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An approach to system reliability analysis with fuzzy random variables

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A B S T R A C T

Nondeterministic variables of certain distributions are employed to represent uncertainties, which are usually treated as the stochastic factors to reliability models. However, model parameters may not be precisely represented due to some factors in engineering practices, such as lack of sufficient data, data with fuzziness and unknown or non-constant reproduction conditions. To address these issues, fuzzy random variables are implemented and two developments are made in this paper. The first development is that the Saddlepoint Approximation (SAP)-simulation is extended to conduct reliability analysis accounting for the time-dependent degradation process and fuzzy random variables, and we attempt to give a method to select a proper saddlepoint. The second development is that two system reliability analysis methods are proposed for different scenarios of reliability modeling processes. It could be suitable for the system consisting of structural components with gradual failure, whose reliability can be obtained by the method in the improved SPA-simulation, also for system consisting of components with sudden failure, whose reliability can be acquired from site field or experiments. An illustrated example is followed to testify the proposed methods.

1. Introduction

Reliability is one of the major concerns in engineering practices since the occurrence of failures may lead to catastrophic consequences. Reliability-based design optimization (RBDO) and maintenances are the two main approaches to system safe operation. RBDO seeks a design which achieves the probability of failure less than an acceptable value. Therefore the likelihood of catastrophic consequences decreases dramatically [1,2]. Maintenance is an important measure to the operation and extension of the product service life. In order to do reasonable RBDO and make proper maintenance decisions, system reliability should be precisely evaluated.

System reliability evaluation methods have been the focus during the past several decades, such as Monte Carlo simulation [3], fault tree analysis [4], Bayesian approach [5], reliability block diagram [6], fuzzy reliability methods [7,8], and multi-state system reliability methods [9,10]. Reliability evaluation is conducted based on field data or experimental data with statistical tools in these methods. It is easy to conduct reliability analysis with these methods when the field data or experimental data are effective and sufficient. However, most structural components will suffer a gradual failure process and it is difficult or impossible to obtain effective and sufficient data from both engineering practices and experiments, even though accelerated life testing (ALT) is implemented. It should therefore recur to the physical model, failure modes and degradation mechanism, by the so-called physics-based reliability method [11].

Many methods have been proposed to conduct physics-based reliability analysis, such as first order reliability method (FORM) [12], second order reliability method (SORM) [13], and Saddlepoint Approximation (SPA) method [14]. SORM is more accurate than FORM, but more computationally intensive. In spite of its usefulness, FORM often could not satisfy the requirement of accuracy in engineering fields. With consideration of the tradeoff between the efficiency and accuracy, the first order saddlepoint approximation (FOSPA) method, which is more accurate without large computational demand, is proposed [15]. With FOSPA, the most likelihood point should be

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searched with an iterative process and need a number of evaluations of limit-state function. Due to the intensive computational expense, FOSPA may not be suitable for large-scale problems. To alleviate the computational demand, a SPA-based simulation method is proposed [16].

Some conditions could be met in engineering practices, where random variables are no longer proper to represent uncertainties. The conditions could include: (1) the available data is so insufficient that the statistical properties could be not expressed properly; (2) the field or experimental data possess fuzziness; (3) field data or experimental data are obtained under such conditions, which is unknown or non-constant reproduction [17]. To address these conditions, some methods have been proposed, such as Bayesian approach [18], evidence theory [19], possibility theory [20] and fuzzy random variable method [21]. Bayesian approach, evidence theory, and possibility theory have exhibited good capability of describing conditions (1) and (2), but weak capability for condition (3). Fuzzy random variable method has been attracting more and more interests due to its strong capability of describing both three conditions.

When considering the merits of both the SPA-based simulation method and the fuzzy random variable, a physics-based reliability analysis method is proposed in this paper. In this method, fuzzy random variable is implemented to deal with uncertainty in three conditions of engineering practices and SPA-based simulation method is used to guarantee the computational efficiency and accuracy. Therefore, the method would extend the current physics-based reliability analysis method. The other development is that we attempt to present two system reliability analysis methods for complex systems. The proposed methods could be suitable for systems consist of structural components with gradual failure, whose reliability could be evaluated with the method in fundamental development by considering the actual conditions, and suitable for systems consist of components with sudden failure, whose reliability can be acquired by experimental and field data. Hence the second development would be helpful for the progress of system reliability theory.

The organization of this paper is as follows. In Section 2, a brief description on fuzzy random variable and SPA-based simulation method is introduced. In Section 3, the proposed physics-based reliability analysis method will be illustrated in detail. In Section 4, the system reliability analysis methods will be provided. An example is followed to demonstrate the proposed methods in Section 5. A conclusion is arrived in Section 6.

2. Fuzzy random variable and SPA-based simulation method

2.1. Fuzzy random variable

Randomness and fuzziness are usually two alternative representations of uncertainties. Only randomness has been considered due to the maturation of probability theory in many studies. However, fuzziness in the randomness exists in engineering practices, because of lack of sufficient data, data with fuzziness, and unknown or non-constant reproduction conditions. Fuzzy random variable has attracting more attention, for its capacity of uncertainty representation when engineering problems are handled.

A fuzzy random variable $\tilde{X}$ can be defined on a fuzzy probability space $((\Omega,\mu(\Omega)),(F,\mu(F)),(P,\mu(P)))$, where $(\Omega,\mu(\Omega))$ is the fuzzy random sample space and $\mu(\Omega)$ is corresponding membership function of Ω; a $\sigma-$algebra $(F,\mu(F))$ is the subsets of $(\Omega,\mu(\Omega))$ and $\mu(F)$ is the corresponding membership function of F; $(P,\mu(P))$ is the fuzzy probability measure and $\mu(P)$ is the corresponding membership function of P. Accordingly, $\tilde{X}$ is a mapping relationship from $(\Omega,\mu(\Omega))$ to $(R^n,\mu(R^n))$, namely $\tilde{X}:(\Omega,\mu(\Omega))\rightarrow\left(R^n,\mu\left(R^n\right)\right)$.

