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Analysis of maintenance policies for finite life-cycle multi-state systems

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ABSTRACT

Maintenance policies for multi-state systems (MSS) are often analyzed under infinite horizon assumptions. In practice, it is important to consider maintenance policies under a finite horizon because the life cycles of most systems are finite. In this paper, we consider a finite life-cycle MSS that is subject to both degradation and Poisson failures. We study two classes of maintenance policies – preventive replacements and corrective replacements, and their effectiveness in controlling the customer's expected discounted maintenance cost (EDMC). For both policies, replacement decisions are modelled via two control parameters – a threshold on the current system state and a threshold on the residual life cycle, which is measured as the time span from present to the end of life cycle. We derive close-to-explicit forms of the cost models under each of the policy. Methodologies for optimizing the maintenance thresholds are further proposed. Computational results verify that preventive replacements outperform corrective replacements typically when the downtime cost per failure is relatively high compared to the repair cost.

1. Introduction

The ageing processes of a system are often modelled as continuous and deterministic functions of time (e.g. failure rate function). However, this might not be realistic for many systems when the processes depend not only on the elapsed operational time, but also the system status, such as vibration level, efficiency, number of random shocks on the system, etc., any of which causes performance degradation. For such multi-state degradation systems, the sojourn times at each state are typically uncertain and therefore result in uncertainty in the state-dependent failure rates. The basic concepts of stochastic multi-state degradation models can be found in Barlow and Wu (1978), El-Neweihi, Proschan, and Sethuraman (1978) and Ross (1979), which defined the system structure function and its properties. The corresponding performance analysis (e.g. reliability, availability, mean time-to-failure, redundancy, etc.) were addressed by Xue and Yang (1995), Pham, Suprasad, and Misra (1997), Wu (2005), Zuo and Tian (2006), Tian, Yam, Zuo, and Huang (2008a, 2008b) and Tai and Chan (2010).

Optimization of maintenance policies for multi-state systems (MSS) is a natural extension of the maintenance studies for the binary systems which utilize many results from the reliability modelling of MSS. The majority of the current literatures assume that maintenance actions for MSS are planned based on an infinite operating horizon and after any replacement or restoration, the system is renewed and the same process is assumed to repeat indefinitely. Characteristics of a system, such as the current state, the age and the elapsed operating time during each state, are often selected as the optimality criterions and used to minimize the long-run average maintenance cost rate function. Reviews of work in this area can be found in Kao (1973), Sim and Endrenyi (1993), Yeh (1996), Levitin and Lisnianski (2000), Grall, Berenguer, and Dieulle (2002), Moustafa, Abdel Maksoud, and Sadek (2004) and Kim and Makis (2009).

In practice, however, the useful life cycle of most systems are finite in nature. For instance, in military applications, a missile launching system is only required to be functioning within the designated mission time. Different from an infinite-horizon maintenance problem, residual life cycle for such system, which is measured from the present time to the end of the mission, is typically finite and decreases over time. When the mission is close to end, replacement of a functioning system becomes less necessary and traditional maintenance strategies, such as those merely relying on the information of the current system state, could turn out to be very costly to the stakeholders. Considering the improper planning horizon, though bringing technical convenience, may not be realistic under these circumstances (Nakagawa & Mizutani, 2009). On the other hand, compared to the vast amount of literature in infinite-horizon maintenance planning, existing work showed very limited options for maintaining MSS with finite life cycles. Su and Chang (2000) proposed a periodic maintenance policy for MSS and derived the optimal number of maintenance activities that minimized the total life-cycle cost. Zuo, Liu, and Murthy

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(2000) investigated the optimal replacement policy for multi-state products under warranty such that the manufacturer's expected warranty cost was minimized. Ivy and Pollock (2005) and Maillart and Zheltova (2007) analyzed maintenance and inspection policies for a discrete-time Markov system over a finite horizon given that perfect observations of systems states were not available. Ding, Lisnianski, Frenkel, and Khvatskin (2009) studied the optimal corrective maintenance planning for the MSS subject to availability constraints. Among these studies, very few further considered maintenance optimization with multiple optimality criterions.

In this paper, we assume that the ageing process of the system is modelled as a continuous-time Markov process that is subject to both degradation and Poisson failures. We assume that the system can fail randomly from any of the operational states (Poisson failures) and can be rectified by minimal repair which returns the system to its previous working state. Any unexpected (Poisson) failure is assumed to result in an extra downtime cost that is borne by the customer. We propose two MSS maintenance policies for controlling the customer's *expected discounted maintenance cost* (EDMC) over a finite system life cycle. The first policy conducts preventive system replacement, i.e. a system may be replaced while still operational. In contrast the second policy allows only corrective replacements, i.e. system replacements are only made when the system suffers a random failure. For both policies, the EDMC is derived as a function of two control parameters, namely, a threshold level on the current state of the system, and a threshold level on the residual life cycle (measured from present time to the end of life cycle). We further propose two different methodologies for the optimization of maintenance thresholds. The first method utilizes the Laplace transform and inversion techniques, while the second method directly approximates the EDMC and optimizes the maintenance thresholds on the time domain. The applications of both methods are illustrated using a numerical case. Through computational examples, we demonstrate that preventive replacement outperforms corrective replacement when the downtime cost per failure is relatively high compared to the repair cost. Unlike past works, our studies incorporate many realistic factors, i.e. multiple system states, discounted economic values, finite planning horizon, and easy-to-implement maintenance policies. As such, our work should be of interest to both theoreticians and practitioners.

The rest of the paper is organized as follows. In Section 2, we present the system descriptions and propose the maintenance policies for the MSS. Section 3 derives the EDMC for the customer under both Policies A and B. Methodologies for analyzing the optimal maintenance policies are further proposed in Section 4. Section 5 demonstrates the applicability of the foregoing analysis with numerical examples. Conclusions are made in Section 6.

2. Model formulation

Acronym	
MSS	multi-state system
EDMC	expected discounted maintenance cost
Policies O, A, B	maintenance options to the customer
Notation	
L	length of the finite life cycle
t	residual life cycle
y	elapsed system lifetime
δ	discounted factor
N, $2N+1$	number of degradation stages, number of system states
Ω	$\{1, 2, 3, \ldots, N\}$
$S_1 = \{2i-1, i \in \Omega\}$	operational states of the MSS
$S_2 = \{2i, i \in \Omega\}$	(Poisson) failure states of the MSS
$\{2N+1\}$	complete failure state
$S_1 \cup S_2 \cup \{2N+1\}$	entire set of the system sates
α_i, λ_i $(i \in \Omega)$	degradation rate from State $2i-1$ to $2i+1$, failure rate from State $2i-1$ to $2i$
m_i, d_i $(i \in \Omega)$	minimal repair and downtime cost for the Poisson failures from State $2i-1$
r_i $(i \in \Omega)$	replacement cost at State $2i-1$ and $2i$
d_{2N+1}, r_{2N+1}	downtime and replacement cost when the system reaches a complete failure
$p_i(y)(g_i(y))$	probability density function of the time to first failure (degradation) for a 1-stage degradation system that initially works under State $2i-1$
(J, τ)	maintenance thresholds on the current system state and the residual life cycle
$C_i^{(O)}(t)$, $\widetilde{C}_i^{(O)}(t)$	the exact value and numerical approximation of EDMC for Policy O when the system is in operational State $2i-1$ and the residual life cycle is t
$C_i^{(A)}(t\vert J,\tau)$, $\widetilde{C}_i^{(A)}(t\vert J,\tau)$	the exact value and numerical approximation of EDMC for Policy A under (J,τ) when the system is in operational State $2i-1$ and the residual life cycle is t
$C_i^{(B)}(t\vert J,\tau)$, $\widetilde{C}_i^{(B)}(t\vert J,\tau)$	the exact value and numerical approximation of EDMC for Policy B under (J,τ) when the system is in operational State $2i-1$ and the residual life cycle is t
(J^*, τ^*)	optimal maintenance thresholds that minimize $\left\{\widetilde{C}_1^{(A)}(L\vert J,\tau), \widetilde{C}_1^{(B)}(L\vert J,\tau)\right\}$

