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A Unified Framework for Integrated Optimization Under Uncertainty

Reliability and robustness are two main attributes of design under uncertainty. Hence, it is necessary to combine reliability-based design and robust design at the design stage. In this paper, a unified framework for integrating reliability-based design and robust design is proposed. In the proposed framework, the probabilistic objective function is converted to a deterministic objective function by the Taylor series expansion or inverse reliability strategy with accounting for the probabilistic characteristic of the objective function. Therefore, with this unified framework, there is no need to deal with a multiobjective optimization problem to integrate reliability-based design and robust design any more. The probabilistic constraints are converted to deterministic constraints with inverse reliability strategy at the same time. In order to solve the unified framework, an improved sequential optimization and reliability assessment method is proposed. Three examples are given to illustrate the benefits of the proposed methods. [DOI: 10.1115/1.4001526]

1 Introduction

Uncertainty is ubiquitous in the engineering design ranging from a simple component to complicated systems. Therefore, design under uncertainty has become growingly important. Many design methods under uncertainty have been developed over the past decades. Among these methods, reliability-based design and robust design are two typical paradigms. The focuses of these two paradigms are different. Reliability-based design achieves a design, which has a probability of failure less than the acceptable level, to ensure that the events lead to a catastrophic result are extremely unlikely [1,2]. On the other hand, robust design seeks a design, which is relatively insensitive to the environmental variation (random parameters), to improve the quality of a product by minimizing the effect of uncertainty on system performance without eliminating the causes [3–5]. Since reliability and robustness are attributes of design under uncertainty, it is necessary to combine them into an integrated framework [6,7].

The wide applications for either reliability-based design and/or robust design are subjected to the restrictions on their costly computation and limited capacities. Under the reliability-based design paradigm, the computational inefficiency is generally derived from the expensive probabilistic analysis. Many methods have been proposed to deal with the probabilistic analysis: (1) most probable point (MPP)-based methods, (2) simulation methods, (3) moment-based methods, and (4) metamodeling methods. MPP-based methods generally include the first order reliability method (FORM) and second order reliability method (SORM) [8–10]. The probabilistic analysis is achieved by simplifying the limit state function with the first order or second order Taylor expansion at the MPP for FORM or SORM. Since SORM is second order gradient-based, it is generally more accurate but more time-consuming than FORM. Simulation methods, generally including Monte Carlo simulation and quasi-Monte Carlo simulation, are easy and feasible to most probabilistic analysis. The computational cost of simulation methods, however, is prohibitively high for high reliability. Moment-based methods, such as point estimate method [11], eigenvalue dimension reduction [12], and saddlepoint approximation method [13], have been used as alternative approaches for probabilistic analysis. In recent years, metamodeling methods, such as response surface method [14], Kriging [15], radial basis function [16], and support vector regression [17], have been used in both academia and industry.

Traditional approaches to reliability-based design require a nested double-loop procedure including the optimization outer loop and the reliability analysis inner loop. The reliability analysis inner loop calculates the reliability for each of probabilistic constraints while the optimization outer loop searches for the optimal design and calls the reliability analysis inner loop repeatedly. The process is computationally intensive under the nested framework. In order to overcome the computational inefficiency of a nested double-loop procedure, single loop methods [18–21], where the reliability analysis inner loop is eliminated by introducing additional variables and constraints, and sequential optimization and reliability assessment (SORA) methods [22–24], where the nested framework is decoupled into serial cycles, are developed.

Under the robust design paradigm, mean and variance of the objective function need to be estimated. Commonly used methods can be divided into three categories: Taylor series expansion method [4,25], point estimating methods [26,27], and simulation methods [28,29]. The first order Taylor series expansion is very simplified and commonly used in the robust design. However, its accuracy is not so good when the limit state function is highly nonlinear. Moreover, since it is a gradient-based method, accurate gradient calculation is required.

Some attempts have been made to combine reliability-based design and robust design [6,7,30,31]. In their work, the mean and variance are minimized at the same time and the weighted-sum approach is usually used to convert the multiobjective optimization to a single objective optimization. In this paper, a unified framework for integrating reliability-based design and robust design is proposed based on the work in Ref. [6]. Two major developments are involved. The fundamental development is converting the probabilistic objective function to the deterministic one according to the probabilistic characteristic of the probabilistic objective function by Taylor series expansion or inverse reliability strategy [32–34]. In this development, the unified framework eliminates the need to deal with explicitly a multiobjective optimization problem. In the traditional combination of reliability-based design and robust design, multiobjective optimization problems have to be solved and the weighted-sum method is usually implemented to deal with the multiobjective optimization prob-

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lem. Meanwhile, the probabilistic constraints are converted to deterministic ones by inverse reliability strategy [6]. The other development is that an improved sequential optimization and reliability assessment (ISORA) method is proposed to solve the unified framework.

The organization of the paper is as follows. In Sec. 2, the typical reliability-based design model and robust design model are introduced. A unified framework for integrating reliability-based design and robust design is proposed in Sec. 3. Three examples follow to demonstrate the effectiveness of the unified framework in Sec. 4. Section 5 presents some conclusions.