When a fuzzy number $\tilde{x}_i$ with the membership functions $\mu(x_i)$ is assigned to an elementary event $\omega\in\Omega,\tilde{x}_i(\omega)$ is a realization of the fuzzy random variable $\tilde{X}$. Several realizations for a fuzzy random variable are given in Fig. 1 [17].

In Fig. 1, $\tilde{X}(\omega_1)$, $\tilde{X}(\omega_2)$, ⋯, $\tilde{X}(\omega_6)$ are several realizations of fuzzy random variable $\tilde{X}$, while $X(\omega_1)$, $X(\omega_2)$, ⋯, $X(\omega_6)$ are several realizations of random variable X. Therefore, one more axis should be added for fuzzy random variables compared with random variables due to its fuzziness.

2.2. SPA-based simulation method

SPA, as an effective alternative approach to structural reliability analysis, has been studied widely in the engineering design because of its higher accuracy than FORM, even than SORM for some cases, with the same computational efficiency with FORM. Its potential use in engineering fields has been illustrated by the integration of the SPA with SORM [22]. The recent attempt is the FOSPA for reliability analysis. To alleviate the computational cost, SPA-based simulation method is provided [16]. The flowchart of SPA-based simulation is given in Fig. 2, in which simulation process and analytical process are involved in the flowchart. The results from the simulation process are considered as the inputs of the analytical process to obtain the expression of the cumulant generating function (CGF) analytically. For more details, please refer to [16].

3. Fuzzy physics-based reliability analysis method

The physics-based reliability analysis model based on fuzzy random design variables and fuzzy random parameters are given by:

$$\tilde{R}=\Pr\left\{g_i\left(\mathbf{d},\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)\geq 0\right\}=\int_{g_i\left(\mathbf{d},\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)\geq 0} f_{\tilde{\mathbf{X}}},\tilde{\mathbf{P}}\left(\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)\mathrm{d}X\mathrm{d}P. \tag{1}$$

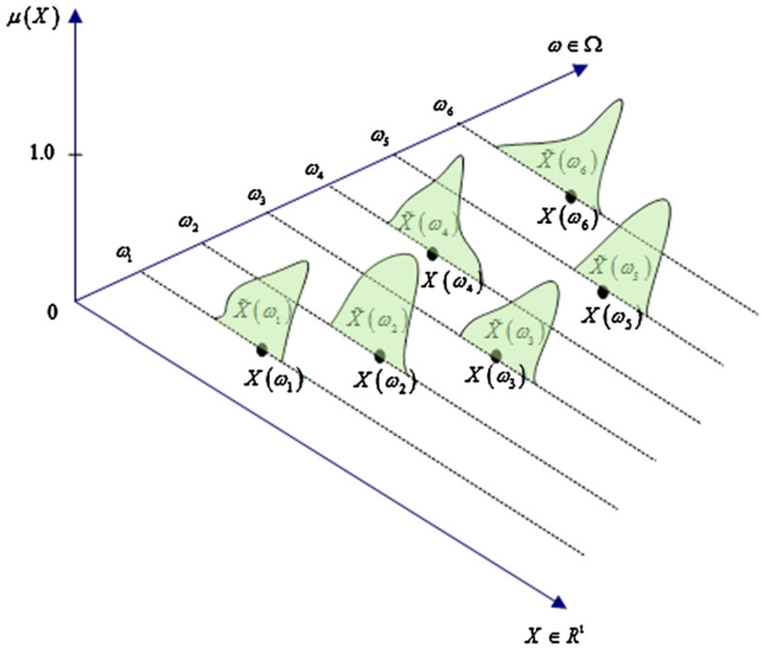

Fig. 1. Several realizations of a one-dimension fuzzy random variable.

It is noted that the reliability is fuzzy and could be obtained by the integration on the fuzzy joint probability density function $f_{\tilde{\mathbf{X}},\tilde{\mathbf{P}}}\left(\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)$ over the fuzzy safe region $g_i\left(\mathbf{d},\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)\geq 0$. $\mathbf{d}$ is the vector of deterministic design variables; $\tilde{\mathbf{X}}$ is the vector of fuzzy random design variables; $\tilde{\mathbf{P}}$ is the vector of fuzzy random parameters; $g_i(\cdot)$ is the limit state function associated with the failure mode; $g_i\left(\mathbf{d},\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)\geq 0$ denotes the safe region, which is a fuzzy region; $f_{\tilde{\mathbf{X}},\tilde{\mathbf{P}}}\left(\tilde{\mathbf{X}},\tilde{\mathbf{P}}\right)$ is the fuzzy joint probability density function of fuzzy random design variables and fuzzy random parameters. The difference between $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{P}}$ is that $\tilde{\mathbf{X}}$ is controllable by the designer during the design process while $\tilde{\mathbf{P}}$ is uncontrollable. Because of the high nonlinear and multidimensional limit state function, there is rarely a close-form solution to Eq. (1). An approximation method is therefore needed to calculate the reliability in Eq. (1). Möller tries to solve Eq. (1) with fuzzy first order reliability method (FFORM) [17], but no formal formula is given. Herein, the extended SPA-simulation method by accounting for fuzzy random variables is proposed to conduct reliability analysis in Eq. (1). Eight steps are involved in the proposed method. The first four steps are simulation process, while the last four steps consist of the analytical process. The flowchart of the proposed method is provided in Fig. 3.