2.1. System description

Consider a multi-state system (MSS) that initially works under a perfect condition. The system can have N stages of degradation before reaching a complete failure and let $\Omega = \{1, 2, 3, \ldots, N\}$ represent the set of all these stages. We define three disjoint sets of the states which fully characterize the MSS – the operational states $S_1 = \{2i-1, i \in \Omega\}$, the (Poisson) failure states $S_2 = \{2i, i \in \Omega\}$ and the complete failure state $\{2N+1\}$. State 1 represents the perfect functioning state and the degree of deterioration increases with each subsequent operational state. In particular, once the system degrades to State $2N+1$, it is considered as completely failed and can only be rectified by a replacement. Here the ith stage degradation is defined as the transition period from State $2i-1$ to $2i+1$ $(i \in \Omega)$ and is characterized by a degradation rate α_i. In addition to the degradation process, the system is also subject to random (Poisson) failure process from any operational state $2i-1$ (i.e. $2i-1 \to 2i, i \in \Omega$) and can be rectified by repair. The failure from operational state $2i-1$ $(i \in \Omega)$ to failure state $2i$ is characterized by a failure rate λ_i.

Let r_i $(i \in \Omega)$ represent the replacement cost for the system when the system is either operating in State $2i-1$ or failed from State $2i-1$, and let $m_i(i \in \Omega)$ represent the corresponding minimal repair cost during this stage. For each (unexpected) Poisson failure, we assume that there is an additional downtime cost d_i associated with m_i which is borne by the customer. Also, let r_{N+1} and d_{N+1} represent the replacement and downtime cost for the system when it reaches a complete failure (i.e. State $2N+1$).

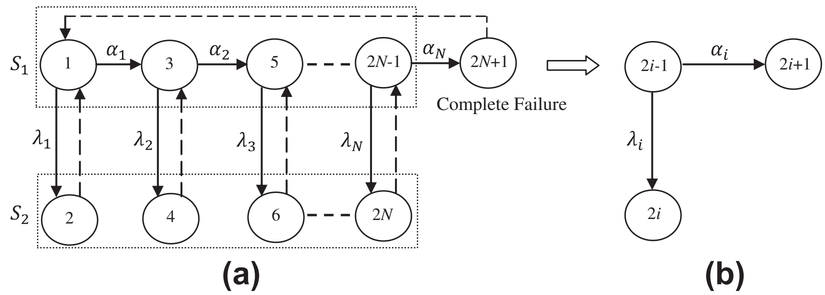

Fig. 1. System description. (a) Description for an N-stage deteriorating MSS; (b) Description for an isolated 3-state MSS.

A graphical description of the above system is given in Fig. 1a. In order to complete our model formulation, the following assumptions are made.

(1) The system is replaced with a new one once it degrades to the complete failure state ($2N+1$).
(2) The system is minimally repaired after Poisson failures. The repair returns the system to the operational state right before the failure.
(3) All the transition rates α_is and λ_is ($i \in \Omega$) are constant but state-dependent. In particular, we assume $\lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \lambda_N$ to describe the ageing of the system.
(4) The average repair and replacement time is very small compared to mean time between failures and therefore is negligible.
(5) The system becomes more costly to repair and replace when it ages, i.e. $m_1 < m_2 < \cdots < m_{N-1} < m_N$ and $r_1 < r_2 < \cdots < r_{N-1} < r_N < r_{N+1}$.
(6) No downtime cost is incurred or associated with preventive replacement when the system is still functioning.
(7) The current state of the system is always known (observed) for certain by continuous monitoring.

2.2. System replacement policies

We denote L as the finite planning horizon, or the system life cycle, and δ as the continuous discounted factor over the cycle. It is important to notice that maintenance cost is not incurred at the stage of maintenance planning but rather spent in future and allocated over the system life cycle. Therefore, incorporating (J,δ) in the cost forecasting will have practical meanings, in particular for those managerial circumstances such as budget allocation and balance-sheet reporting where the accuracy of cost estimation is crucial to the decision makers. As an endeavour to minimize the EDMC for the customer, we propose the following two maintenance policies (A and B), both of which rely on two threshold parameters (J,τ), where $0 \leqslant J < N$ and $0 \leqslant \tau \leqslant L$.

Policy A (preventive replacement): If the system operates in State $2i-1$ ($i \in \Omega$) and the residual life cycle is t ($0 < t \leqslant L$), it is then replaced by a new one if and only if $j+1 \leqslant i \leqslant N$ and $t \geqslant \tau$; otherwise, no replacement is made.

Policy B (corrective replacement): If the system fails from State $2i-1$ ($i \in \Omega$) and the residual life cycle is t ($0 < t \leqslant L$), it is then replaced by a new one if and only if $J+1 \leqslant i \leqslant N$ and $t \geqslant \tau$; otherwise, it is minimally repaired.

Both policies utilize the information of the current system state and the residual life cycle. The system is replaced only when its deterioration level is heavier than the threshold parameter J and the residual life cycle is longer than τ. Such policies can avoid expensive replacements when the system is still relatively healthy or when the system is close to retirement. Policy A requires *preventive replacement* for the system when it is still functioning. On the other hand, Policy B implements *corrective replacement* for the system only upon (Poisson) failures.

For comparison purpose, the base case of no corrective or preventive replacement is also defined (i.e. Policy O). Note that when $\tau = L$, both Policies A and B reduce to Policy O.

Policy O: No corrective or preventive replacement.

3. Model development

In this section, we derive the close-to-explicit forms of the EDMC for the customer under Policies O, A and B. The discounted cost models are presented in recursive forms and solved iteratively.

We present some preliminary results for a 1-stage degradation system before proceeding to the analysis of N-stage degradation system.

3.1. Preliminary results

Consider a 3-state Markov system ($i \in \Omega$) in Fig. 1b that is isolated from Fig. 1a. In contrast to Fig. 1a, we assume that both State $2i$ and $2i+1$ are absorbing states. The objective is to derive the system state transition (degradation) and time-to-failure distributions that are useful in subsequent analysis.