2 Typical Reliability-Based Design and Robust Design

A typical design optimization model under uncertainty is given by [22]

$$\min_{\mathbf{d},\boldsymbol{\mu}_X} f(\mathbf{d},\mathbf{X},\mathbf{P})$$

$$\text{subject to} \quad \Pr\{\, g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0)\} \geq [R_i] \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \boldsymbol{\mu}_X^L \leq \boldsymbol{\mu}_X \leq \boldsymbol{\mu}_X^U \tag{1}$$

In the above model, $f(\bullet)$ is an objective function. In engineering design, an objective function is usually physics-based such as the volume, the weight, and the manufacturing cost of a product. $\mathbf{d}$ is the vector of deterministic design variables. $\mathbf{X}$ is the vector of random design variables. $\mathbf{P}$ is the vector of random parameters. The difference between $\mathbf{X}$ and $\mathbf{P}$ is that the former is changeable and controllable in the design process while the latter is not. $\boldsymbol{\mu}_X$ is the vector of mean values of random design variables. The lower and upper bounds of $\mathbf{d}$ are defined by $\mathbf{d}^L$ and $\mathbf{d}^U$, respectively. Likewise, the lower and upper bounds of $\boldsymbol{\mu}_X$ are defined by $\boldsymbol{\mu}_X^L$ and $\boldsymbol{\mu}_X^U$, respectively. $\mathbf{d}$ and $\boldsymbol{\mu}_X$ are to be determined in the design optimization. $g_i(\mathbf{d},\mathbf{X},\mathbf{P})$ is a limit state function (also called performance function) and n_g is the number of limit state functions. $\Pr\{\bullet\}$ denotes a probability and $\Pr\{\, g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0)\} \geq [R_i]$ means that the probability of constraint satisfaction $g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0$ should not be less than the desired reliability $[R_i]$. Such a probability is obviously the reliability associated with limit state function $g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0$.

It is apparent that the objective function in Eq. (1) is a function of deterministic design variables $\mathbf{d}$, random design variables $\mathbf{X}$ and random parameters $\mathbf{P}$. In the traditional reliability-based design, the mean value of the objective function in Eq. (1) is generally treated as the new objective function. Then a typical reliability-based design model is provided by

$$\min_{\mathbf{d},\boldsymbol{\mu}_X} f(\mathbf{d},\boldsymbol{\mu}_X)$$

$$\text{subject to} \quad \Pr\{\, g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0)\} \geq [R_i] \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \boldsymbol{\mu}_X^L \leq \boldsymbol{\mu}_X \leq \boldsymbol{\mu}_X^U \tag{2}$$

This is a design problem with deterministic objective function and probabilistic constraints. In this model, the main computational expense is to calculate the probability $\Pr\{\, g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0)\}$. Theoretically, the probability can be calculated by integrating the joint probability density $f_{\mathbf{x},\mathbf{P}}(\mathbf{x},\mathbf{p})$ function of $(\mathbf{X},\mathbf{P})$ over the safe region defined by $g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0$. The integral is given by [35]

$$P\{g(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0\} = \int_{g(\mathbf{d},\mathbf{X},\mathbf{P})\geq 0} f_{\mathbf{x},\mathbf{P}}(\mathbf{x},\mathbf{p})\,d\mathbf{x}\,d\mathbf{p} \tag{3}$$

It is usually difficult or even impossible to obtain the analytical solution to the probability integral if the limit state function is highly nonlinear and multidimensional. In order to deal with the difficulty in computing the probability, many approximate methods are developed. FORM and SORM are two main approximate methods. An approximate solution to Eq. (3) is achieved by simplifying the performance function at the MPP in FORM or SORM with the first order or second order Taylor series expansion, respectively. The MPP is the point with the shortest distance from the origin to the constraint boundary $g_i(\mathbf{d},\mathbf{U}_X,\mathbf{U}_P)=0$ in the standard normal space. Because of the shortest distance to the origin, the MPP has the highest probability density on the constraint boundary $g_i(\mathbf{d},\mathbf{U}_X,\mathbf{U}_P)=0$. Monte Carlo simulation [36], as a direct simulation method, can be used. However, simulation is very time-consuming for high reliability. Other methods such as moment-based method [11–13] and metamodeling method [14–17] can be used as alternatives to estimate the probability in Eq. (3). For a good balance between accuracy and efficiency, FORM is usually used.