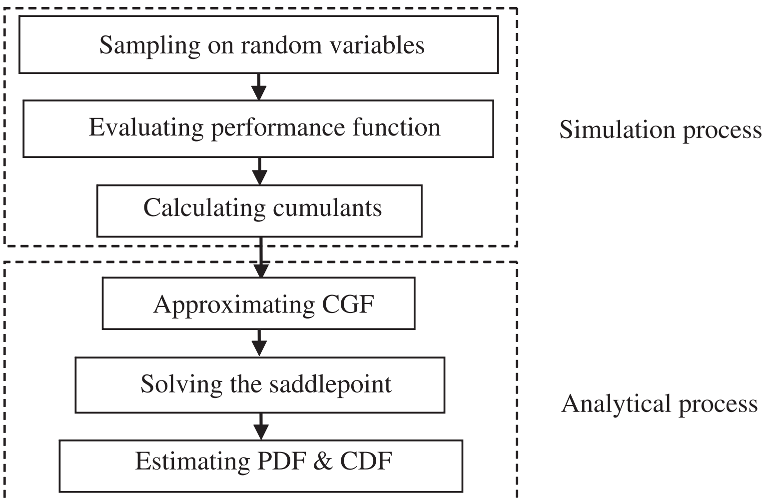

Fig. 2. The flowchart of SPA-based simulation method.

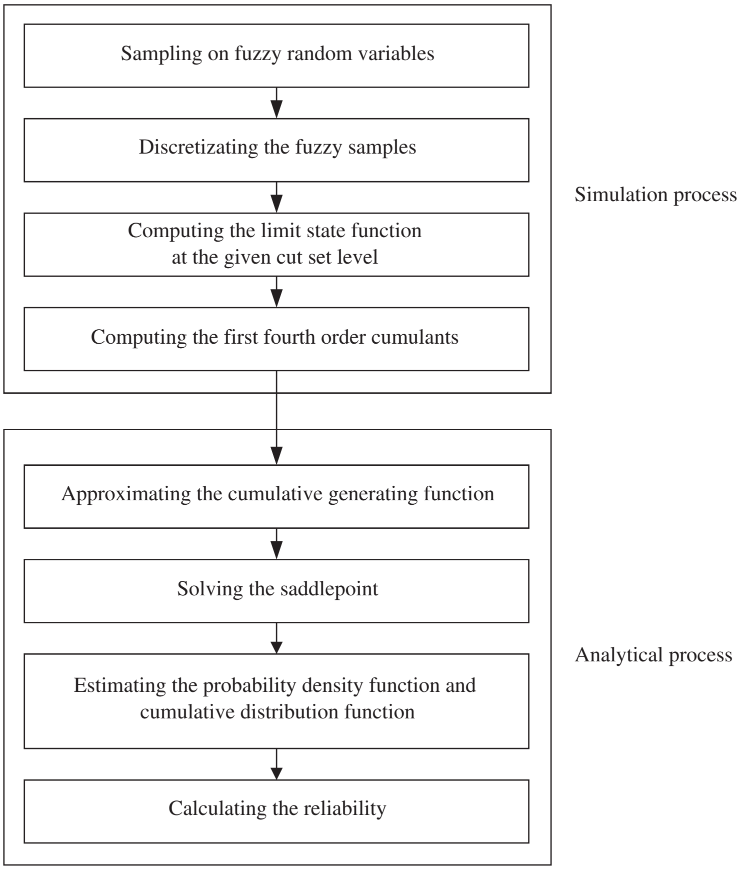

Fig. 3. The flowchart of the proposed method.

3.1. Step 1: sampling on fuzzy random variables

A general model with fuzzy model parameters in the probability distribution could be suitable for representing all three conditions: lack of sufficient data, data with fuzziness, and non-constant reproduction conditions, which is an important representation of fuzzy random variables [17]. Hence it is a fuzzy number sampled from the fuzzy random variables. The samples for the vector of fuzzy random design variables $\tilde{\mathbf{X}}$ and the vector of fuzzy random parameters $\tilde{\mathbf{P}}$ could be provided by:

$$\tilde{\mathbf{X}} = \left[\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n\right] = \left[\tilde{x}_1^1, \tilde{x}_1^2, \cdots, \tilde{x}_1^N; \tilde{x}_2^1, \tilde{x}_2^2, \cdots, \tilde{x}_2^N; \cdots; \tilde{x}_n^1, \tilde{x}_n^2, \cdots, \tilde{x}_n^N\right] \tag{2}$$

$$\tilde{\mathbf{P}} = \left[\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_m\right] = \left[\tilde{p}_1^1, \tilde{p}_1^2, \cdots, \tilde{p}_1^N; \tilde{p}_2^1, \tilde{p}_2^2, \cdots, \tilde{p}_2^N; \cdots; \tilde{p}_m^1, \tilde{p}_m^2, \cdots, \tilde{p}_m^N\right] \tag{3}$$

where n is the number of fuzzy random variables in the vector $\tilde{\mathbf{X}}$; m is the number of fuzzy random variables in the vector $\tilde{\mathbf{P}}$; N is the number of samples for every $\tilde{X}_i$ and $\tilde{P}_i$, which is usually taken to be 500 in the proposed method.