Let $I(Y)$ represent the system state after an elapsed lifetime y. We assume that the system initially operates at State $2i-1$, i.e. $I(0) = 2i-1$. Define $Q_i(y) = \Pr\{I(y) = 2i-1 | I(0) = 2i-1\}$, $P_i(y) = \Pr\{I(y) = 2i | I(0) = 2i-1\}$ and $G_i(y) = \Pr[I(y) = 2i+1 | I(0) = 2i-1]$. Also, define $p_i(y) = dP_i(y)/dy$ and $g_i(y) = dG_i(y)/dy$ as the corresponding probability densities of system failure and degradation at time y. The Chapman–Kolmogorov equations for such a simple Markov system can be written as

$$\begin{cases} \frac{dQ_i(y)}{dy} = -(\alpha_i + \lambda_i)Q_i(y) \\ \frac{dP_i(y)}{dy} = \lambda_i Q_i(y) \\ \frac{dG_i(y)}{dy} = \alpha_i Q_i(y) \end{cases} \tag{1}$$

with the initial conditions satisfying $Q_i(0) = 1$, $P_i(0) = 0$ and $G_i(0) = 0$ ($i \in \Omega$). Solutions for (1) are explicitly given as $Q_i(y) = e^{-(\alpha_i+\lambda_i)y}$, $P_i(y) = \lambda_i[1 - e^{-(\alpha_i+\lambda_i)y}]/(\alpha_i + \lambda_i)$ and $G_i(y) = \alpha_i[1 - e^{-(\alpha_i+\lambda_i)y}]/(\alpha_i + \lambda_i)$. Therefore, $p_i(y) = \lambda_i e^{-(\alpha_i+\lambda_i)y}$ and $g_i(y) = \alpha_i e^{-(\alpha_i+\lambda_i)y}$.

3.2. The EDMC model for Policy O

Here we investigate the EDMC model for the N-stage degradation system under Policy O.

Let $C_i^{(0)}(t)$ represent the EDMC under Policy O when the system is in State $2i-1$ ($i \in \Omega$) and the residual life cycle is t. The objective is therefore to obtain $C_1^{(0)}(L)$. Note that without preventive or corrective replacement, the system is automatically replaced by a new unit when it degrades to State $2N+1$, until the end of its life cycle. Using the expressions for $p_i(t)$ and $g_i(t)$ and incorporating the discounted factor δ, the recursive form of the cost model is presented as follow:

$$\begin{cases} C_i^{(O)}(t) = \int_0^t \left[m_i + d_i + C_i^{(O)}(x)\right] e^{-\delta(t-x)} p_i(t-x)dx \\ \quad + \int_0^t C_{i+1}^{(O)} e^{-\delta(t-x)} g_i(t-x)dx \\ \quad = \int_0^t \left[m_i + d_i + C_i^{(O)}(x)\right] \lambda_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \quad + \int_0^t C_{i+1}^{(O)}(x)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \quad \text{for } i = 1,2,\ldots,N-1 \\ C_N^{(O)}(t) = \int_0^t \left[m_N + d_N + C_N^{(O)}(x)\right] \lambda_N e^{-(\alpha_N+\lambda_N+\delta)(t-x)}dx \\ \quad + \int_0^t \left[r_{N+1} + d_{N+1} + C_1^{(O)}(x)\right] \alpha_N e^{-(\alpha_N+\lambda_N+\delta)(t-x)}dx \end{cases} \tag{2}$$

The analytical form of $C_1^{(0)}(L)$ can be obtained by solving (2) iteratively using Laplace transform. We present the results in the following proposition. Define $\alpha_0 = 1$, and

$$e_i = \begin{cases} \lambda_i(m_i + d_i) & \text{for } i = 1,2,\ldots,N-1 \\ \lambda_N(m_N + d_N) + (r_{N+1} + d_{N+1})\alpha_N & \text{for } i = N \end{cases} \tag{3}$$

Proposition 1. *For an N-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy O is given as*

$$C_1^{(0)}(L) = \left(\mathscr{L}^{-1}\left[\frac{\sum_{j=1}^N \left[\prod_{k=0}^{j-1}\alpha_k * e_j * \prod_{k=j+1}^N (s+\alpha_k+\delta)\right]}{s\left[\prod_{j=1}^N (s+\alpha_j+\delta) - \prod_{j=1}^N \alpha_j\right]} \right] \right)\Bigg|_{t=L} \tag{4}$$

Proof. Define $C_i^{(0)}(s) = \mathscr{L}\left[C_i^{(0)}(t)\right]$ as the Laplace transform of $C_i^{(0)}(t)$ $(i \in \Omega)$. We then have

$$\begin{cases} C_i^{(0)}(s) = \frac{e_i}{s(s+\alpha_i+\delta)} + C_{i+1}^{(0)}(s)\frac{\alpha_i}{s+\alpha_i+\delta} & \text{for } i = 1,2,\ldots,N-1 \\ C_N^{(0)}(s) = \frac{e_N}{s(s+\alpha_N+\delta)} + C_1^{(O)}(s)\frac{\alpha_N}{s+\alpha_N+\delta} \end{cases} \tag{5}$$

After simplification,

$$C_1^{(0)}(s) = \frac{\sum_{j=1}^N \left[\prod_{k=0}^{j-1}\alpha_k * e_j * \prod_{k=j+1}^N (s+\alpha_k+\delta)\right]}{s\left[\prod_{j=1}^N (s+\alpha_j+\delta) - \prod_{j=1}^N \alpha_j\right]} \tag{6}$$

From (6), Proposition 1 is easily obtained. □

Remarks: Eq. (6) is considered as close-to-explicit because obtaining the inverse transform for $C_1^{(0)}(t)$ requires a numerical solver (e.g. Matlab) except for some simple cases. The application of such inversion techniques for the maintenance optimization will be illustrated shortly. Note that when $C_1^{(0)}(t)$ is obtained, we can further obtain $C_i^{(0)}(t)$ for a system starting at any degraded state (i.e. $i > 1$) using the following:

$$C_i^{(O)}(t) = \frac{1}{\alpha_{i-1}}\left[\frac{dC_{i-1}^{(0)}(t)}{dt} + (\alpha_{i-1}+\delta)C_{i-1}^{(0)}(t) - e_{i-1}\right], \quad i = 2,3,\ldots,N \tag{7}$$

3.3. The EDMC model for Policy A

In this section, we investigate the EDMC model for Policy A when the maintenance thresholds (J,τ) are given. We assume that when the system is preventively replaced, no downtime cost is incurred. One of the justifications for this is that a warm-standby may be initiated before shutting down the old unit for replacement. Let $C_i^{(A)}(t|J,\tau)$ represent the EDMC under (J,τ) when the system is working in State $2i-1 (i \in \Omega)$ and the residual life cycle is t. Again, $p_i(t) = \lambda_i e^{-(\alpha_i+\lambda_i)t}$ and $g_i(t) = \alpha_i e^{-(\alpha_i+\lambda_i)t}$. Two cases are further analyzed: $t \leqslant \tau$ and $t > \tau$.

- When $t \leqslant \tau$

In this case, no preventive replacement is required under Policy A. $C_i^{(A)}(t|J,\tau)$ is calculated in the same way as Policy O, i.e.

$$C_i^{(A)}(t|J,\tau) = C_i^{(0)}(t), \quad t \leqslant \tau,\ i \in \Omega \tag{8}$$

- When $t > \tau$

Note that for this case, the deterioration of the system is no heavier than $2J-1 (1 \leqslant J < N)$; otherwise, the system should have been preventively replaced under Policy A. Consequently, we have

$$\begin{cases} C_i^{(A)}(t|J,\tau) = e^{-(\alpha_i+\lambda_i+\delta)(t-\tau)}C_i^{(0)}(\tau) \\ \quad + \int_\tau^t \left[m_i + d_i + C_i^{(A)}(x|J,\tau)\right]\lambda_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \quad + \int_\tau^t C_{i+1}^{(A)}(x|J,\tau)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \quad \text{for } i = 1,2,\ldots,J-1 \\ C_J^{(A)}(t|J,\tau) = e^{-(\alpha_J+\lambda_J+\delta)(t-\tau)}C_J^{(0)}(\tau) \\ \quad + \int_\tau^t \left[m_J + d_J + C_J^{(A)}(x|J,\tau)\right]\lambda_J e^{-(\alpha_J+\lambda_J+\delta)(t-x)}dx \\ \quad + \int_\tau^t \left[r_{J+1} + C_1^{(A)}(x|J,\tau)\right]\alpha_J e^{-(\alpha_J+\lambda_J+\delta)(t-x)}dx \end{cases} \tag{9}$$

Since $\tau \leqslant L$, the analytical form of $C_1^{(A)}(L|J,\tau)$ is determined by (4) and (9). We present the results in the following Proposition.