Many methods are developed to deal with design optimization under uncertainty. Reliability-based design and robust design are two major paradigms for design under uncertainty among them [6]. Different from reliability-based design, the task of robust design is to minimize the mean and the variation in objective function simultaneously under the condition that constraints are satisfied. The typical robust design model is given [4]

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mu_f(\mathbf{d},\mathbf{X},\mathbf{P}),\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P}))$$

$$\text{subject to} \quad g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \leq 0 \quad i=1,\ldots,n$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U; \quad \boldsymbol{\mu}_\mathbf{X}^L \leq \boldsymbol{\mu}_\mathbf{X} \leq \boldsymbol{\mu}_\mathbf{X}^U \tag{4}$$

where $\mu_f(\mathbf{d},\mathbf{X},\mathbf{P})$ and $\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P})$ are the mean value and standard deviation of the objective function in Eq. (1), respectively. n is the number of deterministic constraints. This is a multiobjective optimization problem. A common way to deal with multiobjective design optimization is to use weighting factors. When probabilistic constraints are considered, Eq. (4) can be rewritten into [5]

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} w_1\frac{\mu_f(\mathbf{d},\mathbf{X},\mathbf{P})}{\mu_f^*} + w_2\frac{\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P})}{\sigma_f^*}$$

$$\text{subject to} \quad \mu_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})\text{-}k\sigma_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P}) \leq 0, \quad i=1,\ldots,n_g$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U; \quad \boldsymbol{\mu}_\mathbf{X}^L \leq \boldsymbol{\mu}_\mathbf{X} \leq \boldsymbol{\mu}_\mathbf{X}^U \tag{5}$$

w_1 and w_2 are the weighting factors, which are generally determined by the designer during the design process. A restriction is posed that different designer will choose different values of w_1 and w_2. μ_f^* and σ_f^* are the most achievable optimal solutions for the mean value, $\mu_f(\mathbf{d},\mathbf{X},\mathbf{P})$, and standard deviation, $\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P})$, of the objective function, respectively. $\mu_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$ and $\sigma_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$ are the mean value and standard deviation of the limit state function $g_i(\mathbf{d},\mathbf{X},\mathbf{P})$. k is a constant to express the ratio between the mean, $\mu_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$, and the standard deviation, $\sigma_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$. Therefore, k can indicate the probability of constraint satisfaction only if the distribution of performance function $g_i(\mathbf{d},\mathbf{X},\mathbf{P})$ is known. For example, $k=3$ indicates that the probability of constraint satisfaction is 99.87% under the assumption that $g_i(\mathbf{d},\mathbf{X},\mathbf{P})$ is normally distributed.

Du et al. [6] proposed a novel integrated framework for design optimization under uncertainty by taking both the design objective robustness and the probabilistic constraint into account. The integrated framework is given by

$$\min f(\mu_f,\Delta v_{f\alpha_1}{}^{\alpha_2})$$

$$\text{subject to} \quad g_i^{\alpha}(\mathbf{d},\mathbf{X},\mathbf{P}) \leq 0, \quad i=1,2,\ldots,n \tag{6}$$

where $\Delta v_{f\alpha_1}{}^{\alpha_2}$ is the percentile performance difference and is given by

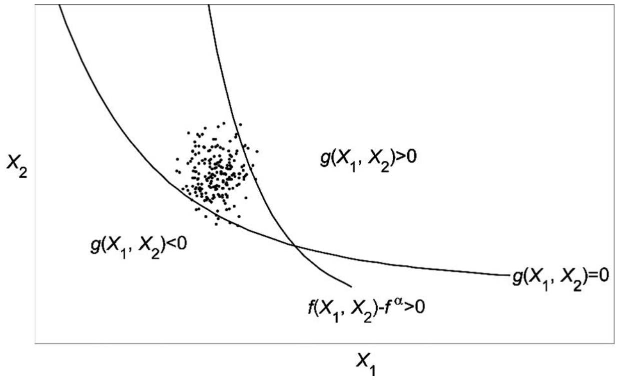

Fig. 1 Optimization under probabilistic objective function and constraints

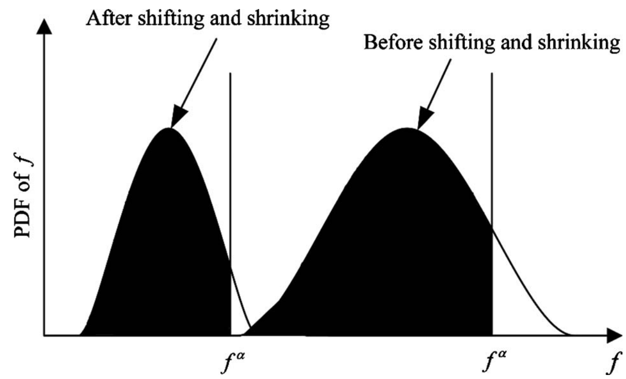

Fig. 2 Optimization of the unified framework

$$\Delta v_{f\alpha_1}{}^{\alpha_2} = \Delta v_f{}^{\alpha_2} - \Delta v_f{}^{\alpha_1} \tag{7}$$

where α_1 and α_2 are the reliability levels or the cumulative distribution functions of f, given by

$$\Pr\{f(\mathbf{d},\mathbf{X},\mathbf{P}) \le v_f^{\alpha_i}\} = \alpha_i \tag{8}$$

In the integrated framework for reliability-based design and robust design, the inverse reliability strategy is used to reformulate the optimization problem under uncertainty. After the reformulation, both the objective function and the probabilistic constraints are deterministic.

In this paper, an improvement will be made for the integrated framework. In the traditional robust design, the weighted-sum method is usually used to deal with the multiobjective optimization. Hence, different designs are made according to the preference of different designers. On one hand, therefore, the improvement is motivated by the need to build a unified framework to overcome the difficulty in choosing the weighting factors to convert the multiobjective optimization problem to a single objective optimization problem. On the other hand, this improvement is motivated by the need to build a relationship between objective function satisfaction and robust design based on the probabilistic objective function. In the reported integrated framework for reliability-based design and robust design, the probability of the probabilistic objective function satisfaction cannot be calculated. In the traditional reliability-based design, the mean of the probabilistic objective function $f(\mathbf{d},\mathbf{X},\mathbf{P})$ in Eq. (1) indicates 50% objective function satisfaction under the assumption that $f(\mathbf{d},\mathbf{X},\mathbf{P})$ is normally distributed. Robustness, however, cannot be expressed in the traditional reliability-based design.

3 Unified Framework for Integrated Reliability-Based and Robust Design

In this section, we will develop a unified framework for integrating reliability-based and robust design. For design optimization with both probabilistic objective function and probabilistic constraints, as shown in Eq. (1), the task is to minimize the probabilistic objective function most probably under the condition that constraints are satisfied. In other words, the required probability of both probabilistic objective function and constraints is satisfied by ensuring the number of probable design points in the region between constraint boundary and objective function boundary. The optimization under both probabilistic objective function and constraints is given in Fig. 1.

In Fig. 1, $g(X_1,X_2)=0$ is the constraint boundary. $g(X_1,X_2)>0$ indicates the safe (feasible) region while $g(X_1,X_2)<0$ indicates the failure (infeasible) region. The probabilistic constraint is satisfied by ensuring the probability that the design points appear in the safe region. $f(X_1,X_2)-f^{\alpha}=0$ is the objective function boundary. f^{α} denotes the α-percentile performance of the objective function f. The objective is to minimize f^{α} under the condition of constraint satisfaction. Then the general unified framework for integrating reliability-based and robust design is provided by

$$\min_{\mathbf{d},\boldsymbol{\mu}_X} f^{\alpha} | \Pr\{f(\mathbf{d},\mathbf{X},\mathbf{P},\mathbf{C}) - f^{\alpha} \le 0\} = \alpha$$

$$\text{subject to}\quad \Pr\{g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \ge 0\} \ge [R_i], \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U; \quad \boldsymbol{\mu}_X^L \le \boldsymbol{\mu}_X \le \boldsymbol{\mu}_X^U \tag{9}$$

In the above model, α denotes a probability. $\mathbf{C}$ is the vector of random parameters, which only appear in the objective function. For example, $\mathbf{C}$ is the cost coefficient in the cost-type objective function, which is stochastic over the market and is not controllable by the designer during the design process. All the random variables are assumed to be independent in this paper. The unified framework is achieved by shifting or shrinking the distribution of the probabilistic objective function or both, as shown in Fig. 2.

As shown in Fig. 2, the standard deviation of the objective function decreases by shrinking the distribution. Overall, minimizing the α-percentile performance f^{α} of the probabilistic objective function f is realized by both shifting and shrinking the distribution. Hence, the main task is to obtain the expression of the α-percentile performance f^{α} in terms of the design variables $\mathbf{d}$ and $\boldsymbol{\mu}_X$. Herein, two methods are given depending on whether the distribution of probabilistic objective function is normally distributed or not.

3.1 Unified Framework With Normal Distribution. When the objective function is normally distributed, the percentile performance f^{α} is easily expressed as a function of the mean value μ_f and standard deviation σ_f.

Then Eq. (9) can be rewritten as

$$\min_{\mathbf{d},\boldsymbol{\mu}_X} \mu_f + k\sigma_f$$

$$\text{subject to}\quad \Pr\{g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \ge 0\} \ge [R_i], \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U; \quad \boldsymbol{\mu}_X^L \le \boldsymbol{\mu}_X \le \boldsymbol{\mu}_X^U \tag{10}$$

In Eq. (10), k is a constant to predict the probability α. For example, $k=3$ indicates that the probability α is equal to 0.9987.

Many approaches could be used to estimate μ_f and σ_f. However, a simple way is the Taylor series expansion of the probabilistic function at the mean value of $(\mathbf{X},\mathbf{P},\mathbf{C})$. The approximation is provided by

$$\mu_f = f(\mathbf{d},\boldsymbol{\mu}_X,\boldsymbol{\mu}_P,\boldsymbol{\mu}_C) \tag{11}$$

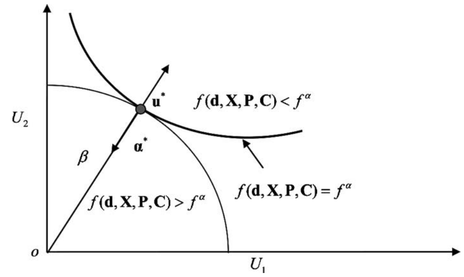

Fig. 3 Inverse MPP search

$$\sigma_f \approx \sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2_{\boldsymbol{\mu}_X,\boldsymbol{\mu}_P,\boldsymbol{\mu}_C}\sigma_{X_i} + \sum_{k=1}^{m}\left(\frac{\partial f}{\partial P_k}\right)^2_{\boldsymbol{\mu}_X,\boldsymbol{\mu}_P,\boldsymbol{\mu}_C}\sigma_{P_k} + \sum_{j=1}^{l}\left(\frac{\partial f}{\partial C_j}\right)^2_{\boldsymbol{\mu}_X,\boldsymbol{\mu}_P,\boldsymbol{\mu}_C}\sigma_{C_j} \tag{12}$$

Since $\mathbf{d}$ and $\boldsymbol{\mu}_X$ in Eqs. (11) and (12) are to be determined, the objective function in Eq. (10) is a deterministic function. Several methods have been developed to solve this design optimization problem such as nested double-loop procedure method, single loop method, and SORA method. Considering the balance between computational efficiency and accuracy, the SORA method will be used to solve the optimization model in Eq. (10) in this paper.

3.2 Unified Framework With Other or Unknown Distribution Types. The approximate reliability index could be calculated in FORM and SORM. Neither FORM nor SORM, however, is suitable for estimating the distribution and probability density function of a probabilistic function. It is also impossible to estimate the probability distribution type by simulation methods such as direct Monte Carlo simulation, quasi-Monte Carlo simulation in which the reliability index, or reliability is calculated by random sampling. In general, it is difficult to estimate the probability distribution type of a probabilistic function even though the expression of the probabilistic function is not so complicated. Thus, a stringent restriction is imposed on the method proposed in Sec. 3.1. An alternative method needs to be developed to deal with the general case of the unified framework for integrating reliability-based and robust design model in Eq. (9). This method needs to be able to deal with other or unknown distributions of probabilistic objective function. An inverse reliability strategy is implemented in this method to convert the probabilistic objective function into the deterministic function. The emphasis of the inverse reliability method is to find the α-percentile performance f^α with the known associated probability α. The formulation is given by [34]

$$\Pr\{f(\mathbf{d},\mathbf{X},\mathbf{P},\mathbf{C}) - f^\alpha \le 0\} = \alpha \tag{13}$$

As illustrated in Fig. 3, the MPP u^* is a point where the performance function $f(\mathbf{d},\mathbf{X},\mathbf{P},\mathbf{C})-f^\alpha=0$ is tangent to the circle with the radius of β.

Moreover, the MPP u^* is also a point with a maximum value of $f(\mathbf{d},\mathbf{X},\mathbf{P},\mathbf{C})$ on the circle with radius β. Then the mathematical model of the inverse MPP search can be stated as: find the maximum value of $f(\mathbf{d},\mathbf{X},\mathbf{P},\mathbf{C})$ under the condition that the MPP remains on the surface of the circle. We first transform random variables $\mathbf{Z}=(\mathbf{X},\mathbf{P},\mathbf{C})$ into the standard normal random variables $\mathbf{U}=(\mathbf{U}_X,\mathbf{U}_P,\mathbf{U}_C)$. Then the mathematical model can be expressed as follows:

$$\max_{\mathbf{u}} f(\mathbf{u})$$

$$\text{subject to} \quad \|\mathbf{u}\| = \beta \tag{14}$$

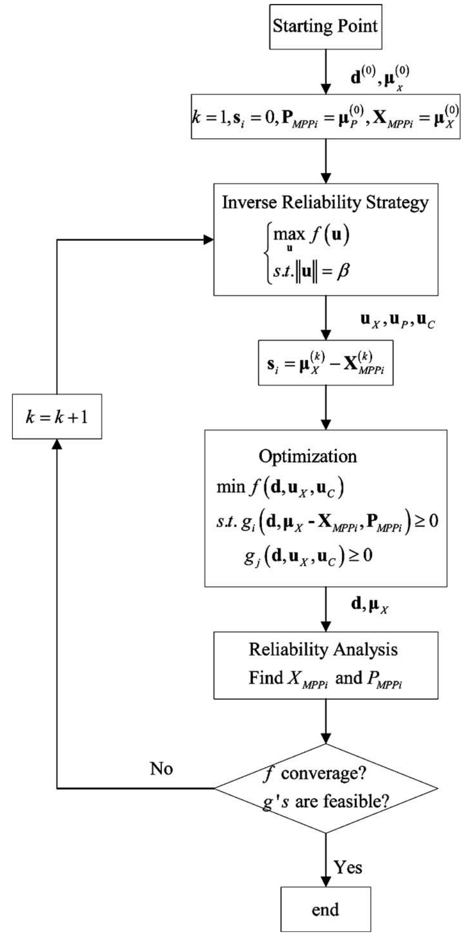

Fig. 4 Flowchart of ISORA

It is obvious that solving this optimization problem requires an iterative procedure. $f(u^*)$ is the performance percentile f^α when the reliability index β is satisfied. In other words, the probability that the objective is less than or equal to f^α is equal to $\Phi(\beta)$. Φ is the standard normal cumulative distribution function.

