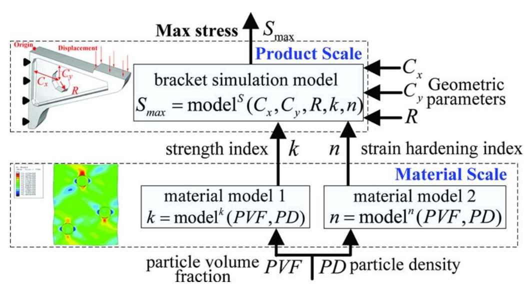


**Fig. 5 Framework of the two-scale system (scale 1 is product scale and scale 2 is material scale)**

Table 2 Updating process of the HSSA-SV method in multiscale design problem

| Scale | Input | Prior stage LSSI | Prior stage Local TSSI | Posterior stage LSSI | Posterior stage Local TSSI |
|---|---|---|---|---|---|
| Product scale | $C_x$ | 0.0020 | 0.0021(0.18%) | 0.0014 | 0.0027 (0.23%) |
| | $C_y$ | 0.0054 | 0.0073(0.63%) | 0.0010 | 0.0074 (0.64%) |
| | $R$ | 0.3469 | 0.5104(43.75%) | 0.3399 | 0.5005 (43.18%) |
| | $U_{kn}$ | 0.4836 | 0.6467(55.44%) | 0.4960 | 0.6486 (55.95%) |
| Material scale | | | | | |
| | | | Material model 1 for $k$ | | |
| | $PVF$ | 0.7307 | 0.7858(74.34%) | / | / |
| | $PD$ | 0.2118 | 0.2712(25.66%) | / | / |
| | | | Material model 2 for $n$ | | |
| | $PVF$ | 0.9134 | 0.9265(91.21%) | / | / |
| | $PD$ | 0.0720 | 0.0893(8.79%) | / | / |

$$\mathrm{Cov}[k,n] = \mathrm{Cov}_{PVF}^{kn} + \mathrm{Cov}_{PD}^{kn} + \mathrm{Cov}_{PVF\ \ PD}^{kn} = (10.1 + 1.26 + 0.44) \times 10^{-4} = 11.8 \times 10^{-4} \quad (33)$$

It is observed that the first-order covariance terms $\mathrm{Cov}_{PVF}^{kn}$ and $\mathrm{Cov}_{PD}^{kn}$ are two major sources of total covariance.

The final GSSIs for the main effects of three product design and two material design variables are plotted in Fig. 7, which illustrates that the diameter $R$ at the product scale and $PVF$ at the material scale are two dominant input variables on the variation in the maximum stress. In this case, the summation of the GSSI for main effects of $R$ and $PVF$ is greater than 0.75 and it means the main effects are sufficient enough to rank the importance of variables without considering the interaction effects. Comparing the results from the AIO method, the proposed method provides identical ranking results.

## 5 Benefits and Limitations of the HSSA-SV Method

While the two example problems in Sec. 4 illustrate the improved accuracy of the HSSA-SV method for capturing dependent responses through comparisons with the original HSSA and the AIO approach, its advantages and limitations over the AIO approach are summarized in this section. Similar to the benefits of the HSSA method, the first benefit of the HSSA-SV method is its ability to concurrently execute the SSA across all levels and submodels, thus saving computation time compared with the AIO method, which has to follow a sequential, bottom-up procedure. The GSSIs are obtained via aggregating the LSSIs of submodels without adding additional samples. This feature of "concurrency" is highly desired in a concurrent multidisciplinary product design environment. For the two examples presented, the computations of integral calculations for updating posterior distributions and the analytical aggregation of submodel LSSIs are negligible compared with those used for local submodel SSAs. Since no submodels are eliminated during the HSSA process, the computational efficiency between using HSSA-SV and AIO would be the same if no concurrent computation is considered for levels 1 and 2 local SSAs. If concurrent computation is considered, the HSSA-SV method will be almost twice as efficient as the AIO method because the two levels of the system are analyzed simultaneously.

The second benefit is the ability of applying the SSA only to critical submodels by following the top-down strategy, thus improving the efficiency of sensitivity analysis compared with the AIO method. More importantly, as SSA is intended to gain knowledge in a design process, the method can better manage the design complexity by fixing the variables from insignificant submodels (see the multilevel suspension design example in our paper presenting the original HSSA method [11]). This benefit of reducing design complexity in a multidisciplinary design environment is not directly illustrated in our examples but is an important motivation of our research.