Proposition 2. *For an N-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy A under maintenance thresholds (J,τ) is given as*

$$C_1^{(A)}(L|J,\tau) = \left(\mathscr{L}^{-1}\left[\frac{\sum_{j=1}^{J-1}\left[\prod_{k=0}^{j-1}\alpha_k\left(e_j + sC_j^{(0)}(\tau)\right)\prod_{k=j+1}^{J}(s+\alpha_k+\delta)\right] + \prod_{k=0}^{J-1}\alpha_k(e_J + r_{J+1}\alpha_J + sC_J^{(0)}(\tau))}{s\left[\prod_{j=1}^J (s+\alpha_j+\delta) - \prod_{j=1}^J \alpha_j\right]} \right] \right)\Bigg|_{t=L-\tau} \tag{10}$$

Proof. Let $u = t - \tau (u \geqslant 0)$ and define $H_i^{(A)}(u|J,\tau) = C_i^{(A)}(t|J,\tau)$ for any $t \geqslant \tau$. Eq. (9) can be rewritten as

$$\begin{cases} H_i^{(A)}(u|J,\tau) = e^{-(\alpha_i+\lambda_i+\delta)u}C_i^{(O)}(\tau) \\ \quad + \int_0^u \left[m_i + d_i + H_i^{(A)}(x|J,\tau)\right]\lambda_i e^{-(\alpha_i+\lambda_i+\delta)(u-x)}dx \\ \quad + \int_0^u H_{i+1}^{(A)}(x|J,\tau)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(u-x)}dx \quad \text{for } i = 1,2,\ldots,J-1 \\ H_J^{(A)}(u|J,\tau) = e^{-(\alpha_J+\lambda_J+\delta)u}C_J^{(O)}(\tau) \\ \quad + \int_0^u \left[m_J + d_J + H_J^{(A)}(x|J,\tau)\right]\lambda_J e^{-(\alpha_J+\lambda_J+\delta)(u-x)}dx \\ \quad + \int_0^u \left[r_{J+1} + H_1^{(A)}(x|J,\tau)\right]\alpha_J e^{-(\alpha_J+\lambda_J+\delta)(u-x)}dx \end{cases} \tag{11}$$

Define $H_i^{(A)}(s|J,\tau) = \mathscr{L}\left[H_i^{(A)}(u|J,\tau)\right]$ as the Laplace transform of $H_i^{(A)}(u|J,\tau)$. We then have

$$\begin{cases} H_i^{(A)}(s|J,\tau) = \frac{e_i + sC_i^{(0)}(\tau)}{s(s+\alpha_i+\delta)} + H_{i+1}^{(A)}(s|J,\tau)\frac{\alpha_i}{s+\alpha_i+\delta} & \text{for } i = 1,2,\ldots,J-1 \\ H_J^{(A)}(s|J,\tau) = \frac{e_J + r_{J+1}\alpha_J + sC_J^{(0)}(\tau)}{s(s+\alpha_J+\delta)} + H_1^{(A)}(s|J,\tau)\frac{\alpha_J}{s+\alpha_J+\delta} \end{cases} \tag{12}$$

After some simplification, $H_1^{(A)}(s|J,\tau)$ is given as

$$H_1^{(A)}(s|J,\tau)=\frac{\sum_{j=1}^{J-1}\left[\prod_{k=0}^{j-1}\alpha_k\left(e_j+sC_j^{(O)}(\tau)\right)\prod_{k=j+1}^{J}(s+\alpha_k+\delta)\right]+\prod_{k=0}^{J-1}\alpha_k\left(e_J+r_{J+1}\alpha_J+sC_J^{(O)}(\tau)\right)}{s\left[\prod_{j=1}^{J}(s+\alpha_j+\delta)-\prod_{j=1}^{J}\alpha_j\right]} \tag{13}$$

Since $C_1^{(A)}(L|J,\tau)=H_1^{(A)}(L-\tau|J,\tau)$, from (13), Proposition 2 is therefore obtained. □

3.4. The EDMC model for Policy B

In this section, we investigate the EDMC model for the customer under Policy B, i.e. corrective replacements. Similarly, let $C_i^{(B)}(t|J,\tau)$ represent the EDMC under (J,τ) when the system is working in State $2i-1$ ($i\in\Omega$) and the residual life cycle is t. Again, two cases are further considered: $t\leqslant\tau$ and $t>\tau$.

- When $t\leqslant\tau$

For this case, no corrective replacement is conducted. We simply have

$$C_i^{(B)}(t|J,\tau)=C_i^{(O)}(t),\quad t\leqslant\tau, i\in\Omega \tag{14}$$

- When $t>\tau$

Under Policy B, the system may deteriorate to any of the operational states. Therefore we have

$$\begin{cases} C_i^{(B)}(t|J,\tau)=e^{-(\alpha_i+\lambda_i+\delta)(t-\tau)}C_i^{(O)}(\tau) \\ \qquad +\int_\tau^t\left[m_i+d_i+C_i^{(B)}(x|J,\tau)\right]\lambda_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \qquad +\int_\tau^t C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \quad \text{for } i=1,2,\ldots,J \\ C_i^{(B)}(t|J,\tau)=e^{-(\alpha_i+\lambda_i+\delta)(t-\tau)}C_i^{(O)}(\tau) \\ \qquad +\int_\tau^t\left[r_i+d_i+C_1^{(B)}(x|J,\tau)\right]\lambda_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \qquad +\int_\tau^t C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(t-x)}dx \\ \qquad \text{for } i=J+1,J+2,\ldots,N-1 \\ C_N^{(B)}(t|J,\tau)=e^{-(\alpha_N+\lambda_N+\delta)(t-\tau)}C_N^{(O)}(\tau) \\ \qquad +\int_\tau^t\left[r_N+d_N+C_1^{(B)}(x|J,\tau)\right]\lambda_N e^{-(\alpha_N+\lambda_N+\delta)(t-x)}dx \\ \qquad +\int_\tau^t\left[r_{N+1}+d_{N+1}+C_1^{(B)}(x|J,\tau)\right]\alpha_N e^{-(\alpha_N+\lambda_N+\delta)(t-x)}dx \end{cases} \tag{15}$$

To further derive the analytical form of $C_i^{(B)}(L|J,\tau)$, we follow similar procedures as Proposition 2.