3.2. Step 2: discretizating fuzzy samples

Fuzzy samples will be implemented for computing the performance based on limit state functions. Because of the high non-linear and multi-dimensional limit state functions, it is very hard to compute the performance with the extension principle of fuzzy sets. Herein, λ level cut set method, as an effective and popular method, is employed. Therefore, fuzzy variables are degenerated into interval variables. For example, with the λ level cut set, a certain sample $\tilde{x}_i^j$ (or $\tilde{p}_i^j$) therefore becomes an interval $\tilde{x}_i^j(\lambda) = \left[\tilde{x}_i^{jL}(\lambda), \tilde{x}_i^{jU}(\lambda)\right]$.

3.3. Step 3: calculating the boundaries of performance

The limit state function at a given level cut set λ could be formulated as

$$\tilde{y}^j(\lambda) = g_i\left(\mathbf{d}, \tilde{\mathbf{X}}^j(\lambda), \tilde{\mathbf{P}}^j(\lambda)\right) \tag{4}$$

$\tilde{y}^j(\lambda)$ is an interval variable because $\tilde{\mathbf{X}}^j(\lambda)$ and $\tilde{\mathbf{P}}^j(\lambda)$ are interval variables. Hence the upper and lower boundaries of the performance could be found with an optimization procedure [23].

$$\tilde{y}^j_U(\lambda) = \max g_i\left(\mathbf{d}, x^j, p^j\right)\Big|\left(x^j, p^j\right)\in\mathbf{X}_\lambda \tag{5}$$

$$\tilde{y}^j_L(\lambda) = \min g_i\left(\mathbf{d}, x^j, p^j\right)\Big|\left(x^j, p^j\right)\in\mathbf{X}_\lambda \tag{6}$$

where $\tilde{y}^j_U(\lambda)$ and $\tilde{y}^j_L(\lambda)$ are the upper and lower boundaries for the jth sampling, respectively. Consequently, $\tilde{y}^j(\lambda)$ represented by $\left[\tilde{y}^j_L(\lambda), \tilde{y}^j_U(\lambda)\right]$ $\lambda\in[0,1]$, could be obtained.

3.4. Step 4: computing the first four order cumulants

CGF is actually a polynomial with cumulants $k_r, r=1,2,\cdots$ as the coefficients [24].

$$K(\xi) = k_1\xi + \frac{1}{2}k_2\xi^2 + \cdots + \frac{1}{r!}k_r\xi^r + \cdots \tag{7}$$

In principle, all the cumulants could be obtained with samples. To make the tradeoff between computational efficiency and accuracy, it is considered suitable to approximate the CGF with the first four order cumulants [25].

$$\begin{cases} \tilde{k}_2 = \dfrac{N\tilde{s}_2 - \tilde{s}_1^2}{N(N-1)} \\ \tilde{k}_3 = \dfrac{2\tilde{s}_1^3 - 3N\tilde{s}_1\tilde{s}_2 + N^2\tilde{s}_3}{N(N-1)(N-2)} \\ \tilde{k}_4 = \dfrac{-6\tilde{s}_1^4 + 12N\tilde{s}_1^2\tilde{s}_2 - 3N(N-1)\tilde{s}_2^2 - 4N(N+1)\tilde{s}_1\tilde{s}_3 + N^2(N+1)\tilde{s}_4}{N(N-1)(N-2)(N-3)} \end{cases} \tag{8}$$

where

$$\tilde{s}_r = \sum_{j=1}^{N}\left(\tilde{y}^j\right)^r. \tag{9}$$

It is noted that $\tilde{k}_1 \sim \tilde{k}_4$ are fuzzy variables since they are the functions of fuzzy variables $\tilde{y}^j$. It is difficult to calculate the membership functions associated with $\tilde{k}_1 \sim \tilde{k}_4$ with the extension principle. The optimization procedure is therefore needed to calculate the boundaries of $\tilde{k}_1 \sim \tilde{k}_4$ at the same λ level as that for calculating $\tilde{y}^j$.

$$k_1^U(\lambda) = \max\frac{\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)}{N}\,|y\in Y_\lambda \tag{10}$$

$$k_1^L(\lambda) = \min\frac{\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)}{N}\,|y\in Y_\lambda \tag{11}$$

$$k_2^U(\lambda) = \max\frac{N\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^2 - \sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2}{N(N-1)}\,|y\in Y_\lambda \tag{12}$$

$$k_2^L(\lambda) = \min\frac{N\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^2 - \sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2}{N(N-1)}\,|y\in Y_\lambda \tag{13}$$

$$k_3^U(\lambda) = \max\frac{2\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^3 - 3N\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2 + N^2\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^3}{N(N-1)(N-2)}\,|y\in Y_\lambda \tag{14}$$

$$k_3^L(\lambda) = \min \frac{2\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^3 - 3N\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2 + N^2\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^3}{N(N-1)(N-2)}\Bigg|y\in Y_\lambda \tag{15}$$

$$\begin{aligned} k_4^U(\lambda) = \max & \frac{-6\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^4 + 12N\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^2\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2 - 3N(N-1)\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2\right]^2}{N(N-1)(N-2)(N-3)} \\ & + \frac{-4N(N+1)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^3 + N^2(N+1)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^4}{N(N-1)(N-2)(N-3)}\Bigg|y\in Y_\lambda \end{aligned} \tag{16}$$

$$\begin{aligned} k_4^L(\lambda) = \min & \frac{-6\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^4 + 12N\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\right]^2\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2 - 3N(N-1)\left[\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^2\right]^2}{N(N-1)(N-2)(N-3)} \\ & + \frac{-4N(N+1)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^3 + N^2(N+1)\sum_{j=1}^{N}\left(\tilde{y}^j(\lambda)\right)^4}{N(N-1)(N-2)(N-3)}\Bigg|y\in Y_\lambda \end{aligned} \tag{17}$$