Then Eq. (9) is rewritten as

$$\min_{\mathbf{d},\boldsymbol{\mu}_X}\{f(u^*):\max f(u)|\|u\| = \beta\}$$

$$\text{subject to} \quad \Pr\{g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \ge 0\} \ge [R_i], \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U; \quad \boldsymbol{\mu}_X^L \le \boldsymbol{\mu}_X \le \boldsymbol{\mu}_X^U \tag{15}$$

Since the transformation from the general design space to the standard normal space and Taylor linear series expansion is employed in the inverse reliability procedure, the objective function is normally distributed. Then the percentile performance of the objective function f^α or $f(u^*)$ can be expressed by the combination of μ_f and σ_f such as $\mu_f+\sigma_f$. After inverse reliability analysis is used on the probabilistic constraints, Eq. (15) can be represented in the unified framework by

$$\min_{\mathbf{d},\boldsymbol{\mu}_X}\{\mu_f + k\sigma_f:\max f(u)|\|u\| = \beta\}$$

$$\text{subject to} \quad g_i^{\alpha_i}(\mathbf{d},\mathbf{X},\mathbf{P}) \le 0, \quad i=1,2,\ldots,n_g$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U; \quad \boldsymbol{\mu}_X^L \le \boldsymbol{\mu}_X \le \boldsymbol{\mu}_X^U \tag{16}$$

Since SORA has a high computational efficiency and accuracy, the ISORA method is proposed to deal with the optimization problem in Eq. (16). The flowchart of the ISORA is given in Fig. 4. As

Table 1 Distribution of random variables

Variables	Mean	Standard deviation	Distribution type
X_1	μ_{X_1}	1	Normal
X_2	μ_{X_2}	1	Normal

illustrated in Fig. 4, the ISORA is a sequential optimization process consisting of three optimization procedures including inverse reliability strategy, deterministic optimization, and reliability assessment. The α-percentile performance f^{α}, which is used as the objective function to be minimized, and the corresponding MPP u^*, which will appear in the deterministic constraints instead of the mean in the traditional SORA, can be obtained from the inverse reliability strategy.

4 Examples

In this section, we will use three examples to demonstrate the proposed unified framework for integrating reliability-based design and robust design. Through the examples, we will discuss the effectiveness of the unified framework.

4.1 Mathematic Example. A unified framework for integrating reliability-based and robust design is given by

$$\min f^* = \mu_f + k\sigma_f$$

$$\text{subject to} \quad \Pr\{g(X) = d_2 X_1^2 X_2/20 - d_1 \geq 0\} \geq [R]$$

$$-10 \leq \mu_{X_1}, \quad \mu_{X_2} \leq 10; \quad 0 \leq d_1, \quad d_2 \leq 2 \tag{17}$$

where

$$f = X_1/d_1 + d_2 X_2 \tag{18}$$

Information on the random design variables $\mathbf{X}=[X_1, X_2]$ is given in Table 1. The required reliability is $[R]=0.9987$. μ_f and σ_f are the mean value and the standard deviation of the function f, respectively. k is a constant and used to express the probability satisfaction of the probabilistic objective function.

Since X_1 and X_2 are normally distributed and d_1 and d_2 are deterministic design variables, f follows the normal distribution. Then we can use the first method proposed in Sec. 3.1 to formulate the unified framework. The results of the mathematical example for different k are given in Table 2.

The last row in Table 2 indicates the probability satisfaction of f for different k values. For example, $k=3$ means that the probability that f is less than or equal to $\mu_f+3\sigma_f$ is 99.87% and $f^* = 11.408$ and $k=0$ corresponds to the probability that f is less than or equal to μ_f. When $k=0$, the objective function in Eq. (17) becomes $f^*=\mu_f$, which is commonly used in the traditional reliability-based design. The objective value f^* for the case $k=3$ is greater than that for the case $k=0$, as shown in Table 2. Therefore, the practical objective function value is usually bigger than the optimal value provided by the traditional reliability-based design.

Using the proposed framework, we can control the satisfaction probability of the probabilistic objective function to ensure that the optimal design is more suitable to the engineering practices. The accuracy and efficiency is similar to that of traditional SORA and the number of function calls during the design optimization is also given in Table 2.

Table 2 Results of the mathematical example

k	0	1.5	2	3
d_1	2	1.9607	1.6611	1.3198
d_2	0.13992	0.12851	0.13159	0.1342
μ_{X_1}	8.4774	8.6646	8.1419	7.5278
μ_{X_2}	10	10	10	10
f^*	5.6386	8.838	9.8028	11.408
Function calls	781	1187	887	894
Probability satisfaction (%)	50	93.32	97.72	99.87

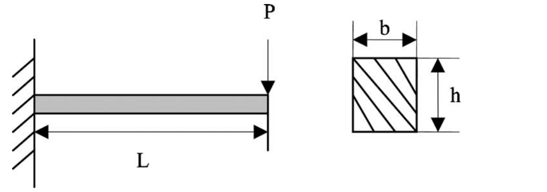

Fig. 5 Cantilever beam

4.2 Cantilever Beam. The cantilever beam shown in Fig. 5 is subjected to the external force P. The performance function is defined by the difference between the yield strength S_y and the maximum tensile stress $S_{\max}$, namely,

$$g(\mathbf{d},\mathbf{X}) = S_y - S_{\max} \tag{19}$$

The maximum tensile stress $S_{\max}$ is given by

$$S_{\max} = \frac{6PL}{bh^2} \tag{20}$$

The cross-sectional area is considered as the objective function, namely,

$$f = bh \tag{21}$$

Then the unified framework for integrating reliability-based and robust design is provided by

$$\min f^* = \mu_f + k\sigma_f$$

$$\text{subject to} \quad \Pr\left\{S_y - \frac{6PL}{bh^2} \geq 0\right\} \geq 0.9995$$

$$\frac{h}{b} - 2 \leq 0$$

$$10 \leq \mu_b \leq 45, \quad 20 \leq \mu_h \leq 80 \tag{22}$$

The distributions of random variables are given in Table 3.

Even though the expression of the probabilistic objective function $f=bh$ is very simple and b and h are normally distributed, it is difficult to estimate the distribution type of f. Then the second method proposed in Sec. 3.2 is employed to solve this problem. The results for different k values are given in Table 4.

Since the standard deviations of the random design variables in the objective function f are very small, e.g., 0.05 mm, and there are no random parameters, the differences among the objective values f^* for different k values are not very big. Hence, we can conclude that the error of the objective function is small if the mean of the probabilistic objective function is used as the deterministic objective function when the standard deviation of design variables in the probabilistic objective function is not big. Actually, when $k=0$, the unified framework reduces to the traditional

Table 3 Distributions of random variables

Variables	Variables	Mean	STD	Distribution
X	b (mm)	μ_b	0.05	Normal
	h (mm)	μ_h	0.05	Normal
P	S_y (MPa)	200	20	Normal
	P (KN)	20	2	Extreme value
	L (mm)	200	1	Normal

Table 4 Results of the cantilever beam

k	0	1	2	3
μ_b	37.236	37.213	37.174	37.168
μ_h	74.471	74.493	74.533	74.538
f^*	2773	2776.3	2782.2	2783.0
Function calls	2215	5441	5417	5569
Probability satisfaction (%)	50	84.13	99.72	99.87

SORA. But when $k=1,2,3$, the unified framework guarantees ISORA. The number of function calls is also given in Table 4. Then we can see that the efficiency of ISORA is lower than that of SORA. The reason is that ISORA includes three loops in a cycle while two loops are included in SORA.

4.3 Single Helical Gear Reducer Design. A single helical gear reducer, shown in Fig. 6, is used in some engines, which allows the engine to rotate at its most efficient speed. This has been used as a testing problem for nonlinear optimization method in literature [37,38]. Since there are some random variables in the speed reducer design, we will consider the speed reducer design as a probabilistic design. In the probabilistic design, there are two deterministic design variables: the tooth module m_n and the number of pinion teeth z_1. Face width b and helix angle β are considered to be random design variables. There are five random parameters P_1-P_5, including the material properties, the rotation speed, the engine power, the allowable fatigue stress and the allowable bending stress, and a random parameter C, which is the cost coefficient and only appears in the objective function. Their distributions are given in Table 5.

The unified framework for integrating reliability-based and robust design is given by

$$\min f^* = \mu_f + k\sigma_f$$

$$\text{subject to} \quad \Pr\{g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0\} \geq [R_i]$$

$$0.3 \leq b/d_1 \leq 0.7$$

$$b \sin \beta/(\pi m_n) \geq 1$$

$$17 \leq z_1 \leq 40; \quad 2 \leq m_n \leq 20$$

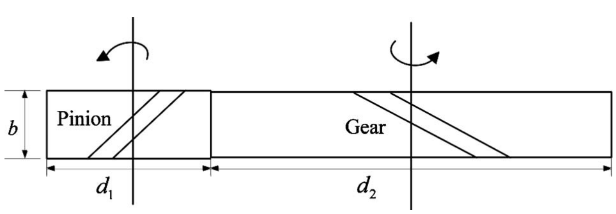

Fig. 6 A single helical gear reducer

Table 5 Distributions of the single helical gear reducer

Variables	Variables	Mean	STD	Distribution
d	z_1	—	—	—
	m_n (mm)	—	—	—
X	b (mm)	μ_b	4	Normal
	β (deg)	μ_β	0.5	Normal
P	P (kW)	2000	200	Normal
	n (rpm)	1000	100	Normal
	Z_E ($\sqrt{\text{MPa}}$)	189.8	18.98	Normal
	$\sigma_{H\min}$ (MPa)	1400	140	Normal
	$\sigma_{F\min}$ (MPa)	480	48	Normal
	C(10^{-4} \$/mm^3)	1	1.3	Normal