The third inherited advantage is that the top-down strategy used in the HSSA methods matches better with a typical design pro-

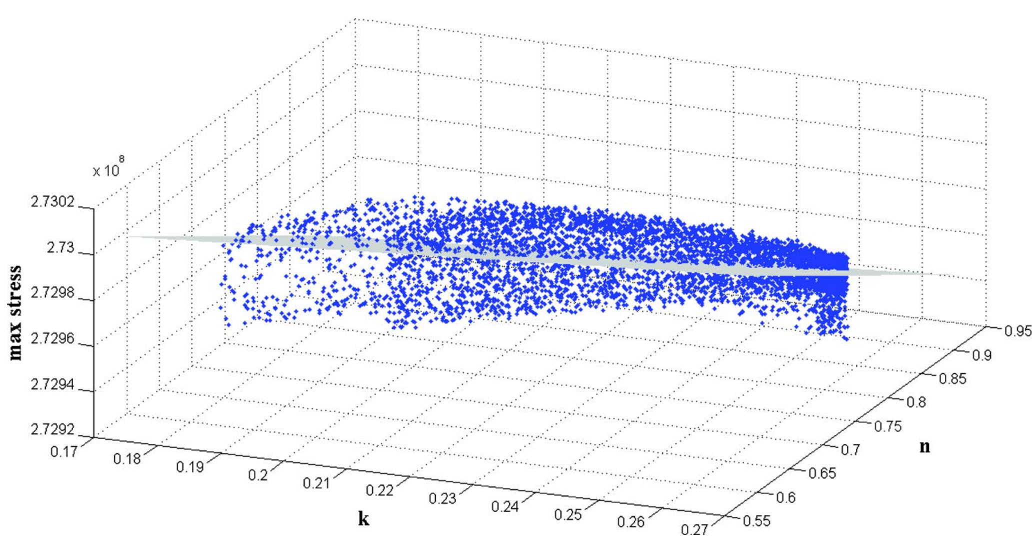


Fig. 6 Weighted linear regression of two dependent responses $\{k, n\}$

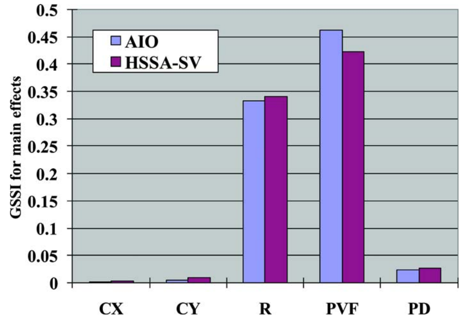


**Fig. 7 The GSSI for Main effect of each input variable**

cess, where exact details of lower level submodels may not be completely known in an early design stage. The HSSA methods have the capability of executing from the top level and then gradually going down to the lower level when more details are explored as design is being further developed. The AIO method, on the other hand, can only be applied when the whole system is exactly known, which is against the purpose of applying SSA to manage the complexity in a design process.

Although the HSSA methods have the aforementioned advantages, designers should be aware of the risk of eliminating "unimportant" submodel responses and analyses using upper level SSA results based on the prior distributions of responses without posterior verifications using the lower level SSA results. It is recommended that designers should cautiously select the threshold for eliminating unimportant submodels. While our empirical study shows that using a large range for a prior distribution encompassing as much of the real performance as possible can improve the posterior estimation, tests with different distribution ranges are highly recommended.

## 6 Conclusion and Future Work

In this paper, an HSSA-SV method is proposed to handle the dependency caused by shared variables in a hierarchical system and to overcome the limitation of the original HSSA method. Following the top-down strategy proposed in the original HSSA method, the concept of local subset SSA is applied in which the dependent responses from lower submodels are considered as a subset. Critical responses from the lower level can be identified according to the rank of the TSSIs at the upper level and SSA is further applied to the lower level submodels with critical responses. With the assistance of the importance sampling technique, LSSI at the upper level is then updated without additional samples after the more accurate posterior joint distributions are available. An extended aggregation approach is introduced to formulate the GSSI for main effects of the input variables at different levels of hierarchy. With the proposed formulation, an accurate GSSI can be achieved when the upper model function is linear with respect to the dependent responses from the lower submodels. A multivariable weighted linear regression approach is employed to capture the trend of the nonlinear effect of the upper level function. As illustrated in the numerical and multiscale design problems, the proposed HSSA-SV method provides sufficiently accurate GSSI compared with the AIO method and importance rankings of all the input variables are exactly identical.