3.5. Step 5: approximation the cumulative generating function

The CGF approximated with the first four order cumulants is expressed by:

$$K(\xi) = \tilde{k}_1(\lambda)\xi + \frac{1}{2}\tilde{k}_2(\lambda)\xi^2 + \frac{1}{3!}\tilde{k}_3(\lambda)\xi^3 + \frac{1}{4!}\tilde{k}_4(\lambda)\xi^4. \tag{18}$$

3.6. Step 6: solving the saddlepoint

The first order derivative of the approximated CGF with respect to ξ is:

$$K'(\xi) = \tilde{k}_1(\lambda) + \tilde{k}_2(\lambda)\xi + \frac{1}{2}\tilde{k}_3(\lambda)\xi^2 + \frac{1}{6}\tilde{k}_4(\lambda)\xi^3. \tag{19}$$

The solution of Eq. (19) is the saddlepoint.

Accounting for the degradation mechanism of a product performance due to aging, erosion and strength reduction, saddlepoint could be obtained by solving a time-dependent equation.

$$K'(\xi) = \tilde{k}_1(\lambda) + \tilde{k}_2(\lambda)\xi + \frac{1}{2}\tilde{k}_3(\lambda)\xi^2 + \frac{1}{6}\tilde{k}_4(\lambda)\xi^3 = \tilde{y}(\lambda;t) \tag{20}$$

where t indicates time, and $\tilde{y}(\lambda;t)$ is a fuzzy random process. For a simplification, the saddlepoint at any time t_i is solved without considering the autocorrelation between different time t_i and t_j. The method proposed in Ref. [26] is employed to solve Eq. (20).

When $\tilde{k}_1$~$\tilde{k}_4$ are crisp variables, there are three solutions for Eq. (20). A method for selecting a proper saddlepoint is given in Ref. [14]. But for Eq. (20), there could be more than three solutions when $\tilde{k}_1$~$\tilde{k}_4$ are fuzzy variables. Herein, the selection method is provided with $\tilde{k}_1(\lambda)$~$\tilde{k}_4(\lambda)$ being interval variables, shown in Table 1.

3.7. Step 7: estimating cumulative distribution function

With the obtained saddlepoint $\xi_s(t)$, the probability density function at λ level could be expressed by:

$$f(y(t)) = \left[\frac{1}{2\pi K''(\xi_s(t))}\right]^{\frac{1}{2}} e^{[K(\xi_s(t)) - \xi_s(t)y(t)]}\Big|_\lambda \tag{21}$$

where $K''(\xi_s(t))$ is the second derivative of $K(\xi(t))$ at $\xi(t) = \xi_s(t)$.

$$K''(\xi_s(t)) = k_2 + k_3\xi_s(t) + \frac{1}{2}k_4\xi_s^2(t) \tag{22}$$

Table 1
The saddlepoint selection method.

Cases	Conditions	Solutions
Case 1	$k_4(\lambda)\in(-\infty,0)$	$\xi_l^L = \min\left\{\frac{k_3(\lambda)-\sqrt{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)}}{-k_4(\lambda)}\right\}$, $\xi_u^U = \max\left\{\frac{k_3(\lambda)+\sqrt{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)}}{-k_4(\lambda)}\right\}$, $\xi\in(\xi_l^L,\xi_u^U)$
Case 2	$k_4(\lambda)\in[0,\infty)$ and $\Delta^U=\max\{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)\}\le 0$	$\xi\in R^n$
Case 3	$k_4(\lambda)\in[0,\infty)$ and $\Delta^L=\min\{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)\}>0$ and $k_3(\lambda)>0$	$\xi\in(\xi_l^L,\infty)$
Case 4	$k_4(\lambda)\in[0,\infty)$ and $\Delta^L=\min\{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)\}>0$ and $k_3(\lambda)<0$	$\xi\in(-\infty,\xi_u^U)$
Case 5	$k_4(\lambda)\in[0,\infty)$ and $\Delta^U=\max\{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)\}>0$ and $\Delta^L=\min\{k_3^2(\lambda)-2k_2(\lambda)k_4(\lambda)\}<0$	If only one solution $\xi\in R^n$ If more than one solutions and $k_3(\lambda)>0$ $\xi\in(\xi_l^L,\infty)$ If more than one solutions and $k_3(\lambda)<0$ $\xi\in(-\infty,\xi_u^U)$

The CGF at λ level is given by:

$$F_Y(y(t)) = \Pr\{Y\le y(t)\}\Big|_\lambda = \left[\Phi(w)+\phi(w)\left(\frac{1}{w}-\frac{1}{v}\right)\right]\Big|_\lambda \tag{23}$$

where

$$w = \operatorname{sgn}(\xi_s(t))\{2[\xi_s(t)y(t)-K(\xi_s(t))]\}^{1/2} \tag{24}$$

$$v = \xi_s\left[K''(\xi_s(t))\right]^{1/2}. \tag{25}$$

3.8. Step 8: calculating the reliability

By accounting for the relationship between the cumulative distribution and reliability, the expression of reliability could be given by:

$$R(\lambda;t) = 1-\left[\Phi(w)+\phi(w)\left(\frac{1}{w}-\frac{1}{v}\right)\right]\Big|_\lambda \tag{26}$$

$K''(\xi_s)\ge 0$ is always satisfied and the detailed proof is illustrated in Ref. [14]. If $0\le\lambda<1$, $R(\lambda)$ is not a deterministic value but an interval while if $\lambda=1$, $R(\lambda)$ is a deterministic value .

4. Reliability analysis methods of complex systems

In engineering practices, a complex system is usually consisted of large numbers of components, including structural components with gradual failure and components with sudden failure. For an easy demonstration, a system with series–parallel configuration is taken as an example.

As shown in Fig. 4, there are three components in the system. Herein, component 1 is assumed to be a structural component with gradual failure, and components 2 and 3 are components with sudden failure. The time-dependent reliability of a structural component could be obtained with the method demonstrated in Section 3. The reliability of components 2 and 3 could be evaluated with failure data $n(t_j)$ at some time t_j. Failure data are usually considered imprecise in handling engineering problems [7]. Two methods are proposed to conduct reliability analysis for the system in terms of different scenarios. There are two steps in

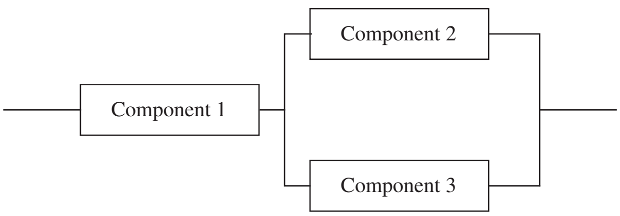

Fig. 4. A system with the series–parallel configuration.

the first method. The system reliability at time t_j by integrating the reliability of components at the same time t_j at a given λ level is provided in the first step.

$$\tilde{R}_S\left(t_j\right)\Big|_\lambda = \tilde{R}_1\left(t_j\right)\left\{1-\left[1-\tilde{R}_2\left(t_j\right)\right]\left[1-\tilde{R}_2\left(t_j\right)\right]\right\}|_\lambda \tag{27}$$

It is noted that $\tilde{R}_S(t_j)|_\lambda$ is an interval $\left[\tilde{R}_S^L(t_j)\Big|_\lambda, \tilde{R}_S^U(t_j)|_\lambda\right] \in [0,1]$. By accounting for the existence of uncertainty during the design, manufacture process in the second step, and the cumulative distribution function, the lifetime of the system are assumed to follow a three-parameter Weibull distribution expressed by:

$$\tilde{R}_S(t)\Big|_\lambda = 1-F(t;\alpha,\beta,\gamma) = e^{-\left(\frac{t+\gamma}{\alpha}\right)^\beta}. \tag{28}$$

Because $\tilde{R}_S(t_j)|_\lambda$ is an interval, the parameters of Weibull distribution are also intervals. Hence $\tilde{R}_S(t)|_\lambda$ is a time-dependent reliability interval bounded by two time-dependent reliability functions $\tilde{R}_S^L(t)|_\lambda$ and $\tilde{R}_S^U(t)|_\lambda$.

The time-dependent system reliability $\tilde{R}_S(t)|_\lambda$ is to integrate the time-dependent reliability of the components in the second method. Then the time-dependent reliability at a given λ level is given:

$$\tilde{R}_S(t)\Big|_\lambda = \tilde{R}_1(t)\left\{1-\left[1-\tilde{R}_2(t)\right]\left[1-\tilde{R}_2(t)\right]\right\}|_\lambda. \tag{29}$$

The same conclusion as that in the first method that $\tilde{R}_S(t)|_\lambda$ is a time-dependent reliability interval bounded by two time-dependent reliability functions $\tilde{R}_S^L(t)|_\lambda$ and $\tilde{R}_S^U(t)|_\lambda$ is arrived. The distribution type of the system reliability is assumed in the first method while the distribution type of all the components is assumed in the second method.

5. An illustrated example

A system consist of three subsystems is employed to illustrate the proposed methods, where two subsystems with sudden failure form a parallel configuration and then serially connected with a subsystem with gradual failure. A single helical gear reducer is considered as the subsystem with gradual failure in this example.

5.1. Reliability evaluation of the single helical gear reducer

Helical gear reducer is widely used in engineering practices, which allows the engine to rotate at its most efficient speed. There are two deterministic design variables: normal module m_n and the number of pinion teeth z_1. Face width b and helix angle β are considered to be random design variables. There are four random parameters P_1~P_4, including the material properties Z_E, the rotation speed n, the engine power P and bending stress fatigue limit $\sigma_{F\,\min}$. Parameters P_1~P_4 are considered as fuzzy random variables by accounting for the three conditions: lack of sufficient data, data with fuzziness, and unknown or non-constant reproduction conditions in engineering practices. The stochastic information of design variables and parameters is given in Table 2.

The bending failure is one of important failure modes for a gear. The limit state function associated with the failure mode is defined as the difference between the allowable bending stress and bending stress:

$$g(\mathbf{d},\mathbf{X},\mathbf{P};t) = \frac{\sigma_{H\,\lim}Z_N}{S_{H\,\min}} - \frac{2\times 9.55\times 10^6 PK}{bd_1 nm_n}Y_{FS}Y_\varepsilon Y_\beta - D(t).$$

The strength of gear usually degrades with time because of some factors, such as wear, fatigue, and erosion. Furthermore, the degradation of strength is testified to follow a Gamma process [27] and $D(t) = \text{Gamma}(0.1, 0.12t)$ is provided in this example.

Table 2
Distribution information of design variables and parameters.