Table 6 Results of the single helical gear reducer

k	0	1	2	3
z_1	22	26	26	26
m_n	8.58	5.14	5.20	5.23
b	153.73	108	108	108
β	10.22	10.89	10.86	10.85
f^*	4730.3	5616.4	11,418.0	12,234.0
Function calls	7198	21,716	31,420	31,852
Probability satisfaction (%)	50	84.13	97.72	99.87

$$8 \text{ deg} \leq \mu_\beta \leq 16 \text{ deg}; \quad 100 \leq \mu_b \leq 240 \tag{23}$$

where

$$f = C\frac{\pi m_n^2}{4}[z_1^2 + (uz_1)^2]b \tag{24}$$

f is the cost-type objective function, which is proportional to the volume of the reducer. Two performance functions are defined by the difference between the allowable fatigue stress and the gear contact stress and the difference between the allowable bending stress and the bending stress. They are given by

$$g_1(\mathbf{d},\mathbf{X},\mathbf{P}) = \sigma_{H\min} Z_N - Z_E Z_H \sqrt{\frac{2000 \times 9.55P}{d_1^2 bn \cos(\beta)} \frac{u+1}{u} K_A} \tag{25}$$

$$g_2(\mathbf{d},\mathbf{X},\mathbf{P}) = \sigma_{F\min} Y_{ST} - \frac{2000 \times 9.55P}{d_1 b m_n n \cos(\beta)} Y_F \tag{26}$$

$Z_H=2.25$, $K_A=1.45$, $Z_N=0.87$, $Y_F=1.98$, $Y_{ST}=2.32$, and $u=4$ are the coefficients and the associated required reliability is $[R_i]=0.9987$.

Since it is impossible to get the distribution of the objective function f, the second method proposed in Sec. 3.2 is implemented to deal with the unified framework in Eq. (23). The results are given in Table 6.

Since the standard deviation of random design variables is relatively small, it is reasonable to treat the mean value of the probabilistic objective function as the objective function when no random parameters appear in the probabilistic objective function, as shown in example 2. In the real word, the probabilistic objective function usually has physical meaning, e.g., cost. Then some random parameters are introduced into the probabilistic objective function. In this example, cost coefficient is introduced into the probabilistic objective function as a random parameter. As shown in Table 6, the difference of the objective values between the case $k=0$ and the case $k=3$ is very big. $k=0$ indicates the mean and $k=3$ indicates that the probability of the probabilistic objective being less than or equal to $\mu_f+3\sigma_f$ is 99.87%. We also give the number of function calls during the optimization in Table 6. As in example 2, when $k=0$, actually the SORA can be used to deal with the unified framework. When $k=1,2,3$, the ISORA is implemented to handle the unified framework. The results also indicate that the efficiency of ISORA is much lower than SORA.

5 Concluding Remarks

In some engineering design, both the objective function and some constraints are stochastic. In the traditional reliability-based design, the probabilistic objective function is converted to a deterministic one by taking the expectation of the probabilistic objective function. A big error will be generated if the standard deviation of the probabilistic objective function is big. Furthermore, robustness is not accounted for. Then the integrated framework for reliability-based design and robust design is proposed. In the framework, the combination of reliability-based design and robust design is considered. Multiobjective optimization problems, however, should be solved. Currently, the weighted-sum method is commonly used to deal with the multiobjective optimization. Hence, different designs may be obtained for the same product due to different designers' preference. A unified framework for integrating reliability-based design and robust design is proposed in this paper. With the Taylor series expansion or inverse reliability strategy, the probabilistic objective function is converted to the deterministic one based on the probabilistic characteristic of the probabilistic objective function. The unified framework eliminates dealing with a multiobjective optimization problem to integrate reliability-based design and robust design. Therefore, the unified framework could have a wider practical application.

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Nomenclature

$\mathbf{d}$ = vector of deterministic design variables
$\mathbf{X}$ = vector of random design variables
$\mathbf{P}$ = vector of random parameters
$f(\bullet)$ = objective function
$\Pr\{\bullet\}$ = probability
$g_i(\bullet)$ = limit state function
n_g = number of limit state functions
$[R_i]$ = required reliability of the ith limit state function
$\mu_{\mathbf{X}}$ = mean value of $\mathbf{X}$
$\mu_{\mathbf{X}}^L$ = lower bound of $\mu_{\mathbf{X}}$
$\mu_{\mathbf{X}}^U$ = upper bound of $\mu_{\mathbf{X}}$
$\mathbf{d}^L$ = lower bound of $\mathbf{d}$
$\mathbf{d}^U$ = upper bound of $\mathbf{d}$
$f_{\mathbf{X},\mathbf{P}}(\mathbf{X},\mathbf{P})$ = joint probability density function of $(\mathbf{X},\mathbf{P})$
$\mu_f(\mathbf{d},\mathbf{X},\mathbf{P})$ = mean value of objective function f
$\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P})$ = standard deviation of objective function f
n = number of deterministic constraints
w_1, w_2 = weighting factors
μ_f^* = most achievable optimal solution for $\mu_f(\mathbf{d},\mathbf{X},\mathbf{P})$
σ_f^* = most achievable optimal solution for $\sigma_f(\mathbf{d},\mathbf{X},\mathbf{P})$
$\mu_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$ = mean value of probabilistic constraint g_i
$\sigma_{g_i}(\mathbf{d},\mathbf{X},\mathbf{P})$ = standard deviation of probabilistic constraint g_i
k = constant
α = reliability level
f^α = α-percentile performance of objective function
$\Delta v_{f\alpha_1}{}^{\alpha_2}$ = percentile performance difference
β = reliability index
$\mathbf{C}$ = vector of random parameters in the objective function

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