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RESEARCH PAPER

Multidisciplinary design optimization with discrete and continuous variables of various uncertainties

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Abstract As a powerful design tool, Reliability Based Multidisciplinary Design Optimization (RBMDO) has received increasing attention to satisfy the requirement for high reliability and safety in complex and coupled systems. In many practical engineering design problems, design variables may consist of both discrete and continuous variables. Moreover, both aleatory and epistemic uncertainties may exist. This paper proposes the formula of RFCDV (Random/Fuzzy Continuous/Discrete Variables) Multidisciplinary Design Optimization (RFCDV-MDO), uncertainty analysis for RFCDV-MDO, and a method of RFCDV-MDO within the framework of Sequential Optimization and Reliability Assessment (RFCDV-MDO-SORA) to solve RFCDV-MDO problems. A mathematical problem and an engineering design problem are used to demonstrate the efficiency of the proposed method.

Keywords Multidisciplinary design optimization · Aleatory uncertainty · Epistemic uncertainty · Continuous/discrete variables · Random/Fuzzy Continuous/Discrete Variables Multidisciplinary Design Optimization (RFCDV-MDO) · Sequential Optimization and Reliability Assessment (SORA)

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1 Introduction

In the last two decades, the consideration of uncertainty has been a focus of engineering design for achieving reliable design of complex and coupled systems. Reliability Based Multidisciplinary Design Optimization (RBMDO) has gained increasing attention because of the desire for high reliability and safety in complex and coupled systems with multiple disciplines (Sues et al. 1995; Sues and Cesare 2000; Koch et al. 2000; Padmanabhan and Batill 2002a, b; Du and Chen 2000, 2005; Du et al. 2008). To release the computational burden in reliability analysis involved in MDO under uncertainty, response surface models which are created at the system level are employed to replace the computationally expensive simulation models (Sues et al. 1995). In Sues and Cesare (2000), a framework for RBMDO is proposed where reliability analysis is decoupled from the optimization loop. Reliabilities are initially computed before the first execution of the optimization loop, and then updated iteratively after the optimization loop during which approximate forms of reliability constraints are used. In Koch et al. (2000), a multi-stage parallel implementation of probabilistic design optimization is utilized with the aim of integrating the existing reliability analysis method into MDO frameworks. To search the Most Probable Point (MPP), the concurrent subsystem optimization was proposed in Padmanabhan and Batill (2002a, b), and Du and Chen (2000) and the collaborative reliability analysis method in Du and Chen (2005). In Du et al. (2008), a Sequential Optimization and Reliability Assessment (SORA) method for RBMDO was proposed. In each optimization loop, the deterministic formulation of the MDO is constructed using the MPP from the previous iteration. Following each optimization loop, reliability analysis is carried out at the optimal solution of the deterministic MDO to check up the feasibilities of the probability constraints. This method was demonstrated as its good capability of dealing with RBMDO problems.

Although a number of works have been done in MDO especially in RBMDO, there are still some issues requiring further exploration. Most urgent two are that both aleatory uncertainty and epistemic uncertainty are associated with design variables and parameters, and also continuous and discrete variables/parameters simultaneously exist in design variables/parameters.

In practical engineering design, aleatory uncertainty and epistemic uncertainty are simultaneously associated with design variables and parameters. Aleatory uncertainty (stochastic uncertainty, irreducible uncertainty, inherent uncertainty, variability) can be modeled with probability theory, and variables with aleatory uncertainty can be treated as random variables. Epistemic uncertainty (reducible uncertainty, subjective uncertainty) caused by lack of knowledge can be modeled with possibility theory (possibility approach can deal with epistemic uncertainty by defining a fuzzy variable corresponding to the limited data). Therefore, variables with epistemic uncertainty can be treated as fuzzy variables (Agarwal et al. 2004; Oberkampf et al. 2000; Du and Choi 2008; Youn et al. 2005; Du et al. 2006). The results of RBMDO, in which variables with epistemic uncertainty are treated as random variables with their probability distributions fitted using available limited data from experiments, may be risky and unreliable because improper modeling of uncertainty could cause greater degree of statistical uncertainty than those of physical uncertainty (Du et al. 2006). Also different kinds of variables, continuous and discrete, exist in practical design including MDO problems. Up to now, almost all existing works only focus on MDO problems with continuous design variables.

MDO problem with random/fuzzy continuous/discrete variables is dealt with here. This paper proposes formula of RFCDV (Random/Fuzzy Continuous/Discrete Variables) MDO (RFCDV-MDO), uncertainty analysis for RFCDV-MDO, and a method of RFCDV-MDO within the framework of Sequential Optimization and Reliability Assessment (RFCDV-MDO-SORA) to deal with RFCDV-MDO problems.

This paper is organized as follows. In Section 2, the mathematical formulation of RFCDV-MDO is provided. In Section 3, the fundamental analysis is proposed including the uncertainty analysis for RFCDV-MDO. In Section 4, the proposed method of RFCDV-MDO-SORA is explained in detail, including the strategy, procedure and formulas. In Section 5, a mathematical example and an engineering practical problem are portrayed to illustrate the efficiency of the proposed method. Finally, the conclusions are given in Section 6.

2 RFCDV-MDO

The mathematical formulation of RFCDV-MDO is given as:

$$
\begin{aligned}
&\min_{DV} f\left(\mathbf{d}_{s,c},\mathbf{d}_{s,d},\mathbf{d}_c,\mathbf{d}_d,\mathbf{X}^M_{s,c},\mathbf{X}^M_{s,d},\mathbf{X}^M_c,\mathbf{X}^M_d,\mathbf{P}^M,\mathbf{Y}^M\right)\\
&s.t.\ \Pi\left[G^{(i)}\left(\mathbf{d}_{s,c},\mathbf{d}_{s,d},\mathbf{d}_{i,c},\mathbf{d}_{i,d},\mathbf{X}_{s,c},\mathbf{X}_{s,d},\mathbf{X}_{i,c},\right.\right.\\
&\qquad\qquad \left.\left.\mathbf{X}_{i,d},\mathbf{P}_{i,c},\mathbf{P}_{i,d},\mathbf{Y}_{\bullet i}\right)>0\right]\le\alpha_t\\
&\quad g^{(i)}\left(\mathbf{d}_{s,c},\mathbf{d}_{s,d},\mathbf{d}_{i,c},\mathbf{d}_{i,d},\mathbf{X}^M_{s,c},\mathbf{X}^M_{s,d},\mathbf{X}^M_{i,c},\mathbf{X}^M_{i,d},\right.\\
&\qquad \left.\mathbf{P}^M_{i,c},\mathbf{P}^M_{i,d},\mathbf{Y}^M_{\bullet i}\right)\le 0\\
&\quad \mathbf{d}^L_{s,c}\le\mathbf{d}_{s,c}\le\mathbf{d}^U_{s,c},\ \mathbf{d}^L_{s,d}\le\mathbf{d}_{s,d}\le\mathbf{d}^U_{s,d},\\
&\qquad \mathbf{d}^L_c\le\mathbf{d}_c\le\mathbf{d}^U_c,\ \mathbf{d}^L_d\le\mathbf{d}_d\le\mathbf{d}^U_d\\
&\quad \mathbf{X}^{M,L}_{s,c}\le\mathbf{X}^M_{s,c}\le\mathbf{X}^{M,U}_{s,c},\mathbf{X}^{M,L}_{s,d}\le\mathbf{X}^M_{s,d}\le\mathbf{X}^{M,U}_{s,d},\\
&\qquad \mathbf{X}^{M,L}_c\le\mathbf{X}^M_c\le\mathbf{X}^{M,U}_c,\mathbf{X}^{M,L}_d\le\mathbf{X}^M_d\le\mathbf{X}^{M,U}_d\\
&\quad i=1,2,\cdots,nd\\
&DV=\left\{\mathbf{d}_{s,c},\mathbf{d}_{s,d},\mathbf{d}_c,\mathbf{d}_d,\mathbf{X}^M_{s,c},\mathbf{X}^M_{s,d},\mathbf{X}^M_c,\mathbf{X}^M_d\right\}
\end{aligned}
\tag{1}
$$

where "DV" denotes design variables in the optimization formulation. The subscript s denotes that the variables are sharing variables to all disciplines while subscript i indicates that the variables or parameters are local ones only for the ith discipline. Subscripts c and d denote that the type of a variable and parameter is continuous and discrete, respectively. $\mathbf{d}$ indicate deterministic design variables, and $\mathbf{d}_c = \left\{\mathbf{d}_{i,c}, i=1\sim nd\right\}, \mathbf{d}_d = \left\{\mathbf{d}_{i,d}, i=1\sim nd\right\}$ where nd is the total number of disciplines. $\mathbf{X}$ denotes a vector of random and fuzzy variables, and $\mathbf{X}_c = \left\{\mathbf{X}_{i,c}, i=1\sim nd\right\}, \mathbf{X}_d = \left\{\mathbf{X}_{i,d}, i=1\sim nd\right\}$. $\mathbf{X}_{s,c} = \left\{\mathbf{X}_{s,rc}, \mathbf{X}_{s,fc}\right\}, \mathbf{X}_{s,d} = \left\{\mathbf{X}_{s,rd}, \mathbf{X}_{s,fd}\right\}$, $\mathbf{X}_{i,c} = \left\{\mathbf{X}_{i,rc}, \mathbf{X}_{i,fc}\right\}, \mathbf{X}_{i,d} = \left\{\mathbf{X}_{i,rd}, \mathbf{X}_{i,fd}\right\}$ in which subscripts rc, fc, rd and fd denote that the type of a variable or parameter is continuous random, continuous fuzzy, discrete random and discrete fuzzy, respectively. $\mathbf{P} = \left\{\left(\mathbf{P}_{i,c}, \mathbf{P}_{i,d}\right), i=1\sim nd\right\}$ is a vector of random and fuzzy parameters, and $\mathbf{P}_{i,c} = \left\{\mathbf{P}_{i,rc}, \mathbf{P}_{i,fc}\right\}, \mathbf{P}_{i,d} = \left\{\mathbf{P}_{i,rd}, \mathbf{P}_{i,fd}\right\}$. $\mathbf{Y} = \{\mathbf{Y}_{i\bullet}, i=1\sim nd\}$ are linking variables, and $\mathbf{Y}_{i\bullet}$ are output linking variables from the ith discipline while $\mathbf{Y}_{\bullet i}$ are input linking variables to discipline i. The superscript M denotes that the mean value of a random variable or parameter, and maximal grade point of a fuzzy variable or parameter, respectively. The maximal grade point of a fuzzy variable X is defined as $X^M =$

$\{x|\max\{\Pi_X(x)\}\}$ where $\prod_X(x)$ is the membership function of X. $\Pi\left[G^{(i)}(\cdot)>0\right]\le\alpha_t$ is the possibility constraint in discipline i with the failure mode defined as $G^{(i)}(\cdot)>0$, and α_t is the allowable possibility of failure. $g^{(i)}$ is the deterministic constraint in discipline i. Superscripts L and U denote the lower and upper bounds, respectively.

During optimization process, when obtained a design point, the feasibilities of possibility constraints should be analyzed at that point. This analysis process is called uncertainty analysis (probability/possibility analysis). In Section 3, the approach for uncertainty analysis for RFCDV-MDO is proposed.

3 Fundamental analysis (uncertainty analysis)

In this section, the uncertainty analysis for the case of only one discipline developed in Huang and Zhang (2009) is introduced in Section 3.2, and uncertainty analysis for RFCDV-MDO will be discussed in Section 3.3.

The conditional possibility of failure proposed in (Du and Choi 2008) is firstly introduced. Suppose there are two continuous fuzzy variables which are mutually non-interactive with their membership functions as $\Pi_{X_1}(x_1)$ and $\Pi_{X_2}(x_2)$. The failure event is $G(x_1,x_2)>0$, the possibility of failure can be computed by

$$\begin{aligned}\Pi_f &= \sup_{G(x_1,x_2)>0}\left[\min\left\{\Pi_{X_1}(x_1),\Pi_{X_2}(x_2)\right\}\right]\\ &=\sup_{x_2}\left[\sup_{x_1:G(x_1,x_2)>0}\left[\min\left\{\Pi_{X_1}(x_1),\Pi_{X_2}(x_2)\right\}\right]\right]\\ &=\sup_{x_2}\left[\min\left\{\sup_{x_1:G(x_1,x_2)>0}\Pi_{X_1}(x_1),\right.\right.\\ &\qquad\left.\left.\sup_{x_1:G(x_1,x_2)>0}\Pi_{X_2}(x_2)\right\}\right]\\ &=\sup_{x_2}\left[\min\left\{\Pi_f\big|\{X_2=x_2\},\Pi_{X_2}(x_2)\right\}\right]\end{aligned}\tag{2}$$

where $\Pi_f\big|\{X_2=x_2\}=\sup_{x_1:G(x_1,x_2)>0}\Pi_{X_1}(x_1)$ is defined as the conditional possibility of failure when $X_2=x_2$ (Du and Choi 2008).

3.1 Transformation

This transformation for fuzzy variables and parameters in X-space into standard non-interactive fuzzy ones in V-space is on: the membership is the same before and after transformation (Du et al. 2006). The standard fuzzy variable has the isosceles triangular membership function as:

$$\begin{aligned}\Pi_V(v)&=\begin{cases}v+1 & if\quad -1\le v\le 0\\ 1-v & if\quad 0\le v\le 1\end{cases}\\ &=1-|v|,\quad |v|\le 1\end{aligned}\tag{3}$$

The transformation is:

$$v=\begin{cases}\Pi_X(x)-1 & if\, x\le X^M\\ 1-\Pi_X(x) & if\, x>X^M\end{cases}\tag{4}$$

where $\prod_X(x)$ is the membership function of a fuzzy variable X, and X^M is the maximal grade point of the fuzzy variable (Du and Choi 2008; Youn et al. 2005; Du et al. 2006).

Suppose there are two mutually non-interactive fuzzy variables X_1, X_2 with their membership functions $\Pi_{X_1}(x_1),\Pi_{X_2}(x_2)$ satisfied unity, strong convexity, and boundedness (detailed definitions of these three properties can be found in Du et al. (2006). After transforming into the standard normalized fuzzy ones, the joint membership function of X_1, X_2 is given as:

$$\begin{aligned}\Pi_{X_1,X_2}(x_1,x_2)&=\min\left\{\Pi_{X_1}(x_1),\Pi_{X_2}(x_2)\right\}\\ &=\min\left\{\Pi_{V_1}(v_1),\Pi_{V_2}(v_2)\right\}\\ &=\min\{1-|v_1|,1-|v_2|\}\\ &=1-\|(v_1,v_2)\|_\infty\end{aligned}\tag{5}$$

3.2 Uncertainty analysis for the case of only one discipline

In this section, the uncertainty analysis for the case of only one discipline developed in Huang and Zhang (2009) is introduced. If only one discipline is considered, there are not any sharing, local and linking variables. When performing uncertainty analysis at a design point, the possibility constraint $\Pi\left[G^{(i)}\left(\mathbf{d}_{s,c},\ \mathbf{d}_{s,d},\ \mathbf{d}_{i,c},\ \mathbf{d}_{i,d},\mathbf{X}_{s,c},\mathbf{X}_{s,d},\mathbf{X}_{i,c},\mathbf{X}_{i,d},\mathbf{P}_{i,c},\mathbf{P}_{i,d},\mathbf{Y}_{\bullet i}\right)>0\right]\le\alpha_t$ in (1) becomes $\prod[G(\mathrm{X}_c,\mathrm{X}_d,\mathrm{P}_c,\mathrm{P}_d)>0]\le\alpha_t$, which is $\prod[G(\mathrm{X}_{rc},\mathrm{X}_{rd},\mathrm{X}_{fc},\mathrm{X}_{fd},\mathrm{P}_{rc},\mathrm{P}_{rd},\mathrm{P}_{fc},\mathrm{P}_{fd})>0]\le\alpha_t$ in detail, since the values of the deterministic variables are known.

Assume the continuous random variables $\mathbf{X}_{rc}$ are subject to a joint probability density function $f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc})$, all discrete random variables $\mathbf{X}_{rd}$ follow a joint probability distribution function $P_{\mathbf{X}_{rd}}$. The continuous random parameters $\mathbf{P}_{rc}$ have the joint probability density function $f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc})$, and all discrete random parameters $\mathbf{P}_{rd}$ have the joint probability distribution function $P_{\mathbf{P}_{rd}}$. Also all fuzzy variables $\mathbf{X}_f = \{\mathbf{X}_{fc}, \mathbf{X}_{fd}\}$ and parameters $\mathbf{P}_f = \{\mathbf{P}_{fc}, \mathbf{P}_{fd}\}$ have the membership function $\Pi_{\mathbf{X}_f,\mathbf{P}_f}\left(\mathbf{x}_f, \mathbf{p}_f\right)$ where $\mathbf{X}_{fc}, \mathbf{X}_{fd}$ are continuous and discrete fuzzy variables while $\mathbf{P}_{fc}, \mathbf{P}_{fd}$ are continuous and discrete fuzzy parameters. The possibility of failure $\prod_f = \left[G\left(\mathrm{X}_{rc}, \mathrm{X}_{rd}, \mathrm{X}_f, \mathrm{P}_{rc}, \mathrm{P}_{rd}, \mathrm{P}_f\right) > 0\right]$ can be calculated by the following steps.

Firstly, temporarily fix the fuzzy variables and parameters at $\mathbf{X}_f = \mathbf{x}_f$, $\mathbf{P}_f = \mathbf{p}_f$, the conditional probability of failure can be calculated by

$$P_f \big| \left\{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\right\} = \sum_{t=1}^{N} \left\{ \int\limits_{\mathbf{x}_{rc},\mathbf{p}_{rc}:G\left(\substack{\mathbf{x}_{rc},\mathbf{x}_{rd}^t,\mathbf{x}_f,\\ \mathbf{p}_{rc},\mathbf{p}_{rd}^t,\mathbf{p}_f}\right)>0} f_{\mathbf{X}_{rc}}\left(\mathbf{x}_{rc}\right) f_{\mathbf{P}_{rc}}\left(\mathbf{p}_{rc}\right) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \times P_{\mathbf{X}_{rd}}\left(\mathbf{x}_{rd}^t\right) \times P_{\mathbf{P}_{rd}}\left(\mathbf{p}_{rd}^t\right) \right\}$$

where N stands for the total number of all possible combinations of $\mathbf{x}_{rd}, \mathbf{p}_{rd}$.

Secondly, set the conditional possibility of failure to be same as the value of conditional probability of failure. It is a reasonable assumption because possibility is an alterative and vague measure of probability, and also possibility of an event can be assigned as the upper boundary of the probability when probability of the event is unknown. If there exists the probability, one can set the possibility to be same as the probability (Du and Choi 2008).

Finally, the possibility of failure can be computed by

$$\begin{aligned} \Pi_f &= \sup_{\mathbf{x}_f,\mathbf{p}_f} \left[\min\left\{\Pi_f \big| \left\{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\right\}, \Pi_{\mathbf{X}_f,\mathbf{P}_f}\left(\mathbf{x}_f, \mathbf{p}_f\right)\right\}\right] \\ &= \sup_{\mathbf{x}_f,\mathbf{p}_f} \left[\min\left\{P_f \big| \left\{\mathbf{X}_f = \mathbf{x}_f, \mathbf{P}_f = \mathbf{p}_f\right\}, \Pi_{\mathbf{X}_f,\mathbf{P}_f}\left(\mathbf{x}_f, \mathbf{p}_f\right)\right\}\right] \\ &= \sup_{\mathbf{x}_f,\mathbf{p}_f} \left[\min\left\{\sum_{t=1}^{N}\left\{ \int\limits_{\mathbf{x}_{rc},\mathbf{p}_{rc}:G\left(\substack{\mathbf{x}_{rc},\mathbf{x}_{rd}^t,\mathbf{x}_f,\\ \mathbf{p}_{rc},\mathbf{p}_{rd}^t,\mathbf{p}_f}\right)>0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc})\, d\mathbf{x}_{rc} d\mathbf{p}_{rc} \times P_{\mathbf{X}_{rd}}\left(\mathbf{x}_{rd}^t\right) \times P_{\mathbf{P}_{rd}}\left(\mathbf{p}_{rd}^t\right)\right\}, \Pi_{\mathbf{X}_f,\mathbf{P}_f}\left(\mathbf{x}_f, \mathbf{p}_f\right)\right\}\right] \end{aligned} \quad (6)$$

When the number of possible combinations of discrete random variables and parameters is large, the computational price of (6) could be unaffordable. To overcome this difficulty, based on possibility-probability consistency, all discrete random variables and parameters are initially transformed into discrete fuzzy ones $\left(\mathbf{x}_{rd} \rightarrow \mathbf{x}_{frd}, \mathrm{p}_{rd} \rightarrow \mathrm{p}_{frd}\right)$ in which the subscript *frd* denotes a discrete fuzzy variable or parameter transformed from the discrete random one. The transformation is: assume $p(y)$ is a distribution on $Y = \{y_1, y_2, \cdots, y_n\}$ in which elements have been indexed in descending order with their probabilities where $p_1 \geq p_2 \geq$

$\cdots \geq p_n$. Then the possibility distribution on Y can be calculated as:

$$\mu_n = n \times p_n$$
$$\mu_j = j\left(p_j - p_{j+1}\right) + \mu_{j+1}$$
$$j = n-1, \cdots, 1 \tag{7}$$

If $p_j = p_{j+1}$, then $\mu_j = \mu_{j+1}$; if $p_j = 0$, then $\mu_j = 0$ (Dubois and Prade 1983).

Because all discrete random variables and parameters have initially been transformed into discrete fuzzy ones $\left(\mathbf{x}_{rd} \to \mathbf{x}_{frd},\ \mathbf{p}_{rd} \to \mathbf{p}_{frd}\right)$, there are no longer any discrete random variables and parameters. Following the same above steps, the possibility of failure can be deduced as:

$$\Pi_f = \sup_{\mathbf{x}_f, \mathbf{x}_{frd}, \mathbf{p}_{frd}, \mathbf{p}_f} \left[\min \left\{ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc} : G\left(\substack{\mathbf{x}_{rc}, \mathbf{x}_{frd}, \mathbf{x}_f, \\ \mathbf{p}_{rc}, \mathbf{p}_{frd}, \mathbf{p}_f}\right) > 0} f_{\mathbf{X}_{rc}}\left(\mathbf{x}_{rc}\right) f_{\mathbf{P}_{rc}}\left(\mathbf{p}_{rc}\right) d\mathbf{x}_{rc} d\mathbf{p}_{rc},\ \Pi\left(\mathbf{x}_f, \mathbf{x}_{frd}, \mathbf{p}_{frd}, \mathbf{p}_f\right) \right\} \right] \tag{8}$$

where $\Pi(\cdot)$ is the membership function of $\mathbf{X}_f, \mathbf{X}_{frd}, \mathbf{P}_{frd}, \mathbf{P}_f$.

All fuzzy variables and parameters are assumed non-interactive. Each fuzzy variable and parameter is assumed with its membership function satisfied: unity, strong convexity, boundedness.

Transform the fuzzy variables and parameters ($\mathbf{X}_f$, $\mathbf{P}_f$, $\mathbf{X}_{frd}$, $\mathbf{P}_{frd}$) including the ones transformed from discrete random variables and parameters into the non-interactive standard fuzzy ones ($\mathbf{V}_f$, $\mathbf{VP}_f$, $\mathbf{V}_{frd}$, $\mathbf{VP}_{frd}$), respectively. Equation (8) can be further written as:

$$\Pi_f = \sup_{\mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f} \left[\min \left\{ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc} : G\left(\substack{\mathbf{x}_{rc}, \mathbf{v}_{frd}, \mathbf{v}_f, \\ \mathbf{p}_{rc}, \mathbf{vp}_{frd}, \mathbf{vp}_f}\right) > 0} f_{\mathbf{X}_{rc}}\left(\mathbf{x}_{rc}\right) f_{\mathbf{P}_{rc}}\left(\mathbf{p}_{rc}\right) d\mathbf{x}_{rc} d\mathbf{p}_{rc},\ 1 - \left\| \mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f \right\|_\infty \right\} \right] \tag{9}$$

There are two ways to judge whether or not $\Pi_f \leq \alpha_t$ is satisfied. The first way whose computation is huge is: directly calculate the possibility of failure utilizing (8) or (9), compare it with α_t and then obtain the results of satisfaction or not.

The second one is:

Among all points satisfied $G(\cdot) > 0$, there are three cases:

(1) $1 - \left\| \mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f \right\|_\infty < \alpha_t$
(2) $1 - \left\| \mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f \right\|_\infty = \alpha_t$
(3) $1 - \left\| \mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f \right\|_\infty > \alpha_t$

When the fuzzy part satisfies cases (1) and (2), the value of $\min \left\{ \int_{\mathbf{x}_{rc}, \mathbf{p}_{rc} : G(\cdot) > 0} f_{\mathbf{X}_{rc}}\left(\mathbf{x}_{rc}\right) f_{\mathbf{P}_{rc}}\left(\mathbf{p}_{rc}\right) d\mathbf{x}_{rc} d\mathbf{p}_{rc},\ 1 - \left\| \mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f \right\|_\infty \right\}$ will not be larger than α_t. So cases (1) and (2) do not affect the final result of $\Pi_f \leq \alpha_t$. If $\int_{\mathbf{x}_{rc}, \mathbf{p}_{rc} : G\left(\substack{\mathbf{x}_{rc}, \mathbf{v}_{frd}, \mathbf{v}_f, \\ \mathbf{p}_{rc}, \mathbf{vp}_{frd}, \mathbf{vp}_f}\right) > 0} f_{\mathbf{X}_{rc}}\left(\mathbf{x}_{rc}\right) f_{\mathbf{P}_{rc}}\left(\mathbf{p}_{rc}\right) d\mathbf{x}_{rc}\, d\mathbf{p}_{rc} \leq \alpha_t$ is satisfied whenever the fuzzy part satisfies case (3), then $\Pi_f \leq \alpha_t$.

Transform the continuous random variables and parameters ($\mathbf{X}_{rc}$, $\mathbf{P}_{rc}$) into the independent standard normal ones ($\mathbf{U}_{rc}$, $\mathbf{UP}_{rc}$) using Rosenblatt transformation (Du et al. 2008), respectively. Given the fuzzy part satisfying $1 - \left\|\mathbf{v}_f, \mathbf{v}_{frd}, \mathbf{vp}_{frd}, \mathbf{vp}_f\right\|_\infty > \alpha_t$, whether or not $\int\limits_{\mathbf{x}_{rc},\mathbf{p}_{rc}:G\left(\substack{\mathbf{x}_{rc},\mathbf{v}_{frd},\mathbf{v}_f,\\ \mathbf{p}_{rc},\mathbf{vp}_{frd},\mathbf{vp}_f}\right)>0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \le \alpha_t$ can be checked out by First Order Reliability Method (FORM) as:

$$\begin{aligned} \max \quad & G\left(\mathbf{U}_{rc}, \mathbf{v}_{frd}, \mathbf{v}_f, \mathbf{UP}_{rc}, \mathbf{vp}_{frd}, \mathbf{vp}_f\right) \\ s.t. \quad & \|(\mathbf{U}_{rc}, \mathbf{UP}_{rc})\|_2 \le -\Phi^{-1}(\alpha_t) \end{aligned} \tag{10}$$

where Φ is the cumulative distribution function of standard normal random variable. If the maximal value $G\left(\mathbf{U}^*_{rc}, \mathbf{v}_{frd}, \mathbf{v}_f, \mathbf{UP}^*_{rc}, \mathbf{vp}_{frd}, \mathbf{vp}_f\right) \le 0$, then $\int\limits_{\mathbf{x}_{rc},\mathbf{p}_{rc}:G\left(\substack{\mathbf{x}_{rc},\mathbf{v}_{frd},\mathbf{v}_f,\\ \mathbf{p}_{rc},\mathbf{vp}_{frd},\mathbf{vp}_f}\right)>0} f_{\mathbf{X}_{rc}}(\mathbf{x}_{rc}) f_{\mathbf{P}_{rc}}(\mathbf{p}_{rc}) d\mathbf{x}_{rc} d\mathbf{p}_{rc} \le \alpha_t$, and vice versa.

So whether or not $\Pi_f \le \alpha_t$ at current design, an optimization formulated as following can be utilized to check out:

$$\begin{aligned} \max \quad & G\left(\mathbf{U}_{rc}, \mathbf{V}_{frd}, \mathbf{V}_f, \mathbf{UP}_{rc}, \mathbf{VP}_{frd}, \mathbf{VP}_f\right) \\ s.t. \quad & \|(\mathbf{U}_{rc}, \mathbf{UP}_{rc})\|_2 \le -\Phi^{-1}(\alpha_t) \\ & \left\|\left(\mathbf{V}_{frd}, \mathbf{V}_f, \mathbf{VP}_{frd}, \mathbf{VP}_f\right)\right\|_\infty < 1 - \alpha_t \end{aligned} \tag{11}$$

The solutions of (11) are the MPPP (Most Probable/Possible Point) $\mathbf{U}^*_{rc}, \mathbf{V}^*_{frd}, \mathbf{V}^*_f, \mathbf{UP}^*_{rc}, \mathbf{VP}^*_{frd}, \mathbf{VP}^*_f$ and $G\left(\mathbf{U}^*_{rc}, \mathbf{V}^*_{frd}, \mathbf{V}^*_f, \mathbf{UP}^*_{rc}, \mathbf{VP}^*_{frd}, \mathbf{VP}^*_f\right)$ which is the value of performance measure at the MPPP. If the value of performance measure at the MPPP is not larger than zero, this indicates that the requirement $\Pi_f \le \alpha_t$ is satisfied; otherwise unsatisfied.

The second way is called Performance Measure Approach (PMA) which is different with the first way in which the actual possibility of failure is calculated.

3.3 Uncertainty analysis for RFCDV-MDO

Different with the case of only one discipline, the consistencies among disciplines should be dealt with and achieved when performing uncertainty analysis in the environment of MDO. In Du et al. (2008), two approaches are adopted: Individual Disciplinary Feasible approach (IDF) and Multidisciplinary Feasible approach (MDF). The first one is that the consistency is considered as extra constraints. The second one is that the consistency is acquired by solving an optimization nested into the probability analysis to obtain the values of linking variables. These two methods can be used during the uncertainty (probability/possibility) analysis, but from the results in Du et al. (2008), the first method is more efficient and stable than the latter. So the first method, IDF, is adopted in this paper.

It is implied that the possibility constraint function $G^{(i)}\left(\mathbf{d}_{s,c}, \mathbf{d}_{s,d}, \mathbf{d}_{i,c}, \mathbf{d}_{i,d}, \mathbf{X}_{s,c}, \mathbf{X}_{s,d}, \mathbf{X}_{i,c}, \mathbf{X}_{i,d}, \mathbf{P}_{i,c}, \mathbf{P}_{i,d}, \mathbf{Y}_{\bullet i}\right)$ includes all design inputs because of the existences of linking variables. In other words, the possibility constraint function includes all design variables and parameters associated with aleatory and epistemic uncertainties.

Followed the IDF method, the consistency is considered as extra constraints. Transform all continuous random variables and parameters into the standard normal ones, and transform all discrete random variables and parameters, all fuzzy variables and parameters into standard fuzzy ones. Based on (11), the mathematical formulation of uncertainty (probability/possibility) analysis under the environment of MDO is given as:

$$\begin{aligned} & \max_{DV} G^{(i)}\begin{pmatrix} \mathbf{d}_{s,c}, \mathbf{d}_{s,d}, \mathbf{d}_{i,c}, \mathbf{d}_{i,d}, \mathbf{U}^{(i)}_{s,rc}, \mathbf{V}^{(i)}_{s,frd}, \mathbf{V}^{(i)}_{s,fc}, \\ \mathbf{V}^{(i)}_{s,fd}, \mathbf{U}^{(i)}_{i,rc}, \mathbf{V}^{(i)}_{i,frd}, \mathbf{V}^{(i)}_{i,fc}, \mathbf{V}^{(i)}_{i,fd}, \\ \mathbf{UP}^{(i)}_{i,rc}, \mathbf{VP}^{(i)}_{i,frd}, \mathbf{VP}^{(i)}_{i,fc}, \mathbf{VP}^{(i)}_{i,fd}, \mathbf{Y}^{(i)}_{\bullet i} \end{pmatrix} \\ & s.t. \ \left\|\left(\mathbf{U}^{(i)}_{s,rc}, \mathbf{U}^{(i)}_{rc}, \mathbf{UP}^{(i)}_{rc}\right)\right\|_2 \le \beta_t \\ & \quad \left\|\left(\mathbf{V}^{(i)}_{s,frd}, \mathbf{V}^{(i)}_{s,fc}, \mathbf{V}^{(i)}_{s,fd}, \mathbf{V}^{(i)}_{frd}, \mathbf{V}^{(i)}_{fc}, \mathbf{V}^{(i)}_{fd}, \right.\right. \\ & \quad \left.\left. \mathbf{VP}^{(i)}_{frd}, \mathbf{VP}^{(i)}_{fc}, \mathbf{VP}^{(i)}_{fd}\right)\right\|_\infty < 1 - \alpha_t \\ & \quad y^{(i)}_{jm} = y^{(i)}_{jm}\begin{pmatrix} \mathbf{d}_{s,c}, \mathbf{d}_{s,d}, \mathbf{d}_{j,c}, \mathbf{d}_{j,d}, \mathbf{U}^{(i)}_{s,rc}, \mathbf{V}^{(i)}_{s,frd}, \mathbf{V}^{(i)}_{s,fc}, \\ \mathbf{V}^{(i)}_{s,fd}, \mathbf{U}^{(i)}_{j,rc}, \mathbf{V}^{(i)}_{j,frd}, \mathbf{V}^{(i)}_{j,fc}, \mathbf{V}^{(i)}_{j,fd}, \\ \mathbf{UP}^{(i)}_{j,rc}, \mathbf{VP}^{(i)}_{j,frd}, \mathbf{VP}^{(i)}_{j,fc}, \mathbf{VP}^{(i)}_{j,fd}, \mathbf{Y}^{(i)}_{\bullet j} \end{pmatrix} \\ & \quad j, m = 1, 2, \cdots, nd \qquad j \ne m; i = 1, 2, \cdots, nd \\ & DV = \left\{\mathbf{U}^{(i)}_{s,rc}, \mathbf{V}^{(i)}_{s,frd}, \mathbf{V}^{(i)}_{s,fc}, \mathbf{V}^{(i)}_{s,fd}, \mathbf{U}^{(i)}_{rc}, \mathbf{V}^{(i)}_{frd}, \mathbf{V}^{(i)}_{fc}, \right. \\ & \quad \left. \mathbf{V}^{(i)}_{fd}, \mathbf{UP}^{(i)}_{rc}, \mathbf{VP}^{(i)}_{frd}, \mathbf{VP}^{(i)}_{fc}, \mathbf{VP}^{(i)}_{fd}, \mathbf{Y}^{(i)}\right\} \end{aligned} \tag{12}$$

where $G^{(i)}$ is the possibility constraint function in the ith discipline. The superscript '(i)' indicates that the variables and parameters correspond to the possibility constraint in the ith discipline since each possibility constraint has its own MPPP. $\mathbf{U}^{(i)}_{s,rc}, \mathbf{V}^{(i)}_{s,frd}, \mathbf{V}^{(i)}_{s,fc}, \mathbf{V}^{(i)}_{s,fd}, \mathbf{U}^{(i)}_{i,rc}, \mathbf{V}^{(i)}_{i,frd}, \mathbf{V}^{(i)}_{i,fc}, \mathbf{V}^{(i)}_{i,fd}, \mathbf{UP}^{(i)}_{i,rc}, \mathbf{VP}^{(i)}_{i,frd}, \mathbf{VP}^{(i)}_{i,fc}, \mathbf{VP}^{(i)}_{i,fd}$ are transformed from $\mathbf{X}_{s,rc}$, $\mathbf{X}_{s,rd}$, $\mathbf{X}_{s,fc}$, $\mathbf{X}_{s,fd}$, $\mathbf{X}_{i,rc}$, $\mathbf{X}_{i,rd}$, $\mathbf{X}_{i,fc}$, $\mathbf{X}_{i,fc}$, $\mathbf{X}_{i,fd}$, $\mathbf{P}_{i,rc}$, $\mathbf{P}_{i,rd}$, $\mathbf{P}_{i,fc}$, $\mathbf{P}_{i,fd}$, respectively. β_t is equal to $-\Phi^{-1}(\alpha_t)$. $\mathbf{U}_{rc}^{(i)} = \left\{\mathbf{U}_{1,rc}^{(i)}, \mathbf{U}_{2,rc}^{(i)}, \cdots, \mathbf{U}_{nd,rc}^{(i)}\right\}$, $\mathbf{UP}_{rc}^{(i)} = \left\{\mathbf{UP}_{1,rc}^{(i)}, \mathbf{UP}_{2,rc}^{(i)}, \cdots, \mathbf{UP}_{nd,rc}^{(i)}\right\}$, $\mathbf{V}_{frd}^{(i)} = \left\{\mathbf{V}_{1,frd}^{(i)}, \mathbf{V}_{2,frd}^{(i)}, \cdots, \mathbf{V}_{nd,frd}^{(i)}\right\}$, $\mathbf{V}_{fc}^{(i)} = \left\{\mathbf{V}_{1,fc}^{(i)}, \mathbf{V}_{2,fc}^{(i)}, \cdots, \mathbf{V}_{nd,fc}^{(i)}\right\}$, $\mathbf{V}_{fd}^{(i)} = \left\{\mathbf{V}_{1,fd}^{(i)}, \mathbf{V}_{2,fd}^{(i)}, \cdots, \mathbf{V}_{nd,fd}^{(i)}\right\}$, $\mathbf{VP}_{frd}^{(i)} = \left\{\mathbf{VP}_{1,frd}^{(i)}, \mathbf{VP}_{2,frd}^{(i)}, \cdots, \mathbf{VP}_{nd,frd}^{(i)}\right\}$, $\mathbf{VP}_{fc}^{(i)} = \left\{\mathbf{VP}_{1,fc}^{(i)}, \mathbf{VP}_{2,fc}^{(i)}, \cdots, \mathbf{VP}_{nd,fc}^{(i)}\right\}$, $\mathbf{VP}_{fd}^{(i)} = \left\{\mathbf{VP}_{1,fd}^{(i)}, \mathbf{VP}_{2,fd}^{(i)}, \cdots, \mathbf{VP}_{nd,fd}^{(i)}\right\}$. The linking variables $\mathbf{Y}^{(i)}$ are as extra variables in this optimization formulation.

In the uncertainty analysis (probability/possibility analysis) formulation, the first and second constraints in (12) include all design inputs associated with uncertainties. The reason is that a possibility constraint function impliedly includes all design inputs associated with uncertainties because of the existences of linking variables. The consistency is considered as extra constraints in which variables and parameters associated with aleatory or epistemic uncertainty should be transformed into standard normal or fuzzy ones.

The solutions of probability/possibility analysis are MPPP $\left(\mathbf{U}_{s,rc}^{*,(i)}, \mathbf{V}_{s,frd}^{*,(i)}, \mathbf{V}_{s,fc}^{*,(i)}, \mathbf{V}_{s,fd}^{*,(i)}, \mathbf{U}_{rc}^{*,(i)}, \mathbf{V}_{frd}^{*,(i)}, \mathbf{V}_{fc}^{*,(i)}, \mathbf{V}_{fd}^{*,(i)}, \mathbf{UP}_{rc}^{*,(i)}, \mathbf{VP}_{frd}^{*,(i)}, \mathbf{VP}_{fc}^{*,(i)}, \mathbf{VP}_{fd}^{*,(i)}\right)$, linking variables $\mathbf{Y}^{*,(i)}$ and value of performance measure at MPPP $G^{(i)}, i = 1 \sim nd$. If the value of performance measure at the MPPP of a possibility constraint function is not larger than zero, then that possibility constraint is satisfied; otherwise unsatisfied. The MPPPs in X-space $\left(\mathbf{X}_{s,rc}^{*,(i)}, \mathbf{X}_{s,rd}^{*,(i)}, \mathbf{X}_{s,fc}^{*,(i)}, \mathbf{X}_{s,fd}^{*,(i)}, \mathbf{X}_{rc}^{*,(i)}, \mathbf{X}_{rd}^{*,(i)}, \mathbf{X}_{fc}^{*,(i)}, \mathbf{X}_{fd}^{*,(i)}, \mathbf{P}_{rc}^{*,(i)}, \mathbf{P}_{rd}^{*,(i)}, \mathbf{P}_{fc}^{*,(i)}, \mathbf{P}_{fd}^{*,(i)}\right)$ can be obtained using the inverse transformations aforementioned.

Directly solving (1) will involve three nested loops: minimizing the objective function in the outer loop; performing uncertainty analysis (probability/possibility analysis) in the middle loop; performing multidisciplinary analysis in the inner loop to obtain the consistency among disciplines. To efficiently solve RFCDV-MDO problem, RFCDV-MDO within the framework of SORA (RFCDV-MDO-SORA) is proposed in next section.

4 RFCDV-MDO within the framework of SORA (RFCDV-MDO-SORA)

Sequential Optimization and Reliability Assessment (SORA) is originally developed to deal with Reliability Based Design Optimization problems, and introduced into RBMDO in Du et al. (2008). In this paper, RFCDV-MDO-SORA is developed to deal with RFCDV-MDO problems utilizing the idea of SORA.

4.1 Strategy of RFCDV-MDO-SORA

To solve the RFCDV-MDO problems efficiently, two technologies are adopted.

1. Performance Measure Approach (PMA). The PMA approach is found more efficient than evaluating the actual probability or possibility of failure directly (Du and Chen 2005; Youn et al. 2005; Du et al. 2006). PMA decreases the computational price through evaluating the value of constraint function corresponding to the target allowable value of probability or possibility of failure, instead of calculating the actual value of probability or possibility of failure which needs more computation. Hence it is adopted in uncertainty analysis when solving the RFCDV-MDO problem.
2. Sequential Optimization and Reliability Assessment (SORA). In this paper, with the idea of SORA, the MDO solution and probability/possibility analysis are decoupled, and the whole solution process is a series of cycles of deterministic MDO and probability/possibility analysis. In each cycle, the probability/possibility analysis is performed after the deterministic MDO. After solving the deterministic MDO, the value of each deterministic design variable and the mean value or maximal grade point of each design variable with uncertainty are obtained. Then the probability/possibility analysis is applied to analyze the feasibility of each possibility constraint. If possibility constraints are not all satisfied, the MPPPs will be used to reconstruct the deterministic constraints in deterministic MDO of the next cycle to improve the feasibility of each possibility constraint. The efficiency is improved obviously since the solution process is a sequential manner but not nested. It is expected that the process will converge in a few cycles.

4.2 Procedure of RFCDV-MDO-SORA

In this section, the procedure of RFCDV-MDO-SORA will be illustrated step by step.

Step 1: Set initial values for design variables as $\mathbf{d}_{s,c}^{(0)}$, $\mathbf{d}_{s,d}^{(0)}, \mathbf{d}_{c}^{(0)}$, $\mathbf{d}_{d}^{(0)}$, $\mathbf{X}_{s,c}^{M,(0)}$, $\mathbf{X}_{s,d}^{M,(0)}$, $\mathbf{X}_{c}^{M,(0)}, \mathbf{X}_{d}^{M,(0)}$, and $k = 1$.

Step 2: Solve the deterministic MDO. The aim of solving the deterministic MDO is to obtain values of

$\mathbf{d}^k_{s,c}$, $\mathbf{d}^k_{s,d}$, $\mathbf{d}^k_c$, $\mathbf{d}^k_d$, $\mathbf{X}^{M,k}_{s,c}$, $\mathbf{X}^{M,k}_{s,d}$, $\mathbf{X}^{M,k}_c$, $\mathbf{X}^{M,k}_d$ where the superscript k denotes the kth cycle. Because there is no information about the MPPPs in the first cycle, the MPPPs are set to be equivalent to $\mathbf{X}^{M,(0)}_{s,c}$, $\mathbf{X}^{M,(0)}_{s,d}$, $\mathbf{X}^{M,(0)}_c$, $\mathbf{X}^{M,(0)}_d$, $\mathbf{P}^M$. Variables in the deterministic constraints are the deterministic variables, the mean value or maximal grade point of each continuous or discrete random or fuzzy variable.

However, from the second cycle, constraints in deterministic MDO are modified with the MPPPs obtained in the previous cycle when requirements on possibility of failure are not all satisfied.

Step 3: Perform probability/possibility analysis. Transform all continuous random variables and parameters into the standard normal ones; and transform all discrete random variables and parameters, all fuzzy variables and parameters into standard fuzzy ones. Then probability/possibility analysis is carried out to check the feasibilities of possibility constraints at the design point which is obtained in Step 2, and the results are MPPP and performance measure corresponding to each possibility constraint.

Step 4: Check convergence. If possibility constraints are all satisfied and the value of the objective is stable $\left(G^{(i)} \le 0, i = 1 \sim nd; |f(k) - f(k-1)| \le \varepsilon\right)$ where ε is an arbitrary small positive constant, stop the process of solution; otherwise set $k = k+1$ and go to Step 2 with the MPPPs obtained in Step 3.

If the possibility constraint $\left(\Pi\left(G^{(i)}(\cdot) > 0\right) \le \alpha_t\right)$ is not satisfied in Cycle k-1 (the value of performance measure at the MPPP satisfies $G^{(i)} > 0$), then the MPPP $\mathbf{X}^{*,(i),(k-1)}_{s,c}$, $\mathbf{X}^{*,(i),(k-1)}_{s,d}$, $\mathbf{X}^{*,(i),(k-1)}_c$, $\mathbf{X}^{*,(i),(k-1)}_d$, $\mathbf{P}^{*,(i),(k-1)}_c$, $\mathbf{P}^{*,(i),(k-1)}_d$ obtained from probability/possibility analysis in Cycle k-1 will be used to reconstruct the constraint in the deterministic MDO in Cycle k. To make sure of feasibility of the possibility constraint, its MPPP in the kth cycle should fall into the deterministic feasible region. Let $\mathbf{S}$ be the shift vector.

Here, two kinds of shift vector are used.

The first is based on the idea of the SORA in Du et al. (2008) as:

$$\mathbf{S}^{(i),k} = \mathbf{X}^{M,(k-1)} - \mathbf{X}^{*,(i),(k-1)}$$

where $\mathbf{S}^{(i),k}$ denotes the shift vector for $\mathbf{X}$ in the ith discipline for Cycle k.

The second is:

If the mean value or maximal grade point of design variable associated with uncertainty is continuous, the shift vector is:

$$S^{(i),k} = X^{M,(k-1)} - X^{*,(i),(k-1)}$$

If the mean value or maximal grade point of design variable associated with uncertainty is discrete, the shift vector is:

$$S^{(i),k} = \begin{cases} \sum_{h=1}^{j} \Delta_h \quad if\ X^{M,(k-1)} - X^{*,(i),(k-1)} > 0\ \ and \\ \sum_{h=1}^{j-1} \Delta_h < X^{M,(k-1)} - X^{*,(i),(k-1)} \le \sum_{h=1}^{j} \Delta_h \\ -\sum_{h=1}^{j} \Delta_h \quad if\ X^{M,(k-1)} - X^{*,(i),(k-1)} < 0\ \ and \\ -\sum_{h=1}^{j} \Delta_h \le X^{M,(k-1)} - X^{*,(i),(k-1)} < -\sum_{h=1}^{j-1} \Delta_h \end{cases}$$

where $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_j$ is the discrete increment from $X^{M,(k-1)}$ when the value of $X^{M,(k-1)} - X^{*,(i),(k-1)}$ is larger than zero; $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_j$ is the discrete decrement from $X^{M,(k-1)}$ when the value of $X^{M,(k-1)} - X^{*,(i),(k-1)}$ is less than zero.

The values of MPPPs $\mathbf{P}^{*,(i),(k-1)}_{i,c}$, $\mathbf{P}^{*,(i),(k-1)}_{i,d}$ directly substitute $\mathbf{P}_{i,c}$, $\mathbf{P}_{i,d}$ in the deterministic constraint. The deterministic constraint in the deterministic MDO of Cycle k is constructed as:

$$\begin{aligned} G^{(i)}_{\Pi}\Big(&\mathbf{d}^k_{s,c}, \mathbf{d}^k_{s,d}, \mathbf{d}^k_{i,c}, \mathbf{d}^k_{i,d}, \mathbf{X}^{M,k}_{s,c} - \mathbf{S}^{(i),k}_{s,c}, \mathbf{X}^{M,k}_{s,d} - \mathbf{S}^{(i),k}_{s,d}, \\ &\mathbf{X}^{M,k}_{i,c} - \mathbf{S}^{(i),k}_{i,c}, \mathbf{X}^{M,k}_{i,d} - \mathbf{S}^{(i),k}_{i,d}, \mathbf{P}^{*,(i),(k-1)}_{i,c}, \\ &\mathbf{P}^{*,(i),(k-1)}_{i,d}, \mathbf{Y}^{*,(i)}_{\bullet i}\Big) \le 0 \end{aligned}$$

where $\mathbf{S}^{(i),k}_{s,c}$, $\mathbf{S}^{(i),k}_{s,d}$, $\mathbf{S}^{(i),k}_{i,c}$, $\mathbf{S}^{(i),k}_{i,d}$ are shift vectors corresponding to $\mathrm{X}_{s,c}$, $\mathrm{X}_{s,d}$, $\mathrm{X}_{i,c}$, $\mathrm{X}_{i,d}$, respectively.

4.3 Formulations of deterministic MDO and probability/possibility analysis

The mathematical formulations of deterministic MDO and probability/possibility analysis in the kth cycle are discussed and provided in this section.

4.3.1 Deterministic MDO in the kth cycle

The mathematical formulation of the deterministic MDO in the kth cycle is given by:

$$
\begin{aligned}
&\min_{DV} f\left(\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{c}^{k}, \mathbf{d}_{d}^{k}, \mathbf{X}_{s,c}^{M,k}, \mathbf{X}_{s,d}^{M,k}, \mathbf{X}_{c}^{M,k}, \mathbf{X}_{d}^{M,k}, \mathbf{P}^{M}, \mathbf{Y}^{M,k}\right)\\
&s.t.\ G_{\Pi}^{(i)}\left(\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{i,c}^{k}, \mathbf{d}_{i,d}^{k}, \mathbf{X}_{s,c}^{M,k}-\mathbf{S}_{s,c}^{(i),k}, \mathbf{X}_{s,d}^{M,k}-\mathbf{S}_{s,d}^{(i),k}, \mathbf{X}_{i,c}^{M,k}-\mathbf{S}_{i,c}^{(i),k}, \mathbf{X}_{i,d}^{M,k}-\mathbf{S}_{i,d}^{(i),k}, \mathbf{P}_{i,c}^{*,(i),(k-1)}, \mathbf{P}_{i,d}^{*,(i),(k-1)}, \mathbf{Y}_{\bullet i}^{*,(i)}\right)\leq 0\\
&g^{(i)}\left(\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{i,c}^{k}, \mathbf{d}_{i,d}^{k}, \mathbf{X}_{s,c}^{M,k}, \mathbf{X}_{s,d}^{M,k}, \mathbf{X}_{i,c}^{M,k}, \mathbf{X}_{i,d}^{M,k}, \mathbf{P}_{i,c}^{M,k}, \mathbf{P}_{i,d}^{M,k}, \mathbf{Y}_{\bullet i}^{M,k}\right)\leq 0\\
&y_{jm}^{*,(i)}=y_{jm}^{*,(i)}\left(\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{j,c}^{k}, \mathbf{d}_{j,d}^{k}, \mathbf{X}_{s,c}^{M,k}-\mathbf{S}_{s,c}^{(i),k}, \mathbf{X}_{s,d}^{M,k}-\mathbf{S}_{s,d}^{(i),k}, \mathbf{X}_{j,c}^{M,k}-\mathbf{S}_{j,c}^{(i),k}, \mathbf{X}_{j,d}^{M,k}-\mathbf{S}_{j,d}^{(i),k}, \mathbf{P}_{j,c}^{*,(i),(k-1)}, \mathbf{P}_{j,d}^{*,(i),(k-1)}, \mathbf{Y}_{\bullet j}^{*,(i)}\right)\\
&i, j, m=1,2,\cdots,nd;\ j\neq m\\
&y_{ij}^{M,k}=y_{ij}\left(\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{i,c}^{k}, \mathbf{d}_{i,d}^{k}, \mathbf{X}_{s,c}^{M,k}, \mathbf{X}_{s,d}^{M,k}, \mathbf{X}_{i,c}^{M,k}, \mathbf{X}_{i,d}^{M,k}, \mathbf{P}_{i,c}^{M}, \mathbf{P}_{i,d}^{M}, \mathbf{Y}_{\bullet i}^{M,k}\right)\\
&\qquad i, j=1,2,\cdots,nd, i\neq j\\
&\mathbf{d}_{s,c}^{L}\leq\mathbf{d}_{s,c}^{k}\leq\mathbf{d}_{s,c}^{U}, \mathbf{d}_{s,d}^{L}\leq\mathbf{d}_{s,d}^{k}\leq\mathbf{d}_{s,d}^{U},\\
&\quad\mathbf{d}_{c}^{L}\leq\mathbf{d}_{c}^{k}\leq\mathbf{d}_{c}^{U}, \mathbf{d}_{d}^{L}\leq\mathbf{d}_{d}^{k}\leq\mathbf{d}_{d}^{U}\\
&\mathbf{X}_{s,c}^{M,L}\leq\mathbf{X}_{s,c}^{M,k}\leq\mathbf{X}_{s,c}^{M,U}, \mathbf{X}_{s,d}^{M,L}\leq\mathbf{X}_{s,d}^{M,k}\leq\mathbf{X}_{s,d}^{M,U},\\
&\quad\mathbf{X}_{c}^{M,L}\leq\mathbf{X}_{c}^{M,k}\leq\mathbf{X}_{c}^{M,U}, \mathbf{X}_{d}^{M,L}\leq\mathbf{X}_{d}^{M,k}\leq\mathbf{X}_{d}^{M,U}\\
&DV=\left\{\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{c}^{k}, \mathbf{d}_{d}^{k}, \mathbf{X}_{s,c}^{M,k}, \mathbf{X}_{s,d}^{M,k}, \mathbf{X}_{c}^{M,k}, \mathbf{X}_{d}^{M,k}, \mathbf{Y}^{M,k}, \mathbf{Y}^{*}\right\}
\end{aligned}
\tag{13}
$$

where $G_{\Pi}^{(i)}$ is the modified deterministic constraint of the ith discipline. $\mathbf{S}_{j,c}^{(i),k}$, $\mathbf{S}_{j,d}^{(i),k}$ are shift vectors respectively corresponding to $\mathbf{X}_{j,c}$, $\mathbf{X}_{j,d}$ in the ith discipline for Cycle k, based on $\mathbf{X}_{j,c}^{M,(k-1)}, \mathbf{X}_{j,d}^{M,(k-1)}, \mathbf{X}_{j,c}^{*,(i),(k-1)}, \mathbf{X}_{j,d}^{*,(i),(k-1)}$. $\mathbf{X}_{j,c}^{*,(i),(k-1)}, \mathbf{X}_{j,d}^{*,(i),(k-1)}, \mathbf{P}_{j,c}^{*,(i),(k-1)}, \mathbf{P}_{j,d}^{*,(i),(k-1)}$ are MPPPs obtained in the ith discipline in Cycle (k-1) corresponding to $\mathbf{X}_{j,c}$, $\mathbf{X}_{j,d}$, $\mathbf{P}_{j,c}$, $\mathbf{P}_{j,d}$, respectively. $\mathbf{Y}_{\bullet i}^{*,(i)}$ is a vector of linking variables, corresponding to the possibility constraint in discipline i.

Because there are linking variables in each deterministic constraint $\left(G_{\Pi}^{(i)}\ i=1,2,\cdots,nd\right)$, the equality constraints for consistency among disciplines should also be modified using the MPPPs of previous cycles. For example, the formulation of $y_{jm}^{*,(i)}$ in the ith disciplinary should be modified as:

$$
\begin{aligned}
y_{jm}^{*,(i)}=y_{jm}^{*,(i)}\Big(&\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{j,c}^{k}, \mathbf{d}_{j,d}^{k}, \mathbf{X}_{s,c}^{M,k}-\mathbf{S}_{s,c}^{(i),k},\\
&\mathbf{X}_{s,d}^{M,k}-\mathbf{S}_{s,d}^{(i),k}, \mathbf{X}_{j,c}^{M,k}-\mathbf{S}_{j,c}^{(i),k}, \mathbf{X}_{j,d}^{M,k}-\mathbf{S}_{j,d}^{(i),k},\\
&\mathbf{P}_{j,c}^{*,(i),(k-1)}, \mathbf{P}_{j,d}^{*,(i),(k-1)}, \mathbf{Y}_{\bullet j}^{*,(i)}\Big)
\end{aligned}
$$

The solution of (13) is the optimal design point $\mathbf{d}_{s,c}^{k}$, $\mathbf{d}_{s,d}^{k}$, $\mathbf{d}_{c}^{k}$, $\mathbf{d}_{d}^{k}$, $\mathbf{X}_{s,c}^{M,k}$, $\mathbf{X}_{s,d}^{M,k}$, $\mathbf{X}_{c}^{M,k}$, $\mathbf{X}_{d}^{M,k}$ in Cycle k.

4.3.2 Probability/possibility analysis in the kth cycle

From (12), the mathematical formulation for probability/possibility analysis in the kth cycle is given as:

$$
\begin{aligned}
&\max_{DV} G^{(i)}\begin{pmatrix}\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{i,c}^{k}, \mathbf{d}_{i,d}^{k}, \mathbf{U}_{s,rc}^{(i),k}, \mathbf{V}_{s,frd}^{(i),k}, \mathbf{V}_{s,fc}^{(i),k},\\ \mathbf{V}_{s,fd}^{(i),k}, \mathbf{U}_{i,rc}^{(i),k}, \mathbf{V}_{i,frd}^{(i),k}, \mathbf{V}_{i,fc}^{(i),k}, \mathbf{V}_{i,fd}^{(i),k}, \mathbf{UP}_{i,rc}^{(i),k},\\ \mathbf{VP}_{i,frd}^{(i),k}, \mathbf{VP}_{i,fc}^{(i),k}, \mathbf{VP}_{i,fd}^{(i),k}, \mathbf{Y}_{\bullet i}^{(i)}\end{pmatrix}\\
&s.t.\ \left\|\left(\mathbf{U}_{s,rc}^{(i),k}, \mathbf{U}_{rc}^{(i),k}, \mathbf{UP}_{rc}^{(i),k}\right)\right\|_{2}\leq\beta_{t}\\
&\quad\left\|\left(\mathbf{V}_{s,frd}^{(i),k}, \mathbf{V}_{s,fc}^{(i),k}, \mathbf{V}_{s,fd}^{(i),k}, \mathbf{V}_{frd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{VP}_{frd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}\right)\right\|_{\infty}<1-\alpha_{t}\\
&\quad y_{jm}^{(i)}=y_{jm}^{(i)}\begin{pmatrix}\mathbf{d}_{s,c}^{k}, \mathbf{d}_{s,d}^{k}, \mathbf{d}_{j,c}^{k}, \mathbf{d}_{j,d}^{k}, \mathbf{U}_{s,rc}^{(i),k}, \mathbf{V}_{s,frd}^{(i),k},\\ \mathbf{V}_{s,fc}^{(i),k}, \mathbf{V}_{s,fd}^{(i),k}, \mathbf{U}_{j,rc}^{(i),k}, \mathbf{V}_{j,frd}^{(i),k}, \mathbf{V}_{j,fc}^{(i),k}, \mathbf{V}_{j,fd}^{(i),k},\\ \mathbf{UP}_{j,rc}^{(i),k}, \mathbf{VP}_{j,frd}^{(i),k}, \mathbf{VP}_{j,fc}^{(i),k}, \mathbf{VP}_{j,fd}^{(i),k}, \mathbf{Y}_{\bullet j}^{(i)}\end{pmatrix}\\
&\quad j, m=1,2,\cdots,nd\quad j\neq m; i=1,2,\cdots,nd\\
&DV=\left\{\mathbf{U}_{s,rc}^{(i),k}, \mathbf{V}_{s,frd}^{(i),k}, \mathbf{V}_{s,fc}^{(i),k}, \mathbf{V}_{s,fd}^{(i),k}, \mathbf{U}_{rc}^{(i),k}, \mathbf{V}_{frd}^{(i),k}, \mathbf{V}_{fc}^{(i),k}, \mathbf{V}_{fd}^{(i),k}, \mathbf{UP}_{rc}^{(i),k}, \mathbf{VP}_{frd}^{(i),k}, \mathbf{VP}_{fc}^{(i),k}, \mathbf{VP}_{fd}^{(i),k}, \mathbf{Y}^{(i)}\right\}
\end{aligned}
\tag{14}
$$

where $\mathbf{d}_{s,c}^{k}$, $\mathbf{d}_{s,d}^{k}$, $\mathbf{d}_{i,c}^{k}$, $\mathbf{d}_{i,d}^{k}$ are optimal values of deterministic design variables obtained after solving the deterministic MDO in Cycle k.

The solutions are MPPP $\left(\mathbf{U}_{s,rc}^{*,(i),k}, \mathbf{V}_{s,frd}^{*,(i),k}, \mathbf{V}_{s,fc}^{*,(i),k}, \mathbf{V}_{s,fd}^{*,(i),k}, \mathbf{U}_{rc}^{*,(i),k}, \mathbf{V}_{frd}^{*,(i),k}, \mathbf{V}_{fc}^{*,(i),k}, \mathbf{V}_{fd}^{*,(i),k}, \mathbf{UP}_{rc}^{*,(i),k}, \mathbf{VP}_{frd}^{*,(i),k}, \mathbf{VP}_{fc}^{*,(i),k}, \mathbf{VP}_{fd}^{*,(i),k}\right)$, linking variables $\mathbf{Y}^{*,(i)}$ and value of performance measure at MPPP $G^{(i)}$, $i = 1 \sim nd$. Then MPPPs in X-space $\left(\mathbf{X}_{s,rc}^{*,(i),k}, \mathbf{X}_{s,rd}^{*,(i),k}, \mathbf{X}_{s,fc}^{*,(i),k}, \mathbf{X}_{s,fd}^{*,(i),k}, \mathbf{X}_{rc}^{*,(i),k}, \mathbf{X}_{rd}^{*,(i),k}, \mathbf{X}_{fc}^{*,(i),k}, \mathbf{X}_{fd}^{*,(i),k}, \mathbf{P}_{rc}^{*,(i),k}, \mathbf{P}_{rd}^{*,(i),k}, \mathbf{P}_{fc}^{*,(i),k}, \mathbf{P}_{fd}^{*,(i),k}\right)$ $(i = 1 \sim nd)$ can be obtained using inverse transformation aforementioned, and will be used to construct deterministic constraints for next cycle if possibility constraints are not all satisfied.

From (13, 14), there are both inequality and equality constraints. Because of the discrete requirements for design variables and the existences of equality constraints, it becomes very difficult to solve the optimization problem. In this paper, it is assumed that the equality constraints can be eliminated by some technologies such as elimination method, and there are only inequality constraints left after elimination.

The discrete-continuous optimization problem is solved by the following steps. First, treating all variables including discrete ones as continuous variables, solve the optimization problem using a continuous optimization algorithm such as sequential quadratic programming (SQP) and so on, and then obtain the optimal solution. Second, eliminating all equality constraints and discretizing the design space, the method of "TRANS" in MDOD (Yu et al. 1997) is used to find a feasible discrete point based on that optimal solution, where the unit vector of the feature vector is used directly for rounding. Then the one-dimension searching and adjacent point-checking in the discrete unit area technologies (Yu et al. 1997; Chen 1989) in the algorithm MDOP (Yu et al. 1997) are utilized to find out the optimal discrete point. Finally, starting from the optimal discrete point, the original discrete-continuous optimization problem is resolved using algorithms for continuous optimization while fixing the discrete part at relevant values of that optimal discrete point. During one-dimension searching and adjacent point-checking in the discrete unit area, when a new point is obtained, the point is firstly compared with those saved. If there exists a same point, set the values of objective function and constraints the same as those saved; otherwise save the point, calculate and save its values of objective function and constraints.

5 Examples

In this section, the proposed RFCDV-MDO and the method of RFCDV-MDO-SORA are demonstrated using a mathematical problem and an engineering problem.

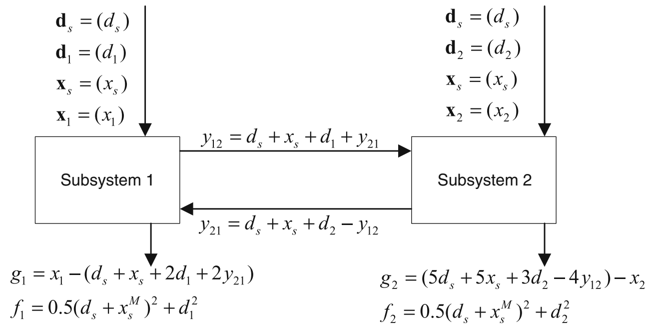

Fig. 1 Mathematical problem

5.1 Mathematical example for RFCDV-MDO

The mathematical example changed from Du et al. (2008) is given as:

$$\min_{(d_s,d_1,d_2)} f\left(\mathbf{d}, \mathbf{x}^M\right) = \left(d_s + x_s^M\right)^2 + d_1^2 + d_2^2$$

$$s.t.\ \Pi\left\{G_1\left(\mathbf{d}, \mathbf{x}\right) = x_1 - d_s - x_s - d_1 - d_2 > 0\right\} \le \alpha_t$$

$$\Pi\left\{G_2\left(\mathbf{d}, \mathbf{x}\right) = d_s + x_s - 2d_1 + d_2 - x_2 > 0\right\} \le \alpha_t$$
$$0 \le d_s, d_1, d_2 \le 5 \tag{15}$$

In this problem, d_s, d_1, d_2 take values as multiple of 0.01. $x_s \sim \begin{cases} -0.24 & p = 0.2 \\ 0 & p = 0.6 \\ 0.24 & p = 0.2 \end{cases}$. $x_1 \sim N(5, 0.5)$ where $N(\mu,\sigma)$ stands for a normal distribution with the mean value μ and the standard deviation σ. The triangular membership of x_2 is (0.7,1,1.3). In $\left(x^M - dt, x^M, x^M + dt\right)$, the value x^M is the maximal grade point of the membership function of x; the value dt is the deviation along each side from the maximal grade point. The problem is decomposed into two subsystems in Fig. 1 as Du et al. (2008).

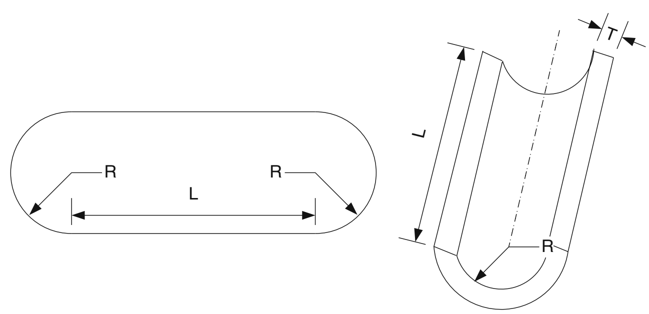

Fig. 2 Pressure vessel

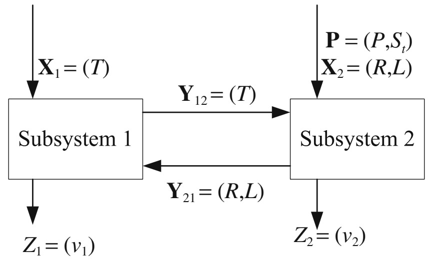

Fig. 3 System structure of pressure vessel design

It should be noted that in this problem x_s, x_1, x_2 are design parameters but not design variables because the mean value or the maximal grade point is known and fixed. The formulation of deterministic MDO is given as:

$$\begin{aligned}
&\min_{DV} f = (f_1 + f_2) = \left(d_s + x_s^M\right)^2 + d_1^2 + d_2^2 \\
&s.t.\ g_1 = x_1^{*,(1)} - \left(d_s + x_s^{*,(1)} + 2d_1 + 2y_{21}^{*,(1)}\right) \le 0 \\
&y_{12}^{*,(1)} = d_s + x_s^{*,(1)} + d_1 + y_{21}^{*,(1)} \\
&y_{21}^{*,(1)} = d_s + x_s^{*,(1)} + d_2 - y_{12}^{*,(1)} \\
&\quad g_2 = \left(5d_s + 5x_s^{*,(2)} + 3d_2 - 4y_{12}^{*,(2)}\right) - x_2^{*,(2)} \le 0 \\
&y_{12}^{*,(2)} = d_s + x_s^{*,(2)} + d_1 + y_{21}^{*,(2)} \\
&y_{21}^{*,(2)} = d_s + x_s^{*,(2)} + d_2 - y_{12}^{*,(2)} \\
&\quad 0 \le d_s, d_1, d_2 \le 5 \\
&DV = \left\{d_s, d_1, d_2, y_{12}^{*,(1)}, y_{21}^{*,(1)}, y_{12}^{*,(2)}, y_{21}^{*,(2)}\right\}
\end{aligned} \tag{16}$$

The probability/possibility analysis is carried out at the optimal point (d_s, d_1, d_2). At first, transform the continuous random parameter into standard normal random one; transform the discrete random parameter and the continuous fuzzy parameter into standard fuzzy ones. The formulation of probability/possibility analysis for G_1 is given as:

$$\begin{aligned}
&\max_{DV} g_1 = \left(5 + 0.5u_1^{(1)}\right) - \left[d_s + \left(0.6v_s^{(1)}\right) + 2d_1 + 2y_{21}^{(1)}\right] \\
&s.t.\ \left\|u_1^{(1)}\right\|_2 \le \beta_t \\
&\quad \left\|\left(v_s^{(1)}, v_2^{(1)}\right)\right\|_\infty < 1 - \alpha_t \\
&\quad y_{12}^{(1)} = d_s + \left(0.6v_s^{(1)}\right) + d_1 + y_{21}^{(1)} \\
&\quad y_{21}^{(1)} = d_s + \left(0.6v_s^{(1)}\right) + d_2 - y_{12}^{(1)} \\
&DV = \left\{v_s^{(1)}, u_1^{(1)}, v_2^{(1)}, y_{12}^{(1)}, y_{21}^{(1)}\right\}
\end{aligned} \tag{17}$$

The variable $v_s^{(1)}$ is discrete and only takes a few allowable values. The solution MPPP $\left(v_s^{*,(1)}, u_1^{*,(1)}, v_2^{*,(1)}\right)$ is then transformed into the MPPP $\left(x_s^{*,(1)}, x_1^{*,(1)}, x_2^{*,(1)}\right)$ in X-space. The formulation for G_2 can be derived in the same way with the solution $\left(x_s^{*,(2)}, x_1^{*,(2)}, x_2^{*,(2)}\right)$. The MPPP will be used to reconstruct the deterministic constraints in the deterministic MDO of the next cycle if possibility constraints are not all satisfied. The judgment of convergence is $g_i \le 0, i = 1 \sim 2$; $|f(k) - f(k-1)| \le 0.0001$. During the solution, $\|\cdot\|_\infty \le 1 - \alpha_t$ is used instead of $\|\cdot\|_\infty < 1 - \alpha_t$ for convenience in computation. Theoretically this will result to a conservative solution.

In this RFCDV-MDO problem with ($\alpha_t = 0.0013$, $\beta_t = 3 = -\Phi^{-1}(\alpha_t)$), the optimal solution of (d_s, d_1, d_2) is (2.2400,2.2600,2.2400), and the objective function value is 15.1428. The values of performance measure at MPPPs of G_1 and G_2 are -8.8818×10^{-16} and -0.5004, respectively. This indicates that both possibility constraints at the optimal design point are satisfied because the values of performance measure at their MPPPs are all not larger than zeros. RFCDV-MDO-SORA efficiently solves this

Table 1 Uncertainty descriptions of design variables and parameters

Variables or parameters	Mean value	Standard deviation	Distribution	Lower bound of mean value	Upper bound of mean value
R		0.01	Normal	0.1	36
T		0.01	Normal	0.5	6.0
L				0.1	140
St	40 Maximal grade point	4 Deviation	Normal Membership function		
P	3.89	1.167	Triangular		

Table 2 Results of pressure vessel design

	T^M	R^M	L^M	V	v_1	v_2	Number	k
First kind of shift	6.0000	33.2300	71.3000	-2.0447×10^5	1.9658×10^5	4.0105×10^5	7,016	3
Second kind of shift	6.0000	33.1600	71.4000	-2.0324×10^5	1.9615×10^5	3.9938×10^5	9,275	4

RFCDV-MDO problem in three cycles with 350 function evaluations including objective and constraints in probability/possibility analysis.

5.2 Design of a pressure vessel

The example of pressure vessel design showed in Fig. 2 is derived from (Lewis and Mistree 1997), in which the example is solved in a multi-player formulation based on game theory. The design variables are radius (R), length (L) and thickness (T). There are two parameters: internal pressure (P) and allowable tensile strength of the material (S_t). The objective is to maximize the internal volume while minimize the weight. In this paper, this problem is modified into an MDO problem.

The pressure vessel is designed by two design groups, and the coupled variables are thickness (T), length (L) and radius (R). Multidisciplinary systems and notations are illustrated in Fig. 3. Here, T, R are continuous random variables, and L is a discrete random variable. Table 1 shows uncertainty descriptions of design variables and parameters.

Due to manufacture practice, mean values of T, R are multiple of 0.01, and that of L is multiple of 0.1. When mean value of T is obtained as T^M, the practical dimension is subjected to $N(T^M, 0.01)$. The case of R is similar as T. The length L is discretely distributed according to the following probability:

$$\Pr|\{L=\tau\} = \begin{cases} 0.1 & \tau = L^M + 0.1 \\ 0.8 & \tau = L^M \\ 0.1 & \tau = L^M - 0.1 \end{cases},\ L^M \text{ is multiple of } 0.1.$$

Sharing design variables: $\mathbf{d}_s = \phi$, ϕ denotes empty set. Sharing random or fuzzy continuous or discrete variables: ϕ.

Subsystem 1:

Random variable: $\mathbf{X}_1 = \{T\}$.

Input linking variables: $\mathbf{Y}_{21} = \left\{y_{21,1}, y_{21,2}\right\} = \{R,\ L\}$.

Output linking variables: $\mathbf{Y}_{12} = \{y_{12}\} = \{T\}$.

Output: $Z_1 = \{v_1\}$. $v_1 = \frac{4}{3}\pi(T^M + y^M_{21,1})^3 + \pi(T^M + y^M_{21,1})^2 y^M_{21,2} - \left[\frac{4}{3}\pi(y^M_{21,1})^3 + \pi(y^M_{21,1})^2 y^M_{21,2}\right]$. In this subsystem, the objective is to minimize the weight, which is equal to minimize the relevant volume.

The constraints in Subsystem 1 are as follows:

The possibility constraints are those:

$$\begin{array}{l} \Pi\left\{G_{11} = 5T - y_{21,1} > 0\right\} \le \alpha_t \\ \Pi\left\{G_{12} = T + y_{21,1} - 40 > 0\right\} \le \alpha_t \end{array}.$$

Subsystem 2:

Random variable: $\mathbf{X}_2 = \{R, L\}$.

Input fuzzy and random parameters: $\mathbf{P} = \{P, S_t\}$.

Input linking variables: $\mathbf{Y}_{12} = \{y_{12}\} = \{T\}$.

Output linking variables: $\mathbf{Y}_{21} = \left\{y_{21,1}, y_{21,2}\right\} = \{R, L\}$.

Output: $Z_2 = \{v_2\}$. $v_2 = \frac{4}{3}\pi(R^M)^3 + \pi(R^M)^2 L^M$. In this subsystem, the objective is to maximize internal volume.

The constraints in Subsystem 2 are as follows:

The possibility constraints are those:

$$\begin{array}{l} \prod\left\{G_{21} = \dfrac{PR}{y_{12}} - S_t > 0\right\} \le \alpha_t \\ \Pi\left\{G_{22} = L + 2R + 2y_{12} - 150 > 0\right\} \le \alpha_t \end{array}.$$

The whole objective v is to minimize $v_1 - v_2$. The target possibility of failure is $\alpha_t = 0.0013 = 1 - 0.9987$.

In the computational process, the starting points in the current cycle are set to be the results of the last cycle to improve efficiency. The optimum results with the first and second kind of shift vector are given in Table 2. The eighth column lists the total number of function evaluations of objective and constraints in probability/possibility analysis. The whole process with the first kind of shift vector converges in three cycles while four cycles with the second kind of shift vector. The values of performance measure of possibility constraint functions at relevant MPPPs with different shift vectors are listed in Table 3. All values of performance

Table 3 Value of performance measure at MPPP

	G_{11}	G_{12}	G_{21}	G_{22}
First kind of shift	−3.0770	−0.7276	-2.0724×10^{-4}	−0.0551
Second kind of shift	−3.0070	−0.7976	−0.0592	−0.0951

measure at MPPPs are less than zeros which indicates that the requirements of possibility of failure are all satisfied.

The optimal design obtained using the first kind of shift vector is better than that of the second kind of shift vector. The reason is that from the second cycle, the feasible area of the reconstructed deterministic MDO with the second kind of shift vector is narrower than that of the first kind because the shiftiness is larger than that of the first one. The aim of the second kind of shift vector is to avoid this situation: when there are discrete requirements on design variables and shiftiness of deterministic constraints, some equality constraints especially with the even power could not be satisfied.

6 Conclusions

This paper proposes the formulation of RFCDV (Random/Fuzzy Continuous/Discrete Variables) Multidisciplinary Design Optimization (RFCDV-MDO), uncertainty analysis for RFCDV-MDO, and a method of RFCDV-MDO within the framework of Sequential Optimization and Reliability Assessment (RFCDV-MDO-SORA) to deal with RFCDV-MDO problems.

Two kinds of uncertainty, Aleatory Uncertainty (AU) and Epistemic Uncertainty (EU), and both continuous and discrete variables and parameters are considered. Based on the conditional possibility of failure, this paper proposes the performance measure approach of probability/possibility analysis by transforming the discrete random variables and parameters into discrete fuzzy ones with the purpose of avoiding the enormous computational price.

In RFCDV-MDO-SORA, the solution of RFCDV-MDO problem is decoupled into deterministic MDO and probability/possibility analysis sequentially. In the proposed RFCDV-MDO-SORA with the second kind of shift vector, the shiftiness of variable whose mean value or maximal grade point is continuous is equal to the value of its mean value or maximal grade point subtracted by relevant MPPP of the previous cycle. The shiftiness of variable whose mean value or maximal grade point is discrete should be expanded according to the value of its mean value or maximal grade point subtracted by relevant MPPP of the previous cycle and discrete increment or decrement. In RFCDV-MDO-SORA with the first kind of shift vector, the shiftiness of variable is equal to the value of its mean value or maximal grade point subtracted by relevant MPPP of the previous cycle. From the examples, RFCDV-MDO-SORA can solve RFCDV-MDO problem efficiently in a few cycles. The RFCDV-MDO-SORA with the first kind of shift vector is more efficient than that with the second one in the second example.

Future works will develop an efficient algorithm to directly deal with the discrete-continuous optimization with equality constraints, more precise measure to simultaneously deal with AU and EU, and also more efficient framework to solve RFCDV-MDO problems.

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