Variables	Variables	Mean	Std	Distribution type
d	z_1	26	–	–
	m_n (mm)	2.5	–	–
X	b (mm)	68	0.05	Normal
	β (degree)	12	0.05	Normal
P	P (kw)	15	1.5	Normal
	n (rpm)	970	97	Normal
	Z_E ($\sqrt{\text{MPa}}$)	189.8	18.98	Normal
	$\sigma_{F\,\min}$ (MPa)	560	56	Normal

Herein, the triangle membership function is employed to represent the fuzziness of the mean values and hence $\tilde{P}\sim N(\tilde{\mu}_P, \sigma_P)$, $\tilde{n}\sim N(\tilde{\mu}_n, \sigma_n)$, $\tilde{Z}_E\sim N\left(\tilde{\mu}_{Z_E}, \sigma_{Z_E}\right)$ and $\tilde{\sigma}_{F\,\mathrm{min}}\sim N\left(\tilde{\mu}_{\sigma_{F\,\mathrm{min}}}, \sigma_{\sigma_{F\,\mathrm{min}}}\right)$. The membership functions for $\tilde{\mu}_P$, $\tilde{\mu}_n$, $\tilde{\mu}_{Z_E}$ and $\tilde{\mu}_{\sigma_{F\,\mathrm{min}}}$ are provided.

$$\mu_P(x) = \begin{cases} \dfrac{x-0.2}{0.2} & x\in[14.8, 15] \\ \dfrac{x+0.2}{0.2} & x\in[15, 15.2] \\ 0 & \text{others} \end{cases}$$

$$\mu_n(x) = \begin{cases} \dfrac{x-20}{20} & x\in[950, 970] \\ \dfrac{x+20}{20} & x\in[970, 990] \\ 0 & \text{others} \end{cases}$$

$$\mu_{Z_E}(x) = \begin{cases} \dfrac{x-10}{10} & x\in[179.8, 189.8] \\ \dfrac{x+10}{10} & x\in[189.8, 199.8] \\ 0 & \text{others} \end{cases}$$

$$\mu_{\sigma_{F\,\mathrm{min}}}(x) = \begin{cases} \dfrac{x-4}{4} & x\in[556, 560] \\ \dfrac{x+4}{4} & x\in[560, 564] \\ 0 & \text{others} \end{cases}$$

The reliability at time t_i under a given level λ could be represented by:

$$R(\lambda; t_i) = \Pr\{g(\mathbf{d}, \mathbf{X}, \mathbf{P}; t_i)\geq 0\}\Big|_\lambda = \Pr\left\{\frac{\sigma_{H\,\mathrm{lim}}Z_N}{S_{H\,\mathrm{min}}} - \frac{2\times 9.55\times 10^6 PK}{bd_1 nm_n} Y_{FS}Y_\varepsilon Y_\beta - D(t_i)\geq 0\right\}|_\lambda.$$

The time-dependent reliability is plotted in Fig. 5. It is known that the initial reliability is not but less than 1. The reason is that uncertainties exist in the products before they are put into operation. $R(\lambda; t)$ is a family of time-dependent reliability functions bounded by the lower and upper boundaries when $\lambda\neq 1$. When $\lambda=1$, $R(\lambda; t)$ is a deterministic time-dependent reliability function.

5.2. System reliability evaluation

For subsystems with sudden failure, statistics-based reliability analysis could be conducted. The center and width of the observed lifetime data for subsystems 2 and 3 are given in Table 3 by accounting for the fuzziness of lifetime data and triangle membership function is used.

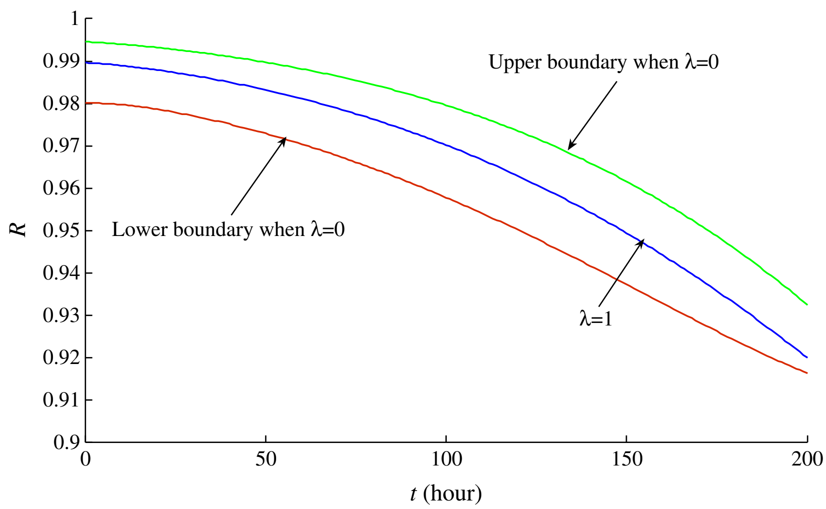

Fig. 5. Time-dependent reliability of the single helical gear reducer.

Table 3
Lifetime data of the subsystems 2 and 3.

Subsystem 2	Center	231	206	242	245	217	263	261	246	243
	Width	2.4	2.1	2.6	2.8	2.4	2.5	2.6	2.4	2.9
Subsystem 3	Center	186	204	178	233	187	192	211	191	188
	Width	2.0	2.2	1.9	2.6	2.1	2.3	2.5	2.4	2.3

When the first method proposed in Section 4 is applied, the system reliability is provided.

$$R_S^U(t) = \exp\left[-\left(\frac{t+19.6915}{3409.8}\right)\right]^{1.0125}$$

$$R_S(t) = \exp\left[-\left(\frac{t+21.24}{4190.7}\right)\right]^{0.8630}$$

$$R_S^L(t) = \exp\left[-\left(\frac{t+22.862}{7917.5}\right)\right]^{0.6673}$$

When the second method is applied and the lifetime of the subsystems is assumed to be normally distributed. The two boundaries of time-dependent reliability for the two subsystems at level 0 and the time-dependent reliability at level 1 are provided respectively.