There are still some important issues in HSSA and HSSA-SV requiring further explorations. First, both the HSSA and the HSSA-SV methods only focus on estimating the main effects with an assumption that the main effect of an input is sufficient to represent its importance. Assessing the global TSSI in the HSSA-SV method needs to be explored even though it is anticipated that it may become computationally intractable. Second, although the proposed HSSA-SV method provides accurate importance rankings, there are several facts that impact the estimation error of the GSSI magnitudes: the nonlinearity of the upper level model with respect to the dependent lower level responses; the simplification of such nonlinearity by using a linear regression; and the error propagation across multiple levels. These factors need to be addressed in order to improve the estimation accuracy in future work.

## Acknowledgment

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## Nomenclature

$V^Y$ = variance of model output $Y$

$V_{X_i}^Y$ = partial variance of output $Y$ contributed by individual input variable $X_i$

$V_{X_{i_1}\ldots X_{i_j}}^Y$ = partial variance of output $Y$ due to the interaction effects between input variables $X_{i_1},\ldots,X_{i_j}$

$\mathrm{SSI}_{X_i}^Y$ = SSI of $X_i$ with respect to $Y$

$\mathrm{SSI}_{X_{i_1}\ldots X_{i_j}}^Y$ = SSI of input variables $X_{i_1},\ldots,X_{i_j}$ with respect to $Y$

$\mathrm{SSI}_{T_{X_i}}^Y$ = TSSI of input variable $X_i$ with respect to $Y$

$\mathrm{SSI}_{X_i\tilde{X}_i}^Y$ = sum of all higher-order SSI involving the input variables $X_i$

$\mathrm{SSI}_{U_i}^Y$ = SSI of a variable subset $U_i$ with respect to $Y$

$\mathrm{SSI}_{U_{i_1}\ldots U_{i_j}}^Y$ = SSI of variable subsets $U_{i_1},\ldots,U_{i_j}$ with respect to $Y$

$\mathrm{Cov}_{X_{si}}^{Y_iY_j}$ = partial covariance of $Y_i$ and $Y_j$ contributed by the shared input variable $X_{si}$

$\mathrm{Cov}_{X_{si_1},\ldots,X_{si_j}}^{Y_iY_j}$ = partial covariance of $Y_i$ and $Y_j$ due to the interaction effect between shared input variables $X_{si_1},\ldots,X_{si_j}$

$\phi_{X_i}(X_i)$ = first effect term in ANOVA

$\phi_{X_{i_1}\ldots X_{i_n}}(X_{i_1},\ldots,X_{i_n})$ = interaction effect term in ANOVA

$\rho_k(X_k)$ = PDF of variable $X_k$

$\rho_{U_{i_k}}(U_{i_k})$ = joint PDF of the subset variables $U_{i_k}$

$\boldsymbol{\rho}_{X_1\ldots X_n}$ = joint PDF of random variables $X_1,\ldots,X_n$

$E(\cdot)$ = expectation operator

$\mathrm{Var}[\cdot]$ = variance operator

$\mathrm{Cov}[\cdot]$ = covariance operator

## Appendix A: Proof of the Proposition

To make the expressions more concise, vectors $\mathbf{X}_1'=\{\mathbf{X}_s,\mathbf{X}_1\}$ and $\mathbf{X}_2'=\{\mathbf{X}_s,\mathbf{X}_2\}$ are used to represent input variables vectors $\{X_{s1},\ldots,X_{sN_S},X_{11},\ldots,X_{1N_1}\}$ and $\{X_{s1},\ldots,X_{sN_S},X_{21},\ldots,X_{2N_2}\}$ of functions $f_1$ and $f_2$, then $\mathbf{X}_1'=\{X_{11}',\ldots,X_{1N_S}',X_{1N_S+1}',\ldots X_{1N_S+N_1}'\}$ $=\{X_{s1},\ldots,X_{sN_S},X_{11},\ldots,X_{1N_1}\}$ and $\mathbf{X}_2'=\{X_{21}',\ldots,X_{2N_S}',X_{2N_S+1}',\ldots X_{2N_S+N_2}'\}=\{X_{s1},\ldots,X_{sN_S},X_{21},\ldots,X_{2N_2}\}$.

According to the ANOVA decomposition, the dependent outputs $Y_1$ and $Y_2$ can be expressed as

$$Y_1=f_1(\mathbf{X}_1')=f_{10}+\sum_i \phi_{X_{1i}'}(X_{1i}')+\sum_{i_1<i_2}\phi_{X_{1i_1}'X_{1i_2}'}(X_{1i_1}',X_{1i_2}')+\ldots+\phi_{X_{1i_1}',\ldots,X_{1i_{N_s+N_1}}'}(X_{1i_1}',\ldots,X_{1i_{N_s+N_1}}') \quad (A1)$$

$$Y_2 = f_2(\mathbf{X_2'}) = f_{20} + \sum_i \varphi_{X'_{2i}}(X'_{2i}) + \sum_{i_1<i_2} \varphi_{X'_{2i_1}X'_{2i_2}}(X'_{i_1},X'_{i_2}) + \dots + \varphi_{X'_{2i_1},\dots,X'_{2i_{N_s+N_2}}}(X'_{2i_1},\dots,X'_{2i_{N_s+N_2}}) \quad (A2)$$

Because of the orthogonal property of each term in Eqs. (A1) and (A2), one has

$$\mathrm{Cov}[Y_1,Y_2] = \mathrm{Cov}\Bigg[\Bigg(f_{10} + \sum_i \phi_{X'_{1i}}(X'_{1i}) + \sum_{i_1<i_2} \phi_{X'_{1i_1}X'_{1i_2}}(X'_{1i_1},X'_{1i_2}) + \dots + \phi_{X'_{1i_1},\dots,X'_{1i_{N_s+N_1}}}(X'_{1i_1},\dots,X'_{1i_{N_s+N_1}})\Bigg), \Bigg(f_{20} + \sum_i \varphi_{X'_{2i}}(X'_{2i}) + \sum_{i_1<i_2} \varphi_{X'_{2i_1}X'_{2i_2}}(X'_{2i_1},X'_{2i_2}) + \dots + \varphi_{X'_{2i_1},\dots,X'_{2i_{N_s+N_2}}}(X'_{2i_1},\dots,X'_{2i_{N_s+N_2}})\Bigg)\Bigg] \quad (A3)$$

Expanding the covariance expression through a mutual multiplying operation and following orthogonal property of decomposed terms, one may obtain

$$\mathrm{Cov}[f_{10},\varphi_{X'_{2i_1},\dots,X'_{2i_m}}(X'_{2i_1},\dots,X'_{2i_m})] = \mathrm{Cov}[\phi_{X'_{1i_1},\dots,X'_{1i_m}}(X'_{1i_1},\dots,X'_{1i_m}),f_{20}] = 0 \quad (A4)$$

$$\mathrm{Cov}[\phi_{X'_{1i}}(X'_{1i}),\varphi_{X'_{2j}}(X'_{2j})] = \begin{cases} \mathrm{Cov}[\phi_{X'_{1i}}(X'_{1i}),\varphi_{X'_{2j}}(X'_{2j})] & \text{if} \quad i=j; \quad 1\le i, \quad j \le N_S \\ 0 & \text{if} \quad i \ne j \end{cases} \quad (A5)$$

$$\mathrm{Cov}[\phi_{X'_{1i_1},\dots,X'_{1i_m}}(X'_{1i_1},\dots,X'_{1i_m}),\varphi_{X'_{2j_1},\dots,X'_{2j_n}}(X'_{2j_1},\dots,X'_{2j_n})] = \begin{cases} \mathrm{Cov}[\phi_{X'_{1i_1},\dots,X'_{1i_j}}(X'_{1i_1},\dots,X'_{1i_j}),\varphi_{X'_{2i_1},\dots,X'_{2i_j}}(X'_{2i_1},\dots,X'_{2i_j})] & \text{if} \quad m=n; \quad i_l=j_l, \quad 1\le l\le m; \quad 1\le j\le N_S \\ 0 & \text{otherwise} \end{cases} \quad (A6)$$

The nonzero expression in Eq. (A5) can be written as

$$\mathrm{Cov}[\phi_{X_{si}}(X_{si}),\varphi_{X_{si}}(X_{si})], \quad 1\le i\le N_s \quad (A7)$$

and it can be simplified into

$$\mathrm{Cov}[E(f_1|X_{si}),E(f_2|X_{si})], \quad 1\le i\le N_s \quad (A8)$$

while the nonzero expression in Eq. (A6) can be written as

$$\mathrm{Cov}[\phi_{X_{si_1},\dots,X_{si_j}}(X_{si_1},\dots,X_{si_j}),\varphi_{X_{si_1},\dots,X_{si_j}}(X_{si_1},\dots,X_{si_j})] \quad 1\le i_1,\dots,i_j\le N_s \quad (A9)$$

and it is equal to

$$\mathrm{Cov}[E(f_1|X_{si_1},\dots,X_{si_j}),E(f_2|X_{si_1},\dots,X_{si_j})] - \sum_{k=1}^{j-1}\sum_{j_1\dots j_k\in(i_1\dots i_j)} \times \mathrm{Cov}[E(f_1|X_{sj_1},\dots,X_{sj_k}),E(f_2|X_{sj_1},\dots,X_{sj_k})], \quad 1\le i_1,\dots,i_j\le N_s \quad (A10)$$

By summing Eqs. (A8) and (A10) recursively, one has

$$\mathrm{Cov}[Y_1,Y_2] = \mathrm{Cov}[E(f_1|X_{s1},\dots,X_{sN_s}),E(f_2|X_{s1},\dots,X_{sN_s})] \quad (A11)$$

## Appendix B: Proof of Eqs. (21)–(23)

Given the linear upper model function with respect to dependent responses $Y_i$ and $Y_j$ from lower level submodels as $Z = h(\mathbf{X_0},\mathbf{Y}) = S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_i + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_j$, there are two dependent responses, $Y_i$ and $Y_j$, and they are regarded as a subset $U_{Y_iY_j}=\{Y_i,Y_j\}$. The main effect for the subset is calculated according the subset SSA method, and thus, the mean of the upper level output is given as

$$h_0 = \int [S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_i + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_j]\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}}d\mathbf{X_0}d\mathbf{Y} = \widetilde{S} + \widetilde{T}_i\cdot E(Y_i) + \widetilde{T}_j\cdot E(Y_j) \quad (B1)$$

where $\widetilde{S}=\int S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\widetilde{ij}}}d\mathbf{X_0}d\mathbf{Y}_{\widetilde{ij}}$, $\widetilde{T}_i=\int T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\tilde{i}}}d\mathbf{X_0}d\mathbf{Y}_{\tilde{i}}$, and $\widetilde{T}_j=\int T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\tilde{j}}}d\mathbf{X_0}d\mathbf{Y}_{\tilde{j}}$. Let $\rho_{Y_iY_j}$ be the joint probability density function. Then

$$\phi_{U_{Y_iY_j}}(U_{Y_iY_j}) = \int [S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_i + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_j]\rho_{\mathbf{X_0}}\rho_{\mathbf{Y}_{\widetilde{ij}}}d\mathbf{X_0}d\mathbf{Y}_{\widetilde{ij}} - h_0 = \widetilde{S} + \widetilde{T}_i\cdot Y_i + \widetilde{T}_j\cdot Y_j - h_0 = \widetilde{T}_i\cdot Y_i + \widetilde{T}_j\cdot Y_j - \widetilde{T}_i\cdot E(Y_i) - \widetilde{T}_j\cdot E(Y_j) \quad (B2)$$

and the first-order partial variance of subset $U_{Y_iY_j}$ is equal to

$$V^Z_{U_{Y_iY_j}} = \int \phi^2_{U_{Y_iY_j}}(U_{Y_iY_j})\rho_{Y_iY_j}dY_idY_j = \widetilde{T}_i^2V^{Y_i} + \widetilde{T}_j^2V^{Y_j} + 2\cdot\widetilde{T}_i\cdot\widetilde{T}_j\cdot\mathrm{Cov}[Y_i,Y_j] \quad (B3)$$

Let $h(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}},\mathbf{X}_i,\mathbf{X}_j,\mathbf{X_s}) = S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_i + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})Y_j = S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\cdot g_i(\mathbf{X_i},\mathbf{X_s}) + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{\mathbf{ij}}})\cdot g_j(\mathbf{X_j},\mathbf{X_s})$ and $X_{ik}\in\mathbf{X_i}$, $X_{jk}\in\mathbf{X_j}$, and $X_{sk}\in\mathbf{X_s}$. The decomposed first-order term of ANOVA for function $h$ with respect to local input variable $X_{ik}$ in the $i$th submodel at the lower level is given as

$$\phi_{X_{ik}}(X_{ik}) = \int (S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\cdot g_i(\mathbf{X_i},\mathbf{X_s}) + T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}})\cdot g_j(\mathbf{X_j},\mathbf{X_s}))\boldsymbol{\rho}_{\mathbf{X_0}}d\mathbf{X_0}\boldsymbol{\rho}_{\mathbf{Y}_{\widetilde{ij}}}d\mathbf{Y}_{\widetilde{ij}}\boldsymbol{\rho}_{\mathbf{X_j}}d\mathbf{X_j}\boldsymbol{\rho}_{\mathbf{X_s}}d\mathbf{X_s} \times \prod_{m\ne k}\rho_{X_{im}}dX_{im} - h_0$$

$$= \widetilde{T}_i \cdot \left( \int g_i(\mathbf{X_i},\mathbf{X_s}) \boldsymbol{\rho}_{\mathbf{X_s}} d\mathbf{X_s} \prod_{m \neq k} \rho_{X_{im}} dX_{im} - E(Y_i) \right) \quad \text{(B4)}$$

Thus, the global main effect contributed by local input variable $X_{ik}$ belonging to the $i$th submodel on the total variance of output $Z$ is equal to

$$V^Z_{X_{ik}} = \int \phi^2_{X_{ik}}(X_{ik}) \rho_{X_{ik}} dX_{ik} = \widetilde{T}_i^2 \cdot V^{Y_i}_{X_{ik}} \quad \text{(B5)}$$

Thus, GSSI for main effect of $X_{ik}$ is expressed as

$$\mathrm{SSI}^Z_{X_{ik}} = \frac{V^Z_{X_{ik}}}{V^Z} = \frac{\widetilde{T}_i^2 \cdot V^{Y_i}_{X_{ik}}}{V^Z} = \frac{\widetilde{T}_i^2 \cdot V^{Y_i}_{X_{ik}}}{V^{Y_i}} \cdot \frac{V^{Y_i}}{V^Z_{U_{Y_iY_j}}} \cdot \frac{V^Z_{U_{Y_iY_j}}}{V^Z}$$

$$= \mathrm{SSI}^Z_{U_{Y_iY_j}} \cdot \mathrm{SSI}^{Y_i}_{X_{ik}} \cdot \frac{\widetilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_iY_j}}} \quad \text{(B6)}$$

When considering the decomposed first-order term of ANOVA for function $f$ with respect to shared variable $X_{sm}$ at the lower level, the term can be written as

$$\phi_{X_{sk}}(X_{sk}) = \int (S(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) + T_i(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) \cdot g_i(\mathbf{X_i},\mathbf{X_s})$$

$$+ T_j(\mathbf{X_0},\mathbf{Y}_{\widetilde{ij}}) \cdot g_j(\mathbf{X_j},\mathbf{X_s})) \boldsymbol{\rho}_{\mathbf{X_0}} d\mathbf{X_0} \boldsymbol{\rho}_{\mathbf{Y}_{\widetilde{ij}}} d\mathbf{Y}_{\widetilde{ij}} \boldsymbol{\rho}_{\mathbf{X_i}} d\mathbf{X_i} \boldsymbol{\rho}_{\mathbf{X_j}} d\mathbf{X_j}$$

$$\times \prod_{m \neq k} \rho_{X_{sm}} dX_{sm} - h_0$$

$$= \widetilde{T}_i \cdot \left( \int g_i(\mathbf{X_i},\mathbf{X_s}) \boldsymbol{\rho}_{\mathbf{X_i}} d\mathbf{X_i} \prod_{m \neq k} \rho_{X_{sm}} dX_{sm} - E(Y_i) \right)$$

$$+ \widetilde{T}_j \cdot \left( \int g_j(\mathbf{X_j},\mathbf{X_s}) \boldsymbol{\rho}_{\mathbf{X_j}} d\mathbf{X_j} \prod_{m \neq k} \rho_{X_{sm}} dX_{sm} - E(Y_j) \right) \quad \text{(B7)}$$

and then, the global main effect contributed by the shared variable $X_{sk}$ at the lower level on the total variance of output $Z$ is equal to

$$V^Z_{X_{sk}} = \int \phi^2_{X_{sk}}(X_{sk}) \rho_{X_{sk}} dX_{sk} = \widetilde{T}_i^2 \cdot V^{Y_i}_{X_{sk}} + \widetilde{T}_j^2 \cdot V^{Y_j}_{X_{sk}}$$

$$+ 2 \cdot \widetilde{T}_i \cdot \widetilde{T}_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sk}} \quad \text{(B8)}$$

Thus, the GSSI for main effect of the shared variable $X_{sm}$ is expressed as

$$\mathrm{SSI}^Z_{X_{sm}} = \frac{V^Z_{X_{sm}}}{V^Z} = \frac{\widetilde{T}_i^2 \cdot V^{Y_i}_{X_{sm}} + \widetilde{T}_j^2 \cdot V^{Y_j}_{X_{sm}} + 2 \cdot \widetilde{T}_i \cdot \widetilde{T}_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sm}}}{V^Z}$$

$$= \mathrm{SSI}^Z_{U_{Y_iY_j}} \cdot \left( \mathrm{SSI}^{Y_i}_{X_{sm}} \cdot \frac{\widetilde{T}_i^2 \cdot V^{Y_i}}{V^Z_{U_{Y_iY_j}}} + \mathrm{SSI}^{Y_j}_{X_{sm}} \cdot \frac{\widetilde{T}_j^2 \cdot V^{Y_j}}{V^Z_{U_{Y_iY_j}}} \right)$$

$$+ \frac{2 \cdot \widetilde{T}_i \cdot \widetilde{T}_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sm}}}{V^Z} \quad \text{(B9)}$$

These formulations can be easily extended to the case with more than two dependent responses from lower level submodels.

## Appendix C: Proof of Eqs. (29) and (30)

Based on the subset decomposition and LSSA at the upper level, the variance of upper level output $Z$ can be decomposed as

$$V^Z = V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}} + V^Z_{U_{Y_1,\ldots,Y_n}} + V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}U_{Y_1,\ldots,Y_n}} \quad \text{(C1)}$$

where $\widetilde{\mathbf{Y}}_{1,\ldots,n} = \{\mathbf{X_0}, \mathbf{Y}_{\widetilde{1,\ldots,n}}\}$ represents all the input variables of the upper level model excluding the subset of dependent responses from the lower level submodels. According to the approximation shown in Eq. (27), the main effect of subset $U_{Y_1,\ldots,Y_n}$ on the total variance of the upper level output $Z$ is formulated as

$$V^Z_{U_{Y_1,\ldots,Y_n}} = \int (h_{U_{Y_1,\ldots,Y_n}}(U_{Y_1,\ldots,Y_n}) - h_0)^2 \boldsymbol{\rho}_{Y_1,\ldots,Y_n} dY_1, \ldots, dY_n$$

$$= \mathrm{Var}\left[ A + \sum_{i=1}^{n} B_i \cdot Y_i \right] = \sum_{i=1}^{n} B_i^2 V^{Y_i}$$

$$+ 2 \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \mathrm{Cov}[Y_i, Y_j] \quad \text{(C2)}$$

where $B_i$ is a linear coefficient of the linearized function. Based on the lower level LSSA and $V^{Y_i}$ can be further decomposed as

$$V^{Y_i} = \sum_m V^{Y_i}_{X_{im}} + \sum_m V^{Y_i}_{X_{sm}} + \sum V^{Y_i}_{\mathrm{high}} \quad \text{(C3)}$$

where the higher-order contributions of the partial variance are represented by $V^{Y_i}_{\mathrm{high}}$. The covariance is also decomposed according to the proposed covariance decomposition method for the shared variables, and is formulated as

$$\mathrm{Cov}[Y_i, Y_j] = \mathrm{Cov}[E(Y_i|\mathbf{X_s}), E(Y_j|\mathbf{X_s})] = \sum_m \mathrm{Cov}^{Y_iY_j}_{X_{sm}} + \sum \mathrm{Cov}^{Y_iY_j}_{\mathrm{high}} \quad \text{(C4)}$$

where $X_{sm} \in \mathbf{X_s}$, and $\mathrm{Cov}^{Y_iY_j}_{X_{sm}}$ represent the main effect contributed by the shared variable $X_{sm}$ on the covariance of $Y_i$ and $Y_j$; $\mathrm{Cov}^{Y_iY_j}_{high}$ stands for the higher-order contributions of partial covariance. $V^Z$ can be expressed as

$$V^Z = V^Z_{U_{Y_1,\ldots,Y_n}} + V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}} + V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}U_{Y_1,\ldots,Y_n}} = \sum_{i=1}^{n} B_i^2 \left[ \sum_m V^{Y_i}_{X_{im}} \right.$$

$$\left. + \sum_m V^{Y_i}_{X_{sm}} + \sum V^{Y_i}_{\mathrm{high}} \right] + 2 \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \left[ \sum_m \mathrm{Cov}^{Y_iY_j}_{X_{sm}} \right.$$

$$\left. + \sum \mathrm{Cov}^{Y_iY_j}_{\mathrm{high}} \right] + V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}} + V^Z_{\widetilde{\mathbf{Y}}_{1,\ldots,n}U_{Y_1,\ldots,Y_n}} \quad \text{(C5)}$$

Thus, the main contribution of local input variable $X_{ik}$ belonging to the $i$th submodel on the variance of $Z$ is given as

$$V^Z_{X_{ik}} = B_i^2 V^{Y_i}_{X_{ik}} \quad \text{(C6)}$$

It can be observed from Eq. (C5), the main (first-order) contribution of a shared variable to its upper level model variance can be regarded as a combination of its contributions to the variance and covariance of the dependent responses at the lower level submodels. Thus, the main contribution of the shared variable $X_{sk}$ on the variance of $Z$ is

$$V^Z_{X_{sk}} = \sum_{i=1}^{n} B_i^2 V^{Y_i}_{X_{sk}} + 2 \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \mathrm{Cov}^{Y_iY_j}_{X_{sk}} \quad \text{(C7)}$$

Given the GSSI in form of aggregation of LSSI, the Eqs. (C6) and (C7) are expressed as

$$\mathrm{SSI}_{X_{ik}}^{Z} = \frac{B_i^2 V_{X_{ik}}^{Y_i}}{V^Z} = \frac{B_i^2 V_{X_{ik}}^{Y_i}}{V^Z} \cdot \frac{V^{Y_i}}{V^{Y_i}} \cdot \frac{V_{U_{Y_1,\ldots,Y_n}}^{Z}}{V_{U_{Y_1,\ldots,Y_n}}^{Z}}$$

$$= \mathrm{SSI}_{U_{Y_1,\ldots,Y_n}}^{Z} \cdot \mathrm{SSI}_{X_{ik}}^{Y_i} \cdot \frac{B_i^2 \cdot V^{Y_i}}{V_{U_{Y_1,\ldots,Y_n}}^{Z}} \tag{C8}$$

and

$$\mathrm{SSI}_{X_{sk}}^{Z} = \frac{\sum_{i=1}^{n} B_i^2 V_{X_{sk}}^{Y_i} + 2 \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \mathrm{Cov}_{X_{sk}}^{Y_i Y_j}}{V^Z}$$

$$= \mathrm{SSI}_{U_{Y_1,\ldots,Y_n}}^{Z} \cdot \sum_{i=1}^{n} \left( \mathrm{SI}_{X_{sk}}^{Y_i} \cdot \frac{B_i^2 \cdot V^{Y_i}}{V_{U_{Y_1,\ldots,Y_n}}^{Z}} \right)$$

$$+ \frac{2 \cdot \sum_{i=1}^{n} \sum_{j<i}^{n-1} B_i \cdot B_j \cdot \mathrm{Cov}_{X_{sk}}^{Y_i Y_j}}{V^Z} \tag{C9}$$

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