Proposition 3. *For an N-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy B under maintenance threshold (J,τ) is given as*

$$C_1^{(B)}(L|J,\tau)=\left(\mathscr{L}^{-1}\left[\frac{U(s|J,\tau)}{V(s|J,\tau)}\right]\right)\Bigg|_{t=L-\tau} \tag{16}$$

where $f_i=\lambda_i(r_i-m_i), u_i(s)=e_i+sC_i^{(0)}(\tau)$ $(i\in\Omega)$, and

$$U(s|J,\tau)=\sum_{j=1}^{J}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right)*u_j(s)*\left(\prod_{k=j+1}^{J}(s+\alpha_k+\delta)\right)\times\left(\prod_{k=J+1}^{N}(s+\alpha_k+\lambda_k+\delta)\right)\right]+\sum_{j=J+1}^{N}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right)*(u_j(s)+f_j)*\left(\prod_{k=j+1}^{N}(s+\alpha_k+\lambda_k+\delta)\right)\right] \tag{17}$$

$$V(s|J,\tau)=s\left[\left(\prod_{j=1}^{J}(s+\alpha_j+\delta)\right)\left(\prod_{j=J+1}^{N}(s+\alpha_j+\lambda_j+\delta)\right)-\sum_{j=J+1}^{N}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right)*\lambda_j*\left(\prod_{k=j+1}^{N}(s+\alpha_k+\lambda_k+\delta)\right)\right]-\prod_{j=0}^{N}\alpha_j\right] \tag{18}$$

Proof. Let $u=t-\tau(u\geqslant 0)$ and define $H_i^{(B)}(u|J,\tau)=C_i^{(B)}(t|J,\tau)$ for any $t\geqslant\tau$. The remaining procedure is identical to (11)–(13) in Proposition 2. □

4. Optimization of the maintenance thresholds

In the foregoing analysis, we have derived the close-to-explicit forms of the EDMC for Policies A and B when the maintenance thresholds (J,τ) are given. Here we further consider the methodologies for optimizing (J,τ) under each of the policies.

4.1. Method 1: optimizing (J,τ) using Laplace inversion

A straightforward way for optimize (J,τ) is to obtain the time-domain functions of the EDMC using the Laplace inversion technique and repeat the same process over the domain of J and τ. Note that the frequency-domain functions in the brackets of (4), (10) and (16) have a simple form that both the numerator and the denominator are rational polynomial functions of s and the degree of the numerator (in terms of s) is smaller than that of the denominator. For such functions, the fundamental theory for conducting the inversion is to apply the *Heaviside's expansion theorem*. Details of the theorem can be found in any textbook of complex analysis. Note that manually implementing the expansion technique is often cumbersome. Alternatively, scientific computing software, such as Matlab, has the embedded function for implementing such technique and is very easy to use. On the other hand, as we will see in the following numerical session, the number of inversions for each policy is entirely determined by the domain of J (i.e. $J=1,2,\ldots,N-1$) ($C_i^{(0)}(\tau), i\in\Omega$ are considered as symbolic values during the inversion). In other words, both Policies A and B requires merely $N-1$ times of inversion. By further optimizing these $N-1$ time-domain functions for each of the policy, optimal maintenance thresholds (J^*,τ^*) can be easily obtained.

Note that nowadays almost all types of the numerical solvers are able to conduct calculations for any pre-specified number of significant digits required. Consequently, the advantage of the above method is that it provides the de facto "analytical" form of the EDMC in a very efficient way, which subsequently guarantees the accuracy of the optimization process. On the other hand, however, the issue of numerical stability associated with existing (Laplace) inversion techniques (Kwok & Barthez, 1989) may surface when the probability of the root-overlapping in the frequency-domain function becomes significantly high, which for our case may only be observed in the MSS with large number of states (reflected by the degree of s in the denominator). Theoretically, such issue can still be addressed by increasing the computational efforts. But clearly this will compromise the efficiency of the numerical inversion. Therefore, in the following we consider an alternative method that can approximate the EDMC and optimize (J,τ) directly on the time domain.

4.2. Method 2: optimizing (J,τ) using discretization in the time domain

We first discretize the integral operator in (2), (9) and (15) before approximating the EDMC in the time domain.

Set $t_j = jh(j = 0,1,2,\ldots,\theta)$ and $L = t_\theta = \theta h$, where h is the minimal step of the approximation. Further set $\tau = lh_\tau(l = 0,1,\ldots,L/h_\tau)$ as the threshold of the residual life cycle. Note that h_τ may not necessarily equal to h. It could be multiples of h and depend on the accuracy requirements of the optimization.

Let $\widetilde{C}_i^{(0)}(t), \widetilde{C}_i^{(A)}(t|J,\tau)$ and $\widetilde{C}_i^{(B)}(t|J,\tau)$ represent the numerical approximations of $C_i^{(0)}(t), C_i^{(A)}(t|J,\tau)$, and $C_i^{(B)}(t|J,\tau)$ respectively.

For Policy O:

$$\begin{cases} \widetilde{C}_i^{(0)}(t_j) = \sum_{k=0}^{j-1}\left(\int_{t_k}^{t_{k+1}}\left(m_i + d_i + C_i^{(0)}(x)\right)\lambda_i e^{-(\alpha_i+\lambda_i+\delta)(t_j-x)}dx \right. \\ \qquad \left. + \int_{t_k}^{t_{k+1}} c_{i+1}^{(0)}(x)\alpha_i e^{-(\alpha_i+\lambda_i+\delta)(t_j-x)}dx\right) \\ \qquad \approx \sum_{k=0}^{j-1} h * e^{-(\alpha_i+\lambda_i+\delta)(t_j-t_k)} * \left[e_i + \lambda_i \widetilde{C}_i^{(0)}(t_k) + \alpha_i \widetilde{C}_{i+1}^{(0)}(t_k)\right] \\ \quad \text{for } i = 1,2,\ldots,N-1 \\ \widetilde{C}_N^{(0)}(t_j) \approx \sum_{k=0}^{j-1} h * e^{-(\alpha_N+\lambda_N+\delta)(t_j-t_k)} * \left[e_N + \lambda_N \widetilde{C}_N^{(0)}(t_k) + \alpha_N \widetilde{C}_1^{(0)}(t_k)\right] \end{cases} \tag{19}$$

To further reduce the computational complexity, Eq. (19) is rewritten in such a linear form that the EDMC with residual time $t = t_j$ only relies on the EDMC at $t = t_{j-1}$:

$$\begin{cases} \widetilde{C}_i^{(0)}(t_j) \approx e^{-(\alpha_i+\lambda_i+\delta)h}\left[\widetilde{C}_i^{(0)}(t_{j-1})(1+h\lambda_i) + \alpha_i \widetilde{C}_{i+1}^{(0)}(t_{j-1})h + e_i h\right] \\ \quad \text{for } i = 1,2,\ldots,N-1 \\ \widetilde{C}_N^{(0)}(t_j) \approx e^{-(\alpha_N+\lambda_N+\delta)h}\left[\widetilde{C}_N^{(0)}(t_{j-1})(1+h\lambda_N) + \alpha_N \widetilde{C}_1^{(0)}(t_{j-1})h + e_N h\right] \end{cases} \tag{20}$$

For Policy A:

For $j = 0,1,2,\ldots,lh_\tau/h$ and any $i \in \Omega$, we have

$$\widetilde{C}_i^{(A)}(t_j|J,\tau) = \widetilde{C}_i^{(0)}(t_j) \tag{21}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2,\ldots,\theta$, we present (9) in a similar form as (20):

$$\begin{cases} \widetilde{C}_i^{(A)}(t_j|J,\tau) \approx e^{-(\alpha_i+\lambda_i+\delta)h}\left[\widetilde{C}_i^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i \widetilde{C}_{i+1}^{(A)}(t_{j-1}|J,\tau)h + e_i h\right] \text{for } i = 1,2,\ldots,J-1 \\ \widetilde{C}_J^{(A)}(t_j|J,\tau) \approx e^{-(\alpha_J+\lambda_J+\delta)h}\left[\widetilde{C}_J^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_J) + \alpha_J \widetilde{C}_1^{(A)}(t_{j-1}|J,\tau)h + (e_J + r_{J+1}\alpha_J)h\right] \end{cases} \tag{22}$$

For Policy B:

For $j = 0,1,2,\ldots,lh_\tau/h$ and any $i \in \Omega$

$$\widetilde{C}_i^{(B)}(t_j|J,\tau) = \widetilde{C}_i^{(0)}(t_j) \tag{23}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2,\ldots,\theta$

$$\begin{cases} \widetilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-(\alpha_i+\lambda_i+\delta)h}\left[\widetilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) \right. \\ \qquad \left. + \alpha_i \widetilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + e_i h\right] \quad \text{for } i = 1,2,\ldots,J \\ \widetilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-(\alpha_i+\lambda_i+\delta)h}\left[\widetilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) \right. \\ \qquad \left. + \alpha_i \widetilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + (e_i + f_i)h\right] \quad \text{for } i = J+1,2,\ldots,N-1 \\ \widetilde{C}_N^{(B)}(t_j|J,\tau) \approx e^{-(\alpha_N+\lambda_N+\delta)h}\left[\widetilde{C}_N^{(B)}(t_{j-1}|J,\tau) + (\lambda_N + \alpha_N) \right. \\ \qquad \left. + \widetilde{C}_1^{(B)}(t_{j-1}|J,\tau)h + (e_N + f_N)h\right] \end{cases} \tag{24}$$

An algorithm for optimizing (J,τ)

Step 1: Select h and let $L = \theta h$. Set $\widetilde{C}_i^{(0)}(t_0) = \widetilde{C}_i^{(0)}(0) = 0$ for any $i \in \Omega$.

Step 2: Calculate $\widetilde{C}_i^{(0)}(t_j)$ for $j = 1,2,\ldots,\theta$ and any $i \in \Omega$ following (20).

Select h_τ. For each $\tau = kh_\tau(k = 0,1,\ldots,L/h_\tau)$ and $J = 1,2,\ldots,N-1$

Step 3: Set $\widetilde{C}_i^{(A)}(t_j|J,\tau) = \widetilde{C}_i^{(B)}(t_j|J,\tau) = \widetilde{C}_i^{(0)}(t_j)$ for any $i \in \Omega$ and $j = 0,1,\ldots,kh_\tau/h$.

Step 4: Calculate $\widetilde{C}_i^{(A)}(t_j|J,\tau)$ and $\widetilde{C}_i^{(B)}(t_j|J,\tau)$ for any $i \in \Omega$ and $j = kh_\tau/h + 1, kh_\tau/h + 2,\ldots,\theta$ following (22) and (24) respectively.

Step 5: Repeat Step 3–4.

Step 6: Select the optimal (J^*,τ^*) that minimize the value of $\min\{\widetilde{C}_1^{(A)}(L|J,\tau), \widetilde{C}_1^{(B)}(L|J,\tau)\}$.

Remarks: (1) The main advantage of the above method is that it is stable and can deal with a wide range of system configurations, which can be subsequently used for the sensitive analysis of the maintenance optimization. In addition, it can be directly applied (without any change) for maintaining a MSS that is not perfect functioning initially since both $\widetilde{C}_i^{(A)}(L|J,\tau)$ and $\widetilde{C}_i^{(B)}(L|J,\tau)(i \neq 1)$ are automatically calculated in the algorithm. (2) The main drawback of the method is that the optimization process is relatively time-consuming and less accurate. Reducing h, though enhancing the accuracy of the optimization, will further increase the computational efforts significantly. Therefore, h needs to be properly selected to balance the accuracy and efficiency of the algorithm. (3) The EDMC is approximated at the lower limit of the integration and therefore is always smaller than the exact result.

5. Numerical example

In this section we illustrate the optimization process with an example using both methods.

Consider the following parameters for a MSS with 4-stage degradation ($N = 4$). Let $L = 5$ year and $\delta = 0.05$/year. The transition rates for degradation are $\alpha_1 = 0.9$/year, $\alpha_2 = 0.8$/year, $\alpha_3 = 0.9$/year and $\alpha_4 = 1.1$/year. The Poisson failure rates are $\lambda_1 = 0.4$/year, $\lambda_2 = 0.6$/year, $\lambda_3 = 1.0$/year and $\lambda_4 = 1.2$/year. The replacement costs are $r_1 = 200$, $r_2 = 240$, $r_3 = 360$, $r_4 = 520$ and $r_5 = 720$. To further investigate the impact of different cost structures on the optimal policies, we consider the following three scenarios.

Scenario 1: $\frac{r_i}{m_i} = 4(i = 1,2,3,4)$ and $d_i = 20(i = 1,2,3,4,5)$, i.e. low repair and downtime cost.

Scenario 2: $\frac{r_i}{m_i} = 4(i = 1,2,3,4)$ and $d_i = 80(i = 1,2,3,4,5)$, i.e. low repair cost and high downtime cost.

Scenario 3: $\frac{r_i}{m_i} = 2(i = 1,2,3,4)$ and $d_i = 80(i = 1,2,3,4,5)$, i.e. high repair and downtime cost.

Results: First of all, the de facto "analytical" forms of the EDMC as functions of (J,τ) are derived and listed in Appendix using Laplace inversion techniques (Method 1). Note that we only show the results for Scenario 1 because it is sufficient for the demonstration purpose. Here $C_i^{(0)}(\tau)$ $(i \in \Omega)$ are treated as symbolic values

Table 1
Selecting a proper minimal interval h.

Scenario	$\frac{r_i}{m_i}$	d_i	$C_1^{(0)}(L)$	$h = 0.1$		$h = 0.01$		$h = 0.001$	
				$\widetilde{C}_1^{(0)}(L)$	Err%	$\widetilde{C}_1^{(0)}(L)$	Err%	$\widetilde{C}_1^{(0)}(L)$	Err%
1	4	20	799.5	429.6	46.3	747.7	6.5	794.1	0.7
2	4	80	1023.2	566.3	44.7	959.5	6.2	1016.7	0.6
3	2	80	1279.6	720.6	43.7	1202.0	6.1	1271.7	0.6

during the inversion for Policies A and B and the number of inversions under each of the policies is simply $3(=N-1)$. For Method 2, we first select a proper minimal interval h to guarantee the accuracy of the approximation. Table 1 illustrates that $h = 0.001$ is potentially a good choice since the error between $\widetilde{C}_1^{(0)}(L)$ (using Method 2) and $C_1^{(0)}(L)$ (using Method 1) is less than 1%. From the efficiency point of view, further reducing h, say, from *0.001* to *0.0001*, barely improves the accuracy of the approximation; for our case, it will increase the system run time from 10 min to hours. Therefore, $h = 0.001$ is considered cost-effective here and is applied thereafter.

