Struct Multidisc Optim (2011) 44:691–705
DOI 10.1007/s00158-011-0652-9

RESEARCH PAPER

Reliability sensitivity analysis for structural systems in interval probability form

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Received: 1 August 2010 / Revised: 20 March 2011 / Accepted: 30 March 2011 / Published online: 7 May 2011

Abstract Reliability sensitivity analysis is used to find the rate of change in the probability of failure (or reliability) due to the changes in distribution parameters such as the means and standard deviations. Most of the existing reliability sensitivity analysis methods assume that all the probabilities and distribution parameters are precisely known. That is, every statistical parameter involved is perfectly determined. However, there are two types of uncertainties, epistemic and aleatory uncertainties that may not be perfectly determined in engineering practices. In this paper, both epistemic and aleatory uncertainties are considered in reliability sensitivity analysis and modeled using P-boxes. The proposed method is based on Monte Carlo simulation (MCS), weighted regression, interval algorithm and first order reliability method (FORM). We linearize original non-linear limit-state function by MCS rather than by expansion as a first order Taylor series at most probable point (MPP) because the MPP search is an iterative optimization process. Finally, we introduce an optimization model for sensitivity analysis under both aleatory and epistemic uncertainties. Four numerical examples are presented to demonstrate the proposed method.

Keywords Reliability sensitivity analysis · Epistemic uncertainty · Aleatory uncertainty · P-box · Interval form · Weighted regression

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1 Introduction

In reliability analysis and reliability-based design, sensitivity analysis identifies the relationship between the change in reliability and the change in the characteristics of uncertain variables. Sensitivity analysis is also used to identify the most significant uncertain variables that have the highest contribution to reliability (Guo and Du 2009). Furthermore, sensitivity analysis could be used to provide information for the reliability-based design. If the reliability of a design does not meet requirements, there are several useful ways to improve the reliability, including (1) change of the mean values of random variables, (2) change of the variances of random variables, and (3) truncation of the distributions of random variables (Du 2005). Sensitivity analysis can provide the information about which random variables are the most significant and therefore, in an effective manner, should be changed in order to improve reliability. Reliability sensitivity analysis has played a key role in structural reliability design. A number of sensitivity analysis methods exist in literature. Among them, Ditlevsen and Madsen (2007) presented an expression based on the first order reliability method (FORM) to evaluate reliability sensitivity for a structural system with linear limit-state and normal random variables. De-Lataliade et al. (2002) developed a method using Monte Carlo simulation (MCS) for reliability sensitivity estimations. Ghosh et al. (2001) proposed a method using the first order perturbation for stochastic sensitivity analysis. For more information, please refer to references (Mundstok and Marczak 2009; Xing et al. 2009; Au 2005; Rahman and Wei 2008; Liu et al. 2006). In the aforementioned literatures, most of methods assume that all the stochastic characteristics of random variables are precisely known. That means that every random parameter involved is perfectly determined. However, in reality, this assumption is unrealistic because both types of uncertainties, epistemic and aleotary uncertainties, often exist in engineering practice (Kiureqhian and Ditlevsen 2009; Kiureghian 2008; Huang and Zhang 2009; Huang and He 2008; Zhang et al. 2010a, b; Zhang and Huang 2010). Epistemic uncertainty is derived from incomplete information, less sampling data or ignorance while aleatory uncertainty comes from inherent variations (Du 2008). Approaches to describe the epistemic uncertainty in structural reliability analysis include the interval analysis (Merlet 2009; Kokkolaras et al. 2006), evidence theory (Christophe and Philippe 2009), possibility theory (Du et al. 2006; Nikolaidis et al. 2004; Zhou and Mourelatos 2008), imprecise probability (Aughenbaugh and Herrmann 2009), fuzzy theory and P-boxes models (Tanrioven et al. 2004; Karanki et al. 2009). Aleatory uncertainty is usually modeled using the probability theory. Data error is inevitable due to multiple contributions from machine errors, human errors or other unexpected situations. For example, a parameter is random subject to a normal distribution while the mean value and standard deviation can not be precisely determined. Therefore, in structural reliability sensitivity analysis, a unified uncertainty analysis method is needed to model both epistemic and aleatory uncertainties.

Research efforts have been made these days on reliability sensitivity analysis when both epistemic and aleatory uncertainties are present in engineering systems (Guo and Du 2007, 2009). In this paper, the parameters with sufficient information are modeled using probability distributions while others are modeled by a pair of upper and lower cumulative distributions (the so-called P-box). P-boxes (Utkin and Destercke 2009; Tucker and Ferson 2003) are one of the simplest and the most popular models of sets of probability distributions, directly extended from cumulative distributions used in the precise case. Assume that the information about a random variable X is represented by a lower bound $\underline{F}$ and an upper bound $\overline{F}$, the cumulative distribution function can be defined based on the P-box bound $[\underline{F}, \overline{F}]$. Thus, the lower bound $\underline{F}$ and the upper bound $\overline{F}$ distributions define a set $\varphi(\underline{F}, \overline{F})$ of precise distributions such that (Utkin and Destercke 2009; Tucker and Ferson 2003)

$$\varphi\left(\underline{F}, \overline{F}\right) = \left\{F | \forall X \in R_v, \underline{F}(X) \le F(X) \le \overline{F}(X)\right\} \tag{1}$$

where R_v denotes a set of real numbers.

In the interval form, F^I is used to denote the P-box $[\underline{F}, \overline{F}]$, i.e.

$$F^I = \left[\underline{F}, \overline{F}\right] = \left\{\underline{F} \le F \le \overline{F}\right\} \tag{2}$$

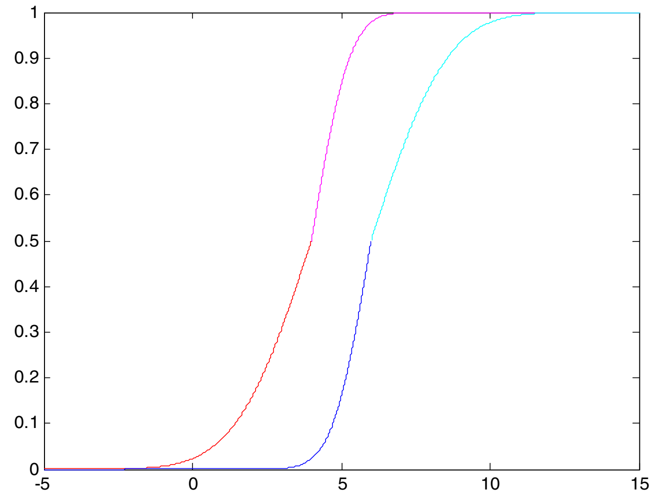


Fig. 1 CDF of parameter X

The P-box of a distribution is a closed-form function or figure. For example, $X \sim N([4, 6], [1, 2])$ represents that X is normally distributed but its mean and standard deviation value is not known precisely. However, its mean value is known to locate in the interval [4, 6] and its standard deviation is within the interval [1, 2]. The cumulative distribution function (CDF) of parameter X is shown in Fig. 1.

In literature, there are studies on sensitivity analysis using P-boxes (Ferson and Tucker 2006a, b; Hall 2006). However, the proposed sensitivity analysis methods are slightly different from reliability sensitivity analysis. Furthermore, Monte Carlo simulation (MCS) based methods were used widely in the reliability sensitivity analysis, for example, De-Lataliade et al. (2002) proposed a sensitivity estimation method based on MCS. Melchers and Ahammed (2004) proposed an efficient method for parameter sensitivity estimation based on MCS. However, these methods can only model aleatory uncertainty rather than both epistemic and aleatory uncertainties. In this paper, by integrating the principle of P-box, interval arithmetic, FORM, MCS, weighted regression, and the works in references (De-Lataliade et al. 2002; Melchers and Ahammed 2004), we propose a unified reliability sensitivity estimation method under both epistemic and aleatory uncertainties.

This paper is organized as follows. Section 2 provides a brief background about the structural reliability and FORM. Structural reliability analysis in the interval form is presented in Section 3. Section 4 proposes a method for system reliability sensitivity analysis in interval form. Four numerical examples are presented in Section 5. Brief discussion and conclusion are presented to close the paper.

2 Reliability analysis by FORM

In the system reliability analysis, the system reliability R and probability of failure P_f are defined as (Melchers 1999)

$$R = P\left[G\left(\mathbf{X}\right) \geq 0\right] \tag{3}$$

and

$$P_f = 1 - R = P\left[G\left(\mathbf{X}\right) < 0\right] \tag{4}$$

respectively, where $P[\cdot]$ denotes a probability, $G(\cdot)$ is a performance function, and $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is the vector of random variables.

f_x denotes the joint probability density function (PDF) of $\mathbf{X}$, the probability of failure is calculated by the integral

$$P_f = P\left[G\left(\mathbf{X}\right) < 0\right] = \int_{G(\mathbf{x})<0} f_{\mathbf{X}}\left(\mathbf{x}\right) d\mathbf{x} \tag{5}$$

The direct evaluation of the probability integration in (5) is extremely difficult. The main reasons are given in (Du 2005), including:

1. Since there are n random variables in the performance function, the probability integration is multidimensional.
2. The performance function $G(\mathbf{X})$ is usually a non-linear function of $\mathbf{X}$, and therefore the integration boundary is also non-linear.

MCS (Ditlevsen and Madsen 2007; Melchers 1999; Melchers and Ahammed 2004) can be used to evaluate the integration in (5). However, the main disadvantage of MCS is that it needs a large number of samples in simulation. So the MCS is time-consuming, especially for small probability of failure. Classical reliability estimation methods, such as FORM (Du 2005; Melchers 1999; Haldar and Mahadevan 2001; Koduru and Haukaas 2010) and second order reliability method (SORM) (Du 2005; Ditlevsen and Madsen 2007; Melchers 1999; Hohenbichler et al. 1987), are widely used for engineering reliability analysis and reliability-based design for the good balance between accuracy and efficiency. The performance function $G(\mathbf{X})$ is transformed into the U-space (standard normal space), denoted as $\hat{G}(\mathbf{U})$. The first order Taylor expansion of $\hat{G}(\mathbf{U})$ can be expressed as (Du 2005; Melchers 1999)

$$\hat{G}\left(\mathbf{U}\right) \approx \hat{G}_L\left(\mathbf{U}\right) = \hat{G}\left(\mathbf{u}^*\right) + \sum_{i=1}^{n} \left.\frac{\partial \hat{G}}{\partial U_i}\right|_{\mathbf{u}^*} \left(U_i - u_i^*\right) \tag{6}$$

where $\hat{G}_L(\mathbf{U})$ is the linearized performance function, and $\mathbf{u}^* = (u_1^*, u_2^*, \cdots, u_n^*)$ is the expansion point. Assume all random variables are statistically independent. Typical FORM involves the three steps to estimate the probability of failure P_f (Guo and Du 2009; Du 2005; Ditlevsen and Madsen 2007).

1. Transform the random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ in X-space into standard normal vector $\mathbf{U} = (U_1, U_2, \cdots, U_n)$ in U-space by Rosenblatt transformation (Ditlevsen and Madsen 2007; Melchers 1999)

$$U_i = \Phi^{-1}\left[F_{X_i}\left(X_i\right)\right] \tag{7}$$

where $\Phi^{-1}[\cdot]$ is the inverse CDF of the standard normal distribution, and $F_{X_i}(X_i)$ is the CDF of X_i.

2. Find the most probable point (MPP). The MPP $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots, u_n^*\right)$ search is an iterative optimization process

$$\begin{cases} \min\limits_{\mathbf{u}} \beta = \|\mathbf{u}\| \\ s.t. \hat{G}\left(\mathbf{u}\right) = 0 \end{cases} \tag{8}$$

where $\|\cdot\|$ denotes the magnitude of a vector and β is called the reliability index.

3. Calculate the probability of failure by the following equation

$$P_f = P\left[G\left(\mathbf{X}\right) < 0\right] \approx \Phi\left(-\beta\right) \tag{9}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

Generally, the main work of FORM is the MPP search. From (6), reliability index β can also be expressed as (Du 2005)

$$\beta = \left(\frac{\mu_{\hat{G}_L(\mathbf{U})}}{\sigma_{\hat{G}_L(\mathbf{U})}}\right) \tag{10}$$

where $\mu_{\hat{G}_L(\mathbf{U})} = -\sum\limits_{i=1}^{n} \left.\frac{\partial \hat{G}}{\partial U_i}\right|_{\mathbf{u}^*} u_i^*$ is the mean value of the function $G_L(\mathbf{U})$, and $\sigma_{\hat{G}_L(\mathbf{U})} = \left[\sum\limits_{i=1}^{n} \left(\left.\frac{\partial \hat{G}}{\partial U_i}\right|_{\mathbf{u}^*}\right)^2\right]^{\frac{1}{2}}$ is the standard deviation of the function $\hat{G}_L(\mathbf{U})$.

3 Structural reliability in interval form

Let the random vector $\mathbf{X} = (X_1, X_2, X_3, \cdots, X_n)$ represent all basic variables which define and characterize the behavior and safety of the structure, and let $G(\mathbf{X})$ represents the performance function of the system. Typical examples of random variables included in the vector of $\mathbf{X}$ are dimensions, densities or unit weights, material, loads, material strengths, and so on. In structural reliability analysis, if there is incomplete information or ignorance about a parameter, the parameter could be modeled using P-boxes as $X_j^I \sim N([\underline{\mu}_{x_j}, \overline{\mu}_{x_j}], [\underline{\sigma}_{x_j}, \overline{\sigma}_{x_j}])$. Then these i parameters can be denoted by $\mathbf{X}_i^I = \left(X_1^I, X_2^I, X_3^I, \cdots, X_i^I\right)$. The other $(n-i)$ parameters $\mathbf{X}_i = (X_{i+1}, X_{i+2}, X_{i+3}, \cdots, X_n)$ which we have sufficient information about are modeled with precise probability distribution such as $X_j \sim N\left(\mu_{X_j}, \sigma_{X_j}\right)$. The parameters of system can be expressed by

$$\mathbf{X}^I = \left(\mathbf{X}_i^I, \mathbf{X}_i\right) = \left(X_1^I,\ X_2^I, \cdots, X_i^I; X_{i+1}, X_{i+2}, \cdots, X_n\right) \tag{11}$$

According to (11), the corresponding performance function can be expressed as $G(\mathbf{X}^I)$. Because both epistemic and aleatory uncertainties are present in a system, the probability of failure is an interval rather than a precise value. From (5), the probability of failure P_f^I in an interval form is given by

$$P_f^I = P\left[G\left(\mathbf{X}^I\right) < 0\right] = \int_{G(\mathbf{x}^I)<0} f_{\mathbf{X}}\left(\mathbf{x}^I\right) d\mathbf{x} \tag{12}$$

where $f_{\mathrm{x}}(\mathbf{x}^I)$ is the joint PDF of the n-dimensional vector $\mathbf{X}^I$ of basic random variables.

From (12), the probability of failure P_f^I also can be calculated as

$$P_f^I = \left[\underline{P}_f, \overline{P}_f\right] = \left[\min\left(\int_{G(\mathbf{x}^I)<0} f_{\mathbf{X}}\left(\mathbf{x}^I\right) d\mathbf{x}\right),\right.$$
$$\left.\max\left(\int_{G(\mathbf{x}^I)<0} f_{\mathbf{X}}\left(\mathbf{x}^I\right) d\mathbf{x}\right)\right] \tag{13}$$

where $\underline{P}_f$ and $\overline{P}_f$ are the lower and upper bounds of P_f^I, respectively.

From (13), in order to keep the consistency, the following constraint must be satisfied

$$0 \le \underline{P}_f \le \overline{P}_f \le 1 \tag{14}$$

As discussed in Section 2, the calculation of the integrals in (5) and (13) are very difficult. The FORM can be used to acquire an approximate solution which assumes the performance function to be linear. However, in practice, the performance function $G(\mathbf{X})$ usually is of a more complex form than linear functions. Furthermore, random variables X_j $(1, 2, \cdots, n)$ are usually not independent or normally distributed. In practice, a dependent non-normal basic random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ can be transformed into an independent standardized normal distributed random vector $\mathbf{U} = (U_1, U_2, \cdots, U_n)$ through the Rosenblatt (Ditlevsen and Madsen 2007; Melchers 1999) transformation $\mathbf{U} = T\mathbf{X}$ by $U_1 = F_i(X_i|X_1, X_2, \cdots, X_{i-1})$, where F_i is the conditional CDF of random vector $\mathbf{X}$. In order to use the FORM, a practical approach is to use linearized $G_L(\mathbf{X})$ by expanding $G(\mathbf{X})$ as a first-order Taylor series at the MPP. However, the expansion of $G(\mathbf{X})$ at MPP is a very challenging because the MPP $\mathbf{u}^* = \left(u_1^*, u_2^*, \cdots, u_n^*\right)$ search is an iterative optimization process (Guo and Du 2009; Du 2005), and sometimes there are more than one MPPs or the MPP search process does not converge. Furthermore, when both P-boxes and precise distributions are presented in a system, the MPP search is more difficult than that has only precise distributions. There is a very limited number of literature in describing how to find the MPP with both precise distribution and interval variables under consideration (Du 2008; Du et al. 2005). In Du et al. (2005), when both precise distributions and interval variables exist in a system, the MPP search is the double optimization process. The derivatives of a non-linear function with a number of variables can be very complicated. Therefore, linearization of $G(\mathbf{X})$ to $G_L(\mathbf{X})$ at MPP could not be the best approach because it is not so robust in such a case. A more robust way to linearize $G(\mathbf{X}) = 0$ may be the sampling method. As being efficient, we should consider those sample points which contribute the most to the probability density or the maximum likelihood of a limit-state function. Those important sample points are usually around or near limit-state function (Melchers 1999; Melchers and Ahammed 2004). The method to linearize $G(\mathbf{X}) = 0$ to $G_L(\mathbf{X}) = 0$ through MCS involves the following steps:

1. Generate k sample points $\mathbf{x}_l (l = 1, 2, \cdots, k)$ by MCS.
2. Calculate the function values of $G(\mathbf{x}_l)$.
3. Give a constraint such as $-\infty < G(\mathbf{x}_l) < 0$.
4. Assume there are m samples which satisfy the constraint, i.e., $-\infty < G(\mathbf{x}_j) < 0$, $(j = 1, 2, \cdots, m)$. Carry out the weighted regression analysis on function values $G(\mathbf{x}_j)$, we can acquire a linear tangent plane $G_L(\mathbf{X}) = 0$ of the original limit-state function $G(\mathbf{X}) = 0$.

Generally, consider a linear function which is given by

$$G_L(\mathbf{X}) = a_0 + \sum_{j=1}^{n} a_j X_j \tag{15}$$

The coefficients of the function determined by the normal linear regression can be expressed as

$$\mathbf{a} = \left(\mathbf{x}_D^T \mathbf{x}_D\right)^{-1} \mathbf{x}_D^T \mathbf{G} \tag{16}$$

where $\mathbf{x}_D$ is the design matrix given by the m samples, i.e.

$$\mathbf{x}_D = \begin{bmatrix} 1 & x_1(1) & \cdots & x_n(1) \\ 1 & x_1(2) & \cdots & x_n(2) \\ \vdots & & & \vdots \\ 1 & x_1(m) & \cdots & x_n(m) \end{bmatrix} \tag{17}$$

In the normal regression analysis, all m samples are equally weighted. However, the sample point near the limit state is more important than others. The coefficients of the function are determined by the weighted linear regression expressed as (Kaymaz and Mcmahon 2005)

$$\mathbf{a} = \left(\mathbf{x}_D^T \mathbf{w} \mathbf{x}_D\right)^{-1} \mathbf{x}_D^T \mathbf{w} \mathbf{G} \tag{18}$$

where $\mathbf{w}$ is an $m \times m$ diagonal matrix of weights which is given by

$$\mathbf{w}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x}_1) & & & \mathbf{0} \\ & w(\mathbf{x}_2) & & \\ & & \ddots & \\ \mathbf{0} & & & w(\mathbf{x}_m) \end{bmatrix} \tag{19}$$

The following expression is suitable to obtain the weight for each sample point (Kaymaz and Mcmahon 2005)

$$w\left(\mathbf{x}_j\right) = \exp\left[-\frac{\left|G\left(\mathbf{x}_j\right)\right| - g_{worst}}{g_{worst}}\right] \tag{20}$$

$$g_{worst} = \max\left|G\left(\mathbf{x}_j\right)\right|, \; j = 1, 2, \cdots, m \tag{21}$$

5. Use the linearized tangent function $G_L(\mathbf{X})$ to replace the original function $G(\mathbf{X})$ in the reliability analysis.

The limit-state function $G(\mathbf{X}) = 0$ and its linearized function $G_L(\mathbf{X}) = 0$ in the two dimensional space are shown in Fig. 2.

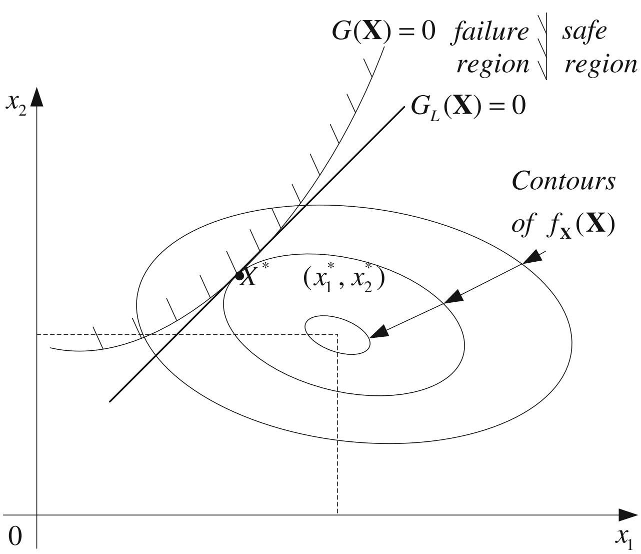


Fig. 2 Limit-state function and its linearized function with two basic variables

For an illustration purpose, we assume that all random variables are normally distributed and mutually independent in this paper. From the weighted regression analysis, the linearized performance function $G_L(\mathbf{X})$ can be written in the same form as (15).

According to (10) and (15), the reliability index is given by

$$\beta = \frac{\mu_{G_L}}{\sigma_{G_L}} \tag{22}$$

where

$$\mu_{G_L} = a_0 + \sum_{j=1}^{n} a_j \mu_{X_j} \tag{23}$$

and

$$\sigma_{G_L} = \sqrt{\sum_{j=1}^{n} \left(a_j \sigma_{X_j}\right)^2} \tag{24}$$

From the discussion above, (10), (11), (15), and (22), (23), (24), for the linearized limit-state function $G_L(\mathbf{X}) = 0$, when the random variables and P-boxes are present in the system. The system probability of failure and reliability index derived from interval-valued probabilities becomes

$$P_f^I = \left[\underline{P}_f, \overline{P}_f\right] = \Phi\left(-\beta^I\right) \tag{25}$$

where

$$\beta^I = \left[\underline{\beta}, \overline{\beta}\right] \tag{26}$$

and

$$\underline{\beta} = \frac{\underline{\mu}_{G_L}}{\overline{\sigma}_{G_L}} = \frac{a_0 + \sum_{j=1}^{i} a_j \underline{\mu}_{X_j} + \sum_{j=i+1}^{n} a_j \mu_{X_j}}{\sqrt{\sum_{j=1}^{i} \left(a_j \overline{\sigma}_{X_j}\right)^2 + \sum_{j=i+1}^{n} \left(a_j \sigma_{X_j}\right)^2}} \tag{27}$$

and

$$\overline{\beta} = \frac{\overline{\mu}_{G_L}}{\underline{\sigma}_{G_L}} = \frac{a_0 + \sum_{j=1}^{i} a_j \overline{\mu}_{X_j} + \sum_{j=i+1}^{n} a_j \mu_{X_j}}{\sqrt{\sum_{j=1}^{i} \left(a_j \underline{\sigma}_{X_j}\right)^2 + \sum_{j=i+1}^{n} \left(a_j \sigma_{X_j}\right)^2}} \tag{28}$$

respectively. Here, for a parameter $X_j \sim N([\underline{\mu}_{X_j}, \overline{\mu}_{X_j}], [\underline{\sigma}_{X_j}, \overline{\sigma}_{X_j}])$, the midpoint values of intervals $[\underline{\mu}_{X_j}, \overline{\mu}_{X_j}]$, $[\underline{\sigma}_{X_j}, \overline{\sigma}_{X_j}]$ are expressed as $\tilde{\mu}_{X_j} = \frac{\underline{\mu}_{X_j} + \overline{\mu}_{X_j}}{2}$, and

$\tilde{\sigma}_{X_j} = \frac{\underline{\sigma}_{X_j}+\overline{\sigma}_{X_j}}{2}$. The interval $[\underline{\mu}_{X_j}, \overline{\mu}_{X_j}]$ of parameter X_j can be expressed by $[\tilde{\mu}_{X_j} - \Delta\mu_{X_j},\ \tilde{\mu}_{X_j} + \Delta\mu_{X_j}]$, where $\Delta\mu_{X_j} = \frac{\overline{\mu}_{X_j}-\underline{\mu}_{X_j}}{2}$. Likewise, the expression of the standard deviation could be obtained. Now, we use two variation coefficients which are defined as

$$\varepsilon_j = \frac{\Delta\mu_{X_j}}{\tilde{\mu}_{X_j}}, \xi_j = \frac{\Delta\sigma_{X_j}}{\tilde{\sigma}_{X_j}} \tag{29}$$

The variation vector of $\mathbf{X}^I = (\mathbf{X}_i^I, \mathbf{X}_i) = (X_1^I,\ X_2^I, \cdots, X_i^I; X_{i+1}, X_{i+2}, \cdots, X_n)$ can be expressed as

$$\varepsilon = [\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3, \cdots, \varepsilon_i, 0, 0, 0, \cdots, 0] \tag{30}$$

and

$$\xi = [\xi_1,\ \xi_2,\ \xi_3, \cdots, \xi_i, 0, 0, \cdots, 0] \tag{31}$$

respectively. From (29), (30), (31), the mean and deviation of a parameter $X_j \sim N([\underline{\mu}_{X_j}, \overline{\mu}_{X_j}], [\underline{\sigma}_{X_j}, \overline{\sigma}_{X_j}])$ in interval form become

$$\mu_{X_j}^I = \left[\underline{\mu}_{X_j}, \overline{\mu}_{X_j}\right] = \left[\tilde{\mu}_{X_j}\left(1-\varepsilon_j\right), \tilde{\mu}_{X_j}\left(1+\varepsilon_j\right)\right] \tag{32}$$

and

$$\sigma_{X_j}^I = \left[\underline{\sigma}_{X_j}, \overline{\sigma}_{X_j}\right] = \left[\tilde{\sigma}_{X_j}\left(1-\xi_j\right), \tilde{\sigma}_{X_j}\left(1+\xi_j\right)\right] \tag{33}$$

respectively.

From (27), (28) and (32), (33), the lower bound and upper bound of reliability indexes can be expressed as

$$\underline{\beta} = \frac{\underline{\mu}_{G_L}}{\overline{\sigma}_{G_L}} = \frac{a_0 + \sum\limits_{j=1}^{i} a_j \tilde{\mu}_{X_j}\left(1-\varepsilon_j\right) + \sum\limits_{j=i+1}^{n} a_j \mu_{X_j}}{\sqrt{\sum\limits_{j=1}^{i}\left[a_j \tilde{\sigma}_{X_j}\left(1+\xi_j\right)\right]^2 + \sum\limits_{j=i+1}^{n}\left(a_j \sigma_{X_j}\right)^2}} \tag{34}$$

and

$$\overline{\beta} = \frac{\overline{\mu}_{G_L}}{\underline{\sigma}_{G_L}} = \frac{a_0 + \sum\limits_{j=1}^{i} a_j \tilde{\mu}_{X_j}\left(1+\varepsilon_j\right) + \sum\limits_{j=i+1}^{n} a_j \mu_{X_j}}{\sqrt{\sum\limits_{j=1}^{i}\left[a_j \tilde{\sigma}_{X_j}\left(1-\xi_j\right)\right]^2 + \sum\limits_{j=i+1}^{n}\left(a_j \sigma_{X_j}\right)^2}} \tag{35}$$

respectively.

4 Reliability sensitivity analysis in interval form

Reliability sensitivity analysis is used to find the rate of change in the probability of failure due to the changes in the parameters, such as means and standard deviations, of distributions (Du 2005). Furthermore, sensitivity analysis could be used to provide information for the reliability-based design. As discussed in previous sections, we can linearize a non-linear limit-state function by simulation.

The reliability sensitivity of a parameter X_j is usually defined as (Guo and Du 2009; Du 2005; Melchers and Ahammed 2004)

$$\left(\frac{\partial P_f}{\partial \mu_{X_j}}, \frac{\partial P_f}{\partial \sigma_{X_j}}\right) \tag{36}$$

where P_f is the probability of failure; μ_{X_j} and σ_{X_j} are the mean value and standard deviation of parameter X_j, respectively.

From (9), (15), and the chain rule of partial derivative, the reliability sensitivities $\left(\frac{\partial P_f}{\partial \mu_{X_j}}, \frac{\partial P_f}{\partial \sigma_{X_j}}\right)$ can be expressed as

$$\frac{\partial P_f}{\partial \mu_{X_j}} = \frac{\partial P\left[G_L\left(\mathbf{X}\right)<0\right]}{\partial \mu_{X_j}} = \frac{\partial \Phi\left(-\beta\right)}{\partial \mu_{X_j}} = \frac{\partial \Phi\left(-\beta\right)}{\partial \beta}\frac{\partial \beta}{\partial \mu_{X_j}} \tag{37}$$

and

$$\frac{\partial P_f}{\partial \sigma_{X_j}} = \frac{\partial P\left[G_L\left(\mathbf{X}\right)<0\right]}{\partial \sigma_{X_j}} = \frac{\partial \Phi\left(-\beta\right)}{\partial \sigma_{X_j}} = \frac{\partial \Phi\left(-\beta\right)}{\partial \beta}\frac{\partial \beta}{\partial \sigma_{X_j}} \tag{38}$$

respectively.

Since all random variables are normally distributed, the partial derivatives of $\frac{\partial \Phi(-\beta)}{\partial \beta}$ can be calculated as

$$\frac{\partial \Phi\left(-\beta\right)}{\partial \beta} = \frac{\partial\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\beta} e^{-\frac{1}{2}x^2}dx\right)}{\partial \beta} = -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\beta^2} \tag{39}$$

The partial derivatives of $\frac{\partial \beta}{\partial \mu_{X_j}}$ and $\frac{\partial \beta}{\partial \sigma_{X_j}}$ can be calculated as (Du 2005)

$$\frac{\partial \beta}{\partial \mu_{X_j}} = \frac{\partial\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)}{\partial \mu_{X_j}} = \frac{a_j}{\sigma_{G_L}} \tag{40}$$

and

$$\frac{\partial \beta}{\partial \sigma_{X_j}} = \frac{\partial\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)}{\partial \sigma_{X_j}} = -\frac{a_j^2 \mu_{G_L}\sigma_{X_j}}{\sigma_{G_L}^3} \tag{41}$$

respectively. Based on (37), (38) and (40), (41), an approximate approach to calculate the system reliability sensitivities $\frac{\partial P_f}{\partial \mu_{X_j}}$ and $\frac{\partial P_f}{\partial \sigma_{X_j}}$ becomes (Melchers and Ahammed 2004)

$$\begin{aligned}\frac{\partial P_f}{\partial \mu_{X_j}} &\approx \frac{\partial P\left[G_L\left(\mathbf{X}\right)<0\right]}{\partial \mu_{X_j}} \\ &= \frac{\partial \Phi\left(-\beta\right)}{\partial \beta}\frac{\partial \beta}{\partial \mu_{X_j}} = -\frac{a_j}{\sqrt{2\pi}\sigma_{G_L}}\exp\left[-\frac{1}{2}\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)^2\right]\end{aligned} \tag{42}$$

Table 1 Distribution details of random variables

Variable	Mean	Standard deviation	Variation coefficient	Distribution
X_1	40	5	(ε_1, ξ_1)	Normal
X_2	50	2.5	(ε_2, ξ_2)	Normal
X_3	1,000	200	(ε_3, ξ_3)	Normal

and

$$\frac{\partial P_f}{\partial \sigma_{X_j}} \approx \frac{\partial P\left[G_L(\mathbf{X}) < 0\right]}{\partial \sigma_{X_j}} = \frac{\partial \Phi(-\beta)}{\partial \beta}\frac{\partial \beta}{\partial \sigma_{X_j}} = \frac{a_j^2 \mu_{G_L} \sigma_{X_j}}{\sqrt{2\pi}\sigma_{G_L}^3}\exp\left[-\frac{1}{2}\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)^2\right] \tag{43}$$

respectively.

As discussed in Section 3, when both epistemic and aleatory uncertainties are considered, reliability sensitivity of X_j is mathematically an interval with its lower and upper bounds. From (25), (42) and (43), and from the interval operation, the lower and upper bounds of reliability sensitivity of each parameter can be expressed as two optimization problems

$$\frac{\partial P_f^I}{\partial \mu_{X_j}} = \left[\min\left(\frac{\partial P_f^I}{\partial \mu_{X_j}}\right), \max\left(\frac{\partial P_f^I}{\partial \mu_{X_j}}\right)\right] \tag{44}$$

$$\min(\max)\left(\frac{\partial P_f^I}{\partial \mu_{X_j}}\right) = \min(\max)\left\{-\frac{a_j}{\sqrt{2\pi}\sigma_{G_L}}\exp\left[-\frac{1}{2}\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)^2\right]\right\}$$
$$\begin{cases} s.t.\, \underline{\mu}_{X_j} \le \mu_{X_j} \le \overline{\mu}_{X_j}\ (j = 1, 2, \cdots, i) \\ \underline{\sigma}_{X_j} \le \sigma_{X_j} \le \overline{\sigma}_{X_j}\ (j = 1, 2, \cdots, i) \\ \underline{\mu}_{G_L} \le \mu_{G_L} \le \overline{\mu}_{G_L} \\ \underline{\sigma}_{G_L} \le \sigma_{G_L} \le \overline{\sigma}_{G_L} \end{cases} \tag{45}$$

and

$$\frac{\partial P_f^I}{\partial \sigma_{X_j}} = \left[\min\left(\frac{\partial P_f^I}{\partial \sigma_{X_j}}\right), \max\left(\frac{\partial P_f^I}{\partial \sigma_{X_j}}\right)\right] \tag{46}$$

Table 2 Results of numerical example 1

Sensitivity type	Variation coefficients	Results provided by the proposed method
$\frac{\partial P_f^I}{\partial \mu_{X_1}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[-2.45 \times 10^{-3}, -9.66 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[-1.46 \times 10^{-3}, -2.12 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[-8.18 \times 10^{-4}, -4.31 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	-5.99×10^{-4} (-6.02×10^{-4})
$\frac{\partial P_f^I}{\partial \mu_{X_2}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[-1.40 \times 10^{-3}, -5.52 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[-8.33 \times 10^{-4}, -1.21 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[-4.68 \times 10^{-4}, -2.47 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	-3.42×10^{-4} (-3.64×10^{-4})
$\frac{\partial P_f^I}{\partial \mu_{X_3}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[2.01 \times 10^{-6}, 5.10 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[4.41 \times 10^{-6}, 3.03 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[8.97 \times 10^{-6}, 1.70 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	1.25×10^{-5} (1.23×10^{-5})
$\frac{\partial P_f^I}{\partial \sigma_{X_1}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[1.67 \times 10^{-4}, 7.27 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[4.11 \times 10^{-4}, 3.89 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[9.31 \times 10^{-4}, 1.97 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	1.36×10^{-3} (1.35×10^{-3})
$\frac{\partial P_f^I}{\partial \sigma_{X_2}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[2.73 \times 10^{-5}, 1.19 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[6.71 \times 10^{-5}, 6.36 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[1.52 \times 10^{-4}, 3.21 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	2.23×10^{-4} (2.76×10^{-4})
$\frac{\partial P_f^I}{\partial \sigma_{X_3}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$	$[2.89 \times 10^{-6}, 1.26 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.03$	$[7.10 \times 10^{-6}, 6.73 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[1.61 \times 10^{-5}, 3.40 \times 10^{-5}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	2.36×10^{-5} (2.32×10^{-5})

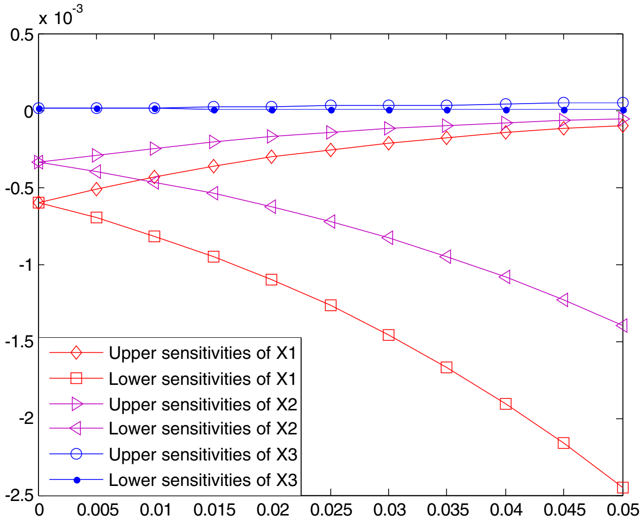


Fig. 3 Parameter mean sensitivities with different coefficients

$$\min\,(\max)\left(\frac{\partial P_f^I}{\partial \sigma_{X_j}}\right) = \min\,(\max)\left\{\frac{a_j^2 \mu_{G_L} \sigma_{X_j}}{\sqrt{2\pi}\sigma_{G_L}^3}\exp\left[-\frac{1}{2}\left(\frac{\mu_{G_L}}{\sigma_{G_L}}\right)^2\right]\right\}$$

$$\begin{cases} s.t.\, \underline{\mu}_{X_j} \le \mu_{X_j} \le \overline{\mu}_{X_j}\ (j = 1, 2, \cdots, i) \\ \underline{\sigma}_{X_j} \le \sigma_{X_j} \le \overline{\sigma}_{X_j}\ (j = 1, 2, \cdots, i) \\ \underline{\mu}_{G_L} \le \mu_{G_L} \le \overline{\mu}_{G_L} \\ \underline{\sigma}_{G_L} \le \sigma_{G_L} \le \overline{\sigma}_{G_L} \end{cases} \tag{47}$$

respectively. Since the two optimization problems are very complicated and time consuming, for the purpose of convenience and simplicity, (42), (43) and the interval operation

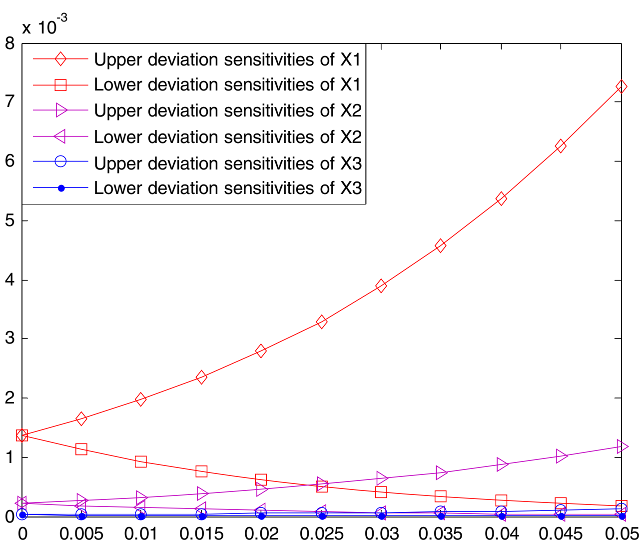


Fig. 4 Parameter deviation sensitivities with different coefficients

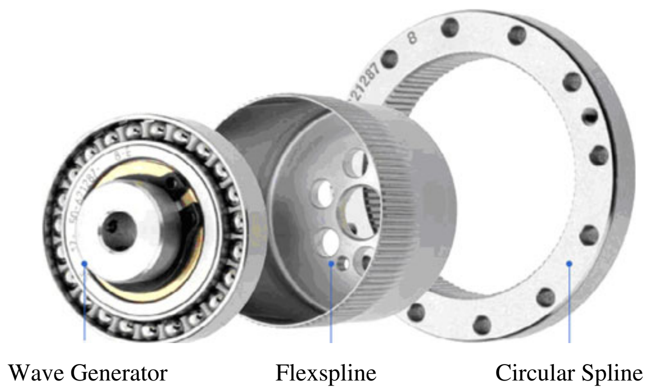


Fig. 5 Schematic of a harmonic drive

can be further used to calculate the approximate interval bounds of parameter sensitivities as follows

$$\frac{\partial P_f^I}{\partial \mu_{X_j}} \approx \left\{-\frac{a_j}{\sqrt{2\pi}\underline{\sigma}_{G_L}}\exp\left[-\frac{1}{2}\left(\frac{\underline{\mu}_{G_L}}{\overline{\sigma}_{G_L}}\right)^2\right],\ -\frac{a_j}{\sqrt{2\pi}\overline{\sigma}_{G_L}}\exp\left[-\frac{1}{2}\left(\frac{\overline{\mu}_{G_L}}{\underline{\sigma}_{G_L}}\right)^2\right]\right\} \tag{48}$$

and

$$\frac{\partial P_f^I}{\partial \sigma_{X_j}} \approx \left\{\frac{a_j^2 \underline{\mu}_{G_L} \underline{\sigma}_{X_j}}{\sqrt{2\pi}\overline{\sigma}_{G_L}^3}\exp\left[-\frac{1}{2}\left(\frac{\overline{\mu}_{G_L}}{\underline{\sigma}_{G_L}}\right)^2\right],\ \frac{a_j^2 \overline{\mu}_{G_L} \overline{\sigma}_{X_j}}{\sqrt{2\pi}\underline{\sigma}_{G_L}^3}\exp\left[-\frac{1}{2}\left(\frac{\underline{\mu}_{G_L}}{\overline{\sigma}_{G_L}}\right)^2\right]\right\} \tag{49}$$

5 Numerical examples

In this section, four examples are provided to demonstrate the application of the proposed method as well as its effectiveness. All parameters are expressed by P-boxes in Example 1. Two parameters are expressed by P-boxes while one is expressed by precise probability distribution in Example 2. Example 3 is used to demonstrate the accuracy of the proposed method when the limit-state function

Table 3 Distribution details of random variables

Variable	Mean	Standard deviation	Variation coefficient	Distribution
T_h(N·m)	350	35	(ε_1, ξ_1)	Normal
N_v(rpm)	0.1	0.01	(ε_2, ξ_2)	Normal
K	1.3	0.1	0	Normal
T(N·m)	2,000	0	0	……

Table 4 Results of example 2

Sensitivity type	Variation coefficient	Results provided by the proposed method
$\frac{\partial P_f^I}{\partial \mu_{T_h}}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	$[-2.356 \times 10^{-3}, -1.585 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	$[-2.199 \times 10^{-3}, -1.727 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	$[-2.029 \times 10^{-3}, -1.875 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	-1.950×10^{-3} (-1.801×10^{-3})
$\frac{\partial P_f^I}{\partial \mu_{N_v}}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	[1.457, 2.165]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	[1.587, 2.016]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	[1.722, 1.864]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	1.791 (1.744)
$\frac{\partial P_f^I}{\partial \mu_K}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	[0.322, 0.478]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	[0.350, 0.444]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	[0.380, 0.412]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	0.396 (0.386)
$\frac{\partial P_f^I}{\partial \sigma_{T_h}}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	$[2.066 \times 10^{-3}, 4.115 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	$[2.385 \times 10^{-3}, 3.603 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	$[2.744 \times 10^{-3}, 3.148 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	2.939×10^{-3} (2.740×10^{-3})
$\frac{\partial P_f^I}{\partial \sigma_{N_v}}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	[0.498, 0.993]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	[0.575, 0.869]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	[0.662, 0.759]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	0.709 (0.669)
$\frac{\partial P_f^I}{\partial \sigma_K}$	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.05$	[0.243, 0.484]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.03$	[0.280, 0.424]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.01$	[0.323, 0.370]
	$\varepsilon_1, \varepsilon_2 = \xi_1, \xi_2 = 0.00$	0.346 (0.352)

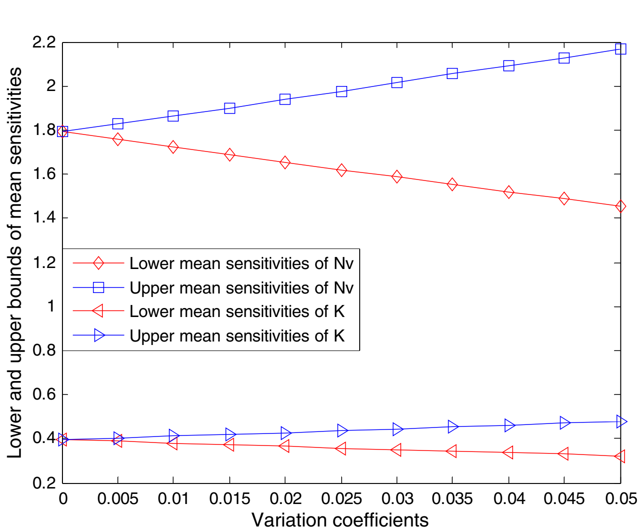


Fig. 6 Parameter mean sensitivities with different variation coefficients

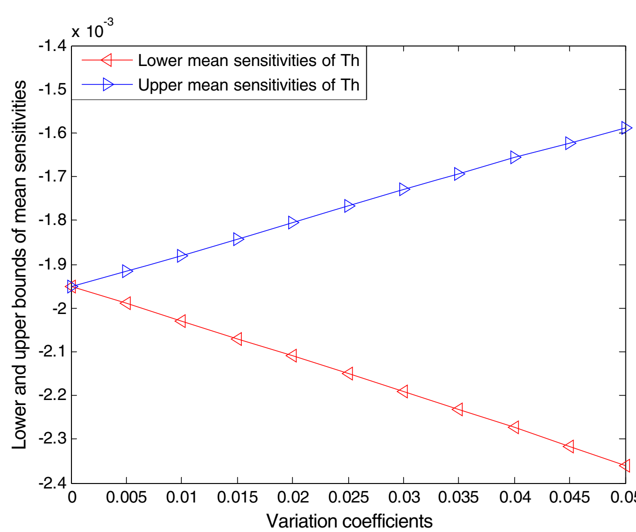


Fig. 7 Parameters mean sensitivities with different variation coefficients

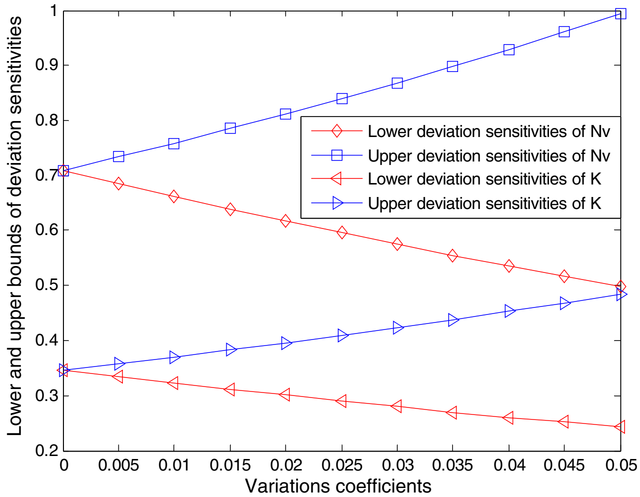


Fig. 8 Parameters deviation sensitivities with different variation coefficients

is a highly non-linear function. In Example 4, the limit-state function is a black-box.

5.1 Example 1: a mathematical problem

Consider a non-linear limit state function with 3 normal random variables. The limit-state function is $G(\mathbf{X}) = X_1X_2 - X_3 = 0$. The distribution details of random variables are given in Table 1 (Melchers and Ahammed 2004).

10^6 samples are used and 1194 samples fall in the constraint domain. With weighted regression analysis, the approximating hyper-plane G_L is

$$G_L(\mathbf{X}) = -1299.77 + 47.13X_1 + 26.94X_2 - 0.980X_3$$

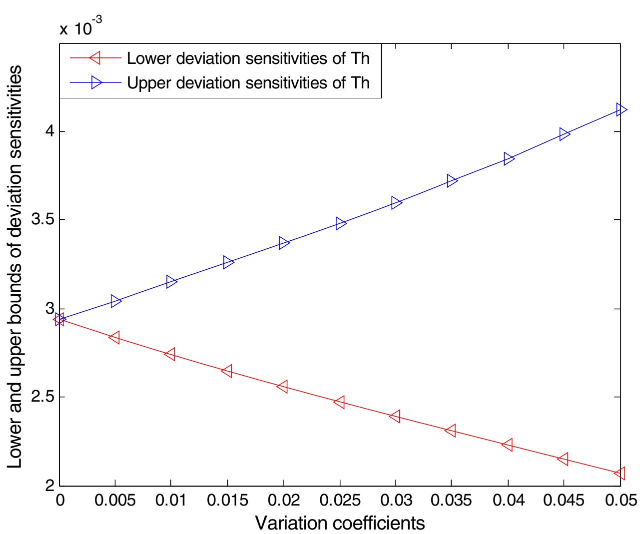


Fig. 9 Parameters deviation sensitivities with different variation coefficients

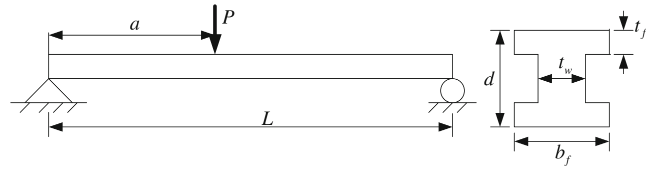


Fig. 10 A beam

The interval-valued reliability sensitivities based on the hyper-plane G_L under different variation coefficients (ε_1, ξ_1) are shown in Table 2, and the values in brackets $(\cdot)$ of Table 2 are calculated by using the MCS-based reliability sensitivity method with the 10^6 samples.

From the results in Table 2, it can be concluded that the reliability sensitivity of each random variable is very sensitive to its distribution parameter. $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$. This means that the variation coefficients of all parameters are equal, that is, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \xi_1 = \xi_2 = \xi_3 = 0.05$. When we have sufficient information about all parameters of the systems, $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0$, that is, there is no epistemic uncertainty in the system. The sensitivity analysis results are precise values rather than intervals. For example, the reliability sensitivity of the variable X_1 is $(-5.99 \times 10^{-4}, 1.36 \times 10^{-3})$. However, in reality, it is impossible to know the parameter probability distributions precisely. If the variation coefficients are $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.05$, the reliability sensitivity of variable X_1 is within an interval $([-2.45 \times 10^{-3}, -9.66 \times 10^{-5}], [1.67 \times 10^{-4}, 7.27 \times 10^{-3}])$. It should be noted that the variation coefficients of parameters may not be all equal, such as, $\varepsilon_1 = 0.05$, $\varepsilon_2 = \varepsilon_3 = 0.03$. The method to handle this case is the same as that used for equal coefficients. For the purpose of simplicity and illustration, we assume that all variation coefficients are equal to each other. The figures of system reliability sensitivities are shown in Figs. 3 and 4. From Figs. 3 and 4, we know that random variables are sensitive to its distribution parameters and a small change to a parameter may lead to a large change to the reliability sensitivity results. The

Table 5 Distribution details of random variables

Variable	Mean	Deviation	Variation coefficient	Distribution type
P	6,070	200	(ε_1, ξ_1)	Normal
L	120	6	(ε_2, ξ_2)	Normal
a	72	6	(ε_3, ξ_4)	Normal
S	170,000	4,760	(ε_4, ξ_4)	Normal
d	2.3	1/24	(ε_5, ξ_5)	Normal
t_w	0.16	1/48	(ε_6, ξ_6)	Normal
t_f	0.26	1/48	(ε_7, ξ_7)	Normal
b_f	2.3	1/24	(ε_8, ξ_8)	Normal

variable X_1 is more sensitive than the other variables in the system. Therefore, in reliability-based design, we need to pay more attention to X_1 than to the other variables.

Furthermore, in this paper, for the purpose of simplicity, we give an accuracy comparison between the proposed method and the MCS-based method for the reliability

Table 6 Results of example 3

Sensitivity type	Variation coefficients	The proposed method
$\frac{\partial P_f^I}{\partial \mu_P}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[-1.11\times10^{-4}, -8.19\times10^{-5}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	-9.59×10^{-5} (-1.01×10^{-4})
$\frac{\partial P_f^I}{\partial \mu_a}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[8.35\times10^{-3}, 1.14\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	9.78×10^{-3} (8.94×10^{-3})
$\frac{\partial P_f^I}{\partial \mu_L}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[-1.40\times10^{-2}, -1.03\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	-1.20×10^{-2} (-1.15×10^{-2})
$\frac{\partial P_f^I}{\partial \mu_d}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	[0.285, 0.387]
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	0.333 (0.352)
$\frac{\partial P_f^I}{\partial \mu_{b_f}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	[0.190, 0.259]
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	0.222 (0.253)
$\frac{\partial P_f^I}{\partial \mu_{t_w}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	[0.156, 0.212]
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	0.183 (0.213)
$\frac{\partial P_f^I}{\partial \mu_{t_f}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	[1.171, 1.593]
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	1.370 (1.445)
$\frac{\partial P_f^I}{\partial \mu_S}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[4.31\times10^{-6}, 5.84\times10^{-6}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	5.05×10^{-6} (5.04×10^{-6})
$\frac{\partial P_f^I}{\partial \sigma_P}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[2.39\times10^{-5}, 3.53\times10^{-5}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	2.92×10^{-5} (4.54×10^{-5})
$\frac{\partial P_f^I}{\partial \sigma_a}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[7.45\times10^{-3}, 1.10\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	9.11×10^{-3} (8.81×10^{-3})
$\frac{\partial P_f^I}{\partial \sigma_L}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[1.13\times10^{-2}, 1.66\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	1.38×10^{-2} (1.34×10^{-2})
$\frac{\partial P_f^I}{\partial \sigma_d}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[6.02\times10^{-2}, 8.88\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	7.35×10^{-2} (8.69×10^{-2})
$\frac{\partial P_f^I}{\partial \sigma_{b_f}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[2.69\times10^{-2}, 3.97\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	3.28×10^{-2} (4.72×10^{-2})
$\frac{\partial P_f^I}{\partial \sigma_{t_w}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[9.05\times10^{-3}, 1.34\times10^{-2}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	1.11×10^{-2} (1.92×10^{-2})
$\frac{\partial P_f^I}{\partial \sigma_{t_f}}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	[0.508, 0.750]
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	0.621 (0.796)
$\frac{\partial P_f^I}{\partial \sigma_S}$	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.01$	$[1.58\times10^{-6}, 2.33\times10^{-6}]$
	$\varepsilon_1,\cdots,\varepsilon_8=\xi_1,\cdots,\xi_8=0.00$	1.93×10^{-6} (2.00×10^{-6})

sensitivity analysis with variation coefficients equal to zeros. From Table 2, we know that the results obtained using the proposed method is almost identical to the results using the MCS-based method.

5.2 Example 2: a harmonic drive

A harmonic drive, shown in Fig. 5, is widely used in the solar array drive mechanism and the antenna pointing mechanism because of its high carrying capacity, light weight, small size, and etc.

The performance function of a harmonic drive for its life estimation is (Du 2010)

$$G\left(T_h, N_V, T, K, m\right) = \frac{75 \times 10^5}{N_V}\left(\frac{T_h}{K\,T}\right)^3 - 8760 \times m$$

where m is the number of years, and T_h, N_V, K, and T are the rated output torque, input speed, condition factor and nominal output torque, respectively. $8760 = (365 \times 24)$ is the total number of hours for one year. When $G > 0$, system is considered safe. When $G < 0$, system falls in the failure domain. The distribution details of random variables are given in Table 3. In this example, we only consider the reliability of the harmonic drive for 10 years.

5×10^4 samples are used and 1,640 samples fall in the constraint domain. By applying the weighted regression analysis, the approximating hyper-plane G_L is

$$\begin{aligned} G_L\left(N_v, K, T_h\right) = {} & 5.8172 \times 10^4 - 6.7030 \times 10^5 N_v \\ & + 7.2950 \times 10^2 T_h - 1.4801 \times 10^5 K \end{aligned}$$

The interval-valued reliability sensitivities based on the hyper-plane G_L under different variation coefficients (ε_1, ξ_1) are listed in Table 4. The values in brackets $(\cdot)$ of Table 4 are calculated using the MCS-based reliability sensitivity method with the 10^6 samples.

In Example 2, the distribution parameters of T_h and N_v have variation coefficients which are modeled using P-boxes. The variation coefficient of the variable K is equal to zero which is modeled using a precise probability distribution. Both epistemic and aleatory uncertainties are considered in this example. From the results in Table 4, we can see that the larger the variation coefficients are, the wider the interval is. The reason is that a large variation coefficient represents a large uncertainty of parameters' influence on the system. When all the variation coefficients are zero, the distributions of all random variables can be precisely determined. The sensitivity analysis results of these variables become precise values. The figures of system reliability sensitivities are shown in Figs. 6, 7, 8, 9. From these figures, a conclusion is reached that the system is very sensitive to the variable N_v, which is a key design variable considered in the reliability-based design.

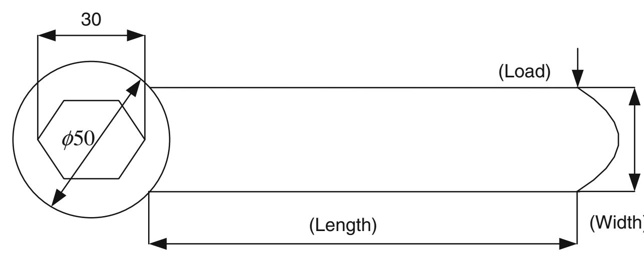


Fig. 11 A wrench

5.3 Example 3: a beam

As shown in Fig. 10, a beam example is used to demonstrate the accuracy of the proposed method. The performance function is given by (Huang and Du 2006)

$$Z = f\left(P, L, a, S, d, b_f, t_w, t_f\right) = \sigma_{\max} - S$$

where

$$\sigma_{\max} = \frac{Pa\left(L - a\right)d}{2LI}$$

and

$$I = \frac{b_f d^3 - \left(b_f - t_w\right)\left(d - 2t_f\right)^3}{12}$$

The distribution details of random variables are given in Table 5. We only consider a case of $Z < -50{,}000$ in the example.

5×10^4 samples are used and 2350 samples fall in the constraint domain. With the weighted regression analysis, the approximating hyper-plane Z_L is

$$\begin{aligned} Z_L \approx {} & 19.220P - 1960.267a + 2409.748L - 66824.231d \\ & - 44658.404b_f - 36644.644t_w \\ & - 274666.167t_f - 1.012S + 275102.329 \end{aligned}$$

The interval-valued reliability sensitivities based on the hyper-plane Z_L under different variation coefficients (ε_1, ξ_1) are shown in Table 6.

In this example, the limit-state function is highly nonlinear. From Table 6 we know that the results calculated

Table 7 Distributions details of random variables

Variable	Mean	Deviation	Variation coefficient	Distribution
Load	500	20	(ε_1, ξ_1)	Normal
Length	330	15	(ε_2, ξ_2)	Normal
Width	30	3	(ε_3, ξ_3)	Normal
σ_s	320	0	0	–

Table 8 Results of example 4

Sensitivity type	Variation coefficients	Results provided by the proposed method
$\frac{\partial P_f^I}{\partial \mu_{X_1}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[6.32 \times 10^{-4}, 6.42 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	6.37×10^{-3} (5.69×10^{-3})
$\frac{\partial P_f^I}{\partial \mu_{X_2}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[-9.73 \times 10^{-2}, -9.88 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	-9.81×10^{-2} (-9.01×10^{-2})
$\frac{\partial P_f^I}{\partial \mu_{X_3}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[3.76 \times 10^{-3}, 3.82 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	3.79×10^{-3} (4.73×10^{-3})
$\frac{\partial P_f^I}{\partial \sigma_{X_1}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[1.25 \times 10^{-3}, 1.32 \times 10^{-3}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	1.28×10^{-3} (1.58×10^{-3})
$\frac{\partial P_f^I}{\partial \sigma_{X_2}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[5.91 \times 10^{-2}, 6.24 \times 10^{-2}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	6.08×10^{-2} (6.62×10^{-2})
$\frac{\partial P_f^I}{\partial \sigma_{X_3}}$	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.01$	$[5.89 \times 10^{-4}, 6.22 \times 10^{-4}]$
	$\varepsilon_1, \varepsilon_2, \varepsilon_3 = \xi_1, \xi_2, \xi_3 = 0.00$	6.06×10^{-4} (3.63×10^{-3})

using the proposed method are not very accurate when compared with the results using the MCS-based method. The values in brackets $(\cdot)$ are calculated using the MCS-based method with 10^6 samples with all the variation coefficients equal to zeros. It is observed that when the limit-state function is highly non-linear, the results calculated using the proposed method are not accurate results.

5.4 Example 4: a wrench

In an example of a wrench as shown in Fig. 11, the limit-state function is given by (Wang et al. 2006)

$$g\left(\sigma_{\max}, \sigma_s, Load, Width, Length\right) = \sigma_{\max} - \sigma_s = 0$$

where σ_{max} and σ_s are the maximum stress and the rated stress. The distribution details of random variables are given in Table 7.

10^3 samples are used and 222 samples fall in the constraint domain. With the weighted regression analysis, the approximating hyper-plane g_L is

$$g_L \approx 207.577 - 1.041X_1 + 16.030X_2 - 0.620X_3$$

where X_1, X_2 and X_3 are used to denote random variables Length, Width and Load, respectively.

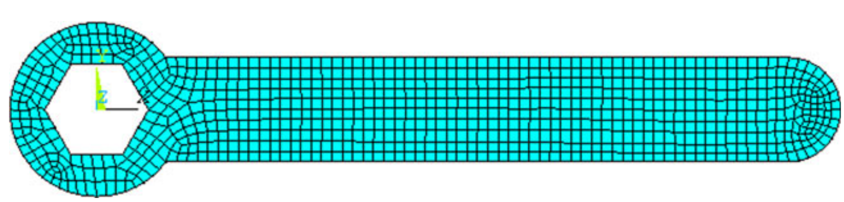
Applying the proposed method, the interval-valued reliability sensitivities based on the hyper-plane g_L for different variation coefficients (ε_i, ξ_i) are given in Table 8.

In this example, the limit-state function is an implicit function or a black-box. In order to calculate $\sigma_{\max}$, the finite element analysis (FEA) method is used. The FEA for wrench is shown in Fig. 12. Generally, the results calculated using the proposed method are not accurate, especially for large-scale real engineering problems. Because FEA is time costly, the results calculated by the MCS-based method with 1,000 samples are used as reference results for accuracy comparison.

6 Conclusions

Based on the P-boxes, interval algorithm, MCS, weighted linear aggression analysis and FORM, a new sensitivity analysis method is proposed. In the structural reliability and sensitivity analysis, it is practically appropriate to obtain a P-box interval constraint for system random variables rather than precise distributions because two types of uncertainties, epistemic and aleatory uncertainties, exist widely in engineering practices. The results of the four examples have

Fig. 12 FEA for wrench



shown that the proposed method is effective because it provides a means of reliability sensitivity analysis under either epistemic uncertainty, aleatory uncertainty or both of them. Generally, it is more robust than the traditional sensitivity method such as the FORM-based ones because it does not require the MPP search. Furthermore, the proposed method is superior to the MCS-based method in terms of computational times. In addition, the proposed method is also applicable for the situation where the limit-state function is a black box. The numerical examples indicate that a small change to a distribution parameter may lead to a large change to the reliability sensitivity results.

It should be noted that there are limitations to the proposed method based on the interval form. The proposed method is an approximation method that is laid on a linearized function instead of its original limit-state function. The linearization may cause a loss of information. The result of the interval bounds in the paper is an approximation rather than an exact solution. Generally, the proposed method is not available for large-scale real engineering problems with highly non-linear performance functions. The more nonlinearity of a limit-state function is, the larger errors will be. Future work involves an accuracy improvement especially when the system failure probability is very small and the computational time is huge.

Acknowledgments This research was partially supported by the National Natural Science Foundation of China under the contract number 51075061 and the Specialized Research Fund for the Doctoral Program of Higher Education of China under the contract number 20090185110019.

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