$$R_2^L(t) = 1-\int_0^t \frac{1}{38.8\sqrt{2\pi}} \exp\left(\frac{x-237.2}{2\times 38.8^2}\right)dx$$

$$R_2(t) = 1-\int_0^t \frac{1}{38.9\sqrt{2\pi}} \exp\left(\frac{x-239.4}{2\times 38.9^2}\right)dx$$

$$R_2^U(t) = 1-\int_0^t \frac{1}{39.0\sqrt{2\pi}} \exp\left(\frac{x-241.7}{2\times 39.0^2}\right)dx$$

$$R_3^L(t) = 1-\int_0^t \frac{1}{37.6\sqrt{2\pi}} \exp\left(\frac{x-193.7}{2\times 37.6^2}\right)dx$$

$$R_3(t) = 1-\int_0^t \frac{1}{37.8\sqrt{2\pi}} \exp\left(\frac{x-195.7}{2\times 37.8^2}\right)dx$$

$$R_3^U(t) = 1-\int_0^t \frac{1}{37.9\sqrt{2\pi}} \exp\left(\frac{x-197.6}{2\times 37.9^2}\right)dx$$

With the proposed method in Section 3, the two boundaries at level 0 and the deterministic one at level 1 of time-dependent reliability for subsystem 1 are provided.

$$R_1^U(t) = \exp\left[-\left(\frac{t+17.8927}{6111.3}\right)^{0.8946}\right]$$

$$R_1(t) = \exp\left[-\left(\frac{t+19.0106}{8857.7}\right)^{0.7423}\right]$$

$$R_1^L(t) = \exp\left[-\left(\frac{t+20.4651}{24225}\right)^{0.5514}\right]$$

Then the two boundaries at level 0 and the deterministic one at level 1 of time-dependent system reliability could be expressed by:

$$R_S^U(t) = \exp\left[-\left(\frac{t+17.8927}{6111.3}\right)^{0.8946}\right] \times\left[1-\int_0^t \frac{1}{39.0\sqrt{2\pi}} \exp\left(\frac{x-241.7}{2\times 39.0^2}\right)dx \int_0^t \frac{1}{37.9\sqrt{2\pi}} \exp\left(\frac{x-197.6}{2\times 37.9^2}\right)dx\right]$$

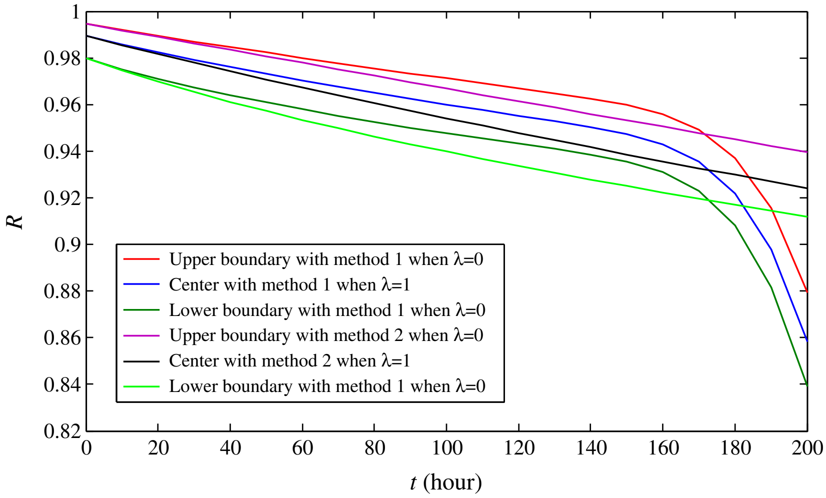

Fig. 6. Time-dependent system reliability.

$$R_S(t) = \exp\left[-\left(\frac{t+19.0106}{24225}\right)^{0.7423}\right] \times\left[1-\int_0^t \frac{1}{38.9\sqrt{2\pi}}\exp\left(\frac{x-239.4}{2\times 38.9^2}\right)dx\int_0^t \frac{1}{37.8\sqrt{2\pi}}\exp\left(\frac{x-195.7}{2\times 37.8^2}\right)dx\right]$$

$$R_S^L(t) = \exp\left[-\left(\frac{t+20.4651}{24225}\right)^{0.5514}\right] \times\left[1-\int_0^t \frac{1}{38.8\sqrt{2\pi}}\exp\left(\frac{x-237.2}{2\times 38.8^2}\right)dx\int_0^t \frac{1}{37.6\sqrt{2\pi}}\exp\left(\frac{x-193.7}{2\times 37.6^2}\right)dx\right].$$

The results with the two methods are plotted in Fig. 6.

It is noted that the initial reliability of a system is less than 1 at time $t=0$, which is different from the statistics-based reliability method in the reliability engineering. The reason is that the initial reliability of structural component is usually less than 1 in terms of the uncertainties of design variables and parameters in the design and manufacturing process.

6. Conclusions

Three conditions could be met when conducting reliability analysis in engineering practices: (1) the available data is so insufficient that the statistical properties could be not expressed properly; (2) the field or experimental data possess fuzziness; (3) field data or experimental data are obtained under such conditions, which is unknown or non-constant reproduction. Fuzzy random variables exhibit a good capacity of representing these three conditions. In this paper, the SPA-simulation is extended to deal with reliability analysis accounting for the time-dependent degradation process and fuzzy random variables. Furthermore, we attempt to provide a method to select a proper saddlepoint. To address the reliability analysis of complex system consisted of subsystems with sudden failure and subsystems with gradual failure, two methods are proposed. To further construct more credible reliability model, more issues should be accounted for: (1) the coefficient of autocorrelation of the stochastic process; (2) more accurate solution and selection method of saddlepoint.

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