Optimal maintenance thresholds (J^*, τ^*) using both methods are presented in Figs. 2–4 with different values of the cost parameters. It is not surprising that the results given by Method 2 when $h = 0.001$ almost overlap with those using Method 1, which always appear to be slightly higher. Based on the results using the Laplace inversion (Method 1), the following observations are made.

For Scenario 1, Policy B is a better choice for the customer and the minimum EDMC over the 5 years is 655.9 under $J^* = 2$ and $\tau^* = 0.9$ year. In other words, the system is always correctively replaced when it fails during the 3rd stage degradation and the residual life cycle is longer than 0.9 year. For Scenario 2, Policy A (or preventive replacement) should be enforced and the minimum EDMC over the 5 years is 804.8 under $J^* = 2$ and $\tau^* = 1.2$ year. The optimal choice for Scenario 3 is Policy B and the minimum EDMC over the interval is 909.1 under $J^* = 1$ and $\tau^* = 0.6$ year. From the above results, we conclude that Policy A outperforms Policy B typically when the downtime cost is considerably high compared to

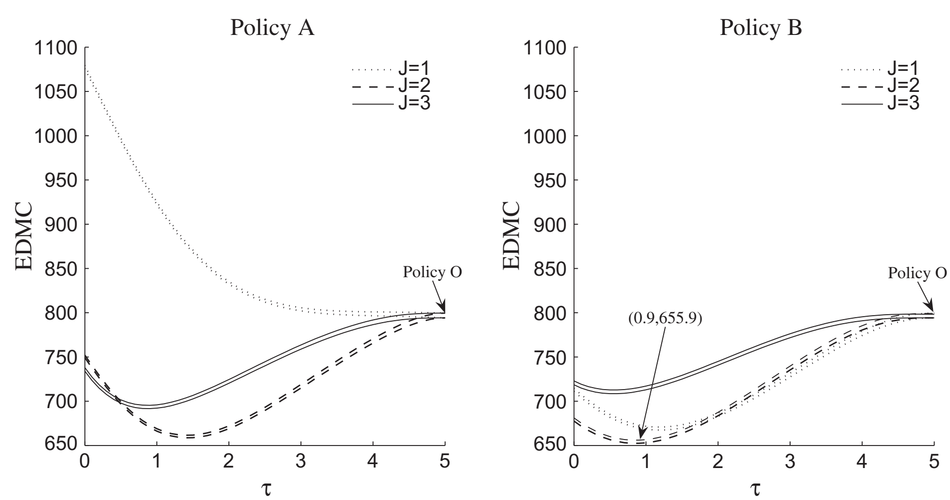

Fig. 2. The EDMC as a function of (J, τ) for Scenario 1.

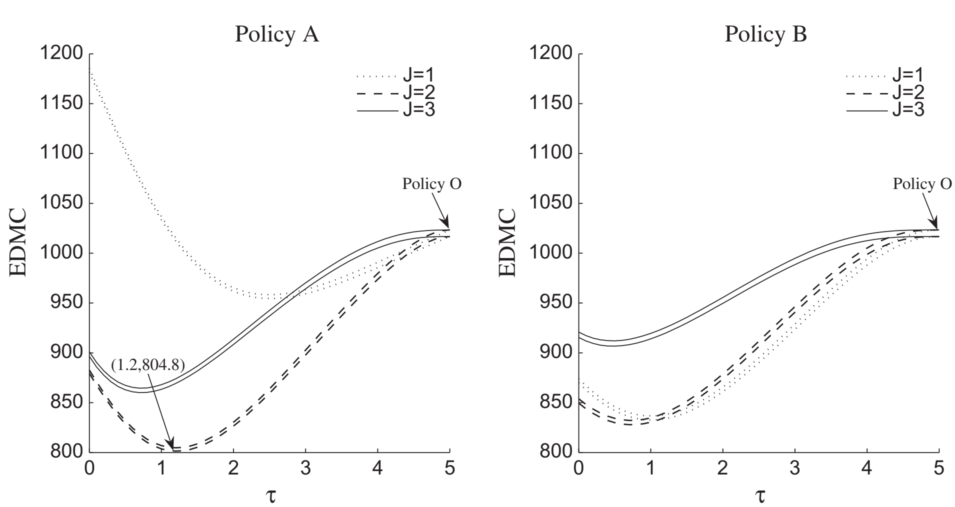

Fig. 3. The EDMC as a function of (J, τ) for Scenario 2.

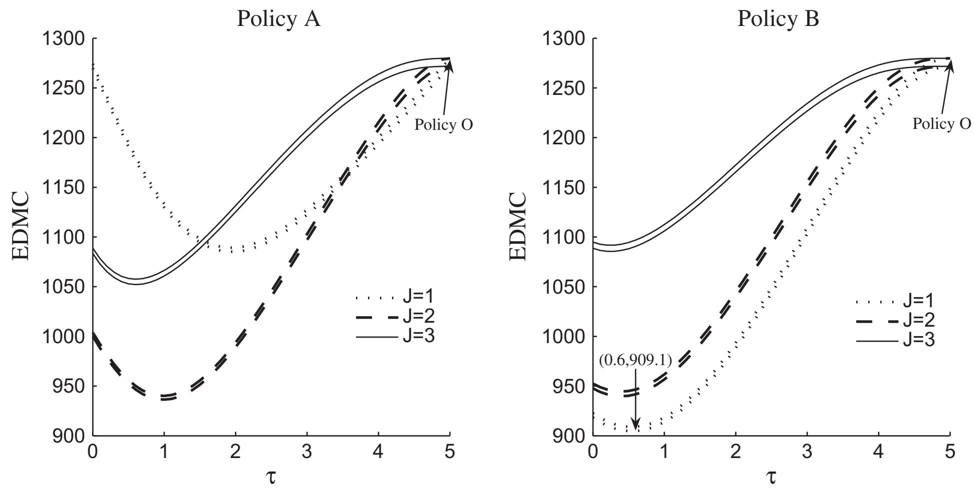

Fig. 4. The EDMC as a function of (J, τ) for Scenario 3.

the minimal repair cost. In addition, the optimal τ^* under Policy A is larger than the value under Policy B. The interpretation is that for Policy A, more stringent requirements on τ^* are necessary as a balance for more aggressive replacement strategies – preventive replacement, when compared with the corrective replacement strategies under Policy B. Furthermore, the results in Scenario 3 also indicate that when both repair and downtime cost are high, the system should be correctively replaced upon failure even when it is working under a relatively good condition (e.g. the 2nd degradation stage for Scenario 3). Finally, we use Method 2 for the sensitivity analysis of another cost parameter – the discounted factor δ. Results do not show a substantial impact on (J^*, τ^*) (we therefore do not list the results here). Alternatively, it could be important in more practical situations, say, when the cost estimation is also crucial to the decision makers. (Note that for the current case, the minimum cost is 10–20% higher if $\delta = 0$.)

6. Conclusion

In this paper, we considered a finite life-cycle MSS that is subject to both degradation and Poisson failures. We study two classes of maintenance policies, that of preventive replacements and corrective replacements. For both policies, the EDMC is derived as a function of two control parameters – a threshold level on the current state of the system, and a threshold level on the residual life cycle. In order to obtain the optimal maintenance thresholds to minimize the EDMC, two different methodologies are proposed which utilizes Laplace transform (inversion) techniques and time-domain numerical approximation respectively. The applications of both methods are illustrated using a numerical case. Through computational examples, we demonstrate that preventive replacements outperform corrective replacements typically when the downtime cost of each failure is relatively high compared to the repair cost. The two proposed replacement policies can effectively detect necessary replacements for the condition when the system has already experienced heavy deterioration and the remaining service time is still long, but can also avoid the excessive replacements for the condition when the system has only experienced minor deterioration.

Our work can be further extended in the following directions. First, rather than using a continuous-time Markov process model to describe the system aging as in this paper, more general models can be developed (e.g. semi-Markov), but this is expected to increase computational efforts substantially. Second, we assumed that imperfect repair is not available to the customer. More general results can be derived when the repair restores the system to one of the previous operational states other than merely 'minimal' (Soro, Nourelfath, & Ait-Kadi, 2010). Third, we assume that the system is under continuous and perfect monitoring. Such services may not be available in practice. More general situations like 'silent failures' or imperfect inspections may be considered. Finally, some systems are under certain forms of service contracts, e.g. maintenance service contract, (extended) warranty service, etc. For these cases, part of the maintenance cost of the customer (e.g. minimal repair cost during warranty) may be shared by the manufacturer or the service provider. As a result, the replacement policies for the customer could be very different when such extra services are considered. The situation may become even more complicated when deciding whether the replacement of the system will renew the service contract or not.

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Appendix A

The "analytical" form of the EDMC for Scenario 1 is given as:
Policy O

$$C_1^{(0)}(\tau) = 4772.3 - 5098.8e^{-0.05\tau} + 100.0e^{-1.90\tau} + (226.5\cos 0.91\tau + 201.6\sin 0.91\tau)e^{-0.97\tau}$$

$$C_2^{(0)}(\tau) = 5006.3 - 5098.8e^{-0.05\tau} - 105.6e^{-1.90\tau} + (198.2\cos 0.91\tau - 235.2\sin 0.91\tau)e^{-0.97\tau}$$

$$C_3^{(0)}(\tau) = 5259.2 - 5098.8e^{-0.05\tau} + 138.7e^{-1.90\tau} - (299.1\cos 0.91\tau + 189.4\sin 0.91\tau)e^{-0.97\tau}$$

$$C_4^{(0)}(\tau) = 5429.1 - 5098.8e^{-0.05\tau} - 146.6e^{-1.90\tau} - (183.8\cos 0.91\tau - 308.4\sin 0.91\tau)e^{-0.97\tau}$$

Policy A

$$C_1^{(A)}(L|1,\tau) = 4880.0 + \left(C_1^{(0)}(\tau) - 4880.0\right)e^{-0.05(L-\tau)}$$

$$C_1^{(A)}(L|2,\tau) = 3728.0 + \left(0.47C_1^{(0)}(\tau) + 0.53C_2^{(0)}(\tau) - 3821.2\right)e^{-0.05(L-\tau)} + \left(0.53C_1^{(0)}(\tau) - 0.53C_2^{(0)}(\tau) + 93.2\right)e^{-1.75(L-\tau)}$$

$$C_1^{(A)}(L|3,\tau) = 4027.8 + \left(0.32C_1^{(0)}(\tau) + 0.36C_2^{(0)}(\tau) + 0.32C_3^{(0)}(\tau) - 4224\right)e^{-0.05(L-\tau)} + \Big[\left(0.68C_1^{(0)}(\tau) - 0.36C_2^{(0)}(\tau) - 0.32C_3^{(0)}(\tau) + 196.2\right)\cos 0.75(L-\tau) - \left(0.02C_1^{(0)}(\tau) - 0.58C_2^{(0)}(\tau) + 0.56C_3^{(0)}(\tau) - 109.2\right)\sin 0.75(L-\tau)\Big]e^{-1.35(L-\tau)}$$

Policy B

$$C_1^{(B)}(L|1,\tau) = 3691.8 + \left(0.50C_1^{(0)}(\tau) + 0.32C_2^{(0)}(\tau) + 0.13C_3^{(0)}(\tau) + 0.05C_4^{(0)}(\tau) - 3822.8\right)e^{-0.05(L-\tau)} + \left(0.13C_1^{(0)}(\tau) - 0.08C_2^{(0)}(\tau) + 0.07C_3^{(0)}(\tau) - 0.12C_4^{(0)}(\tau) + 52.2\right)e^{-2.88(L-\tau)} + \Big[\left(0.38C_1^{(0)}(\tau) - 0.24C_2^{(0)}(\tau) - 0.20C_3^{(0)}(\tau) + 0.06C_4^{(0)}(\tau) + 78.7\right)\cos 1.00(L-\tau) + \left(0.15C_1^{(0)}(\tau) + 0.24C_2^{(0)}(\tau) - 0.18C_3^{(0)}(\tau) - 0.21C_4^{(0)}(\tau) + 135.8\right)\sin 1.00(L-\tau)\Big]e^{-1.88(L-\tau)}$$

$$C_1^{(B)}(L|2,\tau) = 3663.0 + \left(0.36C_1^{(0)}(\tau) + 0.40C_2^{(0)}(\tau) + 0.17C_3^{(0)}(\tau) + 0.07C_4^{(0)}(\tau) - 3828.8\right)e^{-0.05(L-\tau)} + \left(0.10C_1^{(0)}(\tau) - 0.05C_2^{(0)}(\tau) + 0.05C_3^{(0)}(\tau) - 0.10C_4^{(0)}(\tau) + 47.6\right)e^{-2.77(L-\tau)} + \Big[\left(0.23C_1^{(0)}(\tau) + 0.20C_2^{(0)}(\tau) - 0.21C_3^{(0)}(\tau) - 0.22C_4^{(0)}(\tau) + 158.4\right)\sin 1.02(L-\tau) + \left(0.54C_1^{(0)}(\tau) - 0.36C_2^{(0)}(\tau) - 0.22C_3^{(0)}(\tau) + 0.04C_4^{(0)}(\tau) + 118.2\right)\cos 1.02(L-\tau)\Big]e^{-1.64(L-\tau)}$$

$$C_1^{(B)}(L|3,\tau) = 4107.9 + (0.28C_1^{(0)}(\tau) + 0.32C_2^{(0)}(\tau) + 0.28C_3^{(0)}(\tau) + 0.11C_4^{(0)}(\tau) - 4346.0)e^{-0.05(L-\tau)} + (0.11C_1^{(0)}(\tau) - 0.06C_2^{(0)}(\tau) + 0.03C_3^{(0)}(\tau) - 0.08C_4^{(0)}(\tau) + 45.2)e^{-2.64(L-\tau)} + \Big[\left(0.60C_1^{(0)}(\tau) - 0.26C_2^{(0)}(\tau) - 0.31C_3^{(0)}(\tau) - 0.03C_4^{(0)}(\tau) + 192.9\right)\cos 0.96(L-\tau) + \left(0.09C_1^{(0)}(\tau) + 0.47C_2^{(0)}(\tau) - 0.30C_3^{(0)}(\tau) - 0.26C_4^{(0)}(\tau) + 169.8\right)\sin 0.96(L-\tau)\Big]e^{-1.20(L-\tau)}$$

References

Barlow, S. E., & Wu, A. S. (1978). Coherent systems with multi-state components. *Mathematics of Operations Research*, 275–281.

Ding, Y., Lisnianski, A., Frenkel, I., & Khvatskin, L. (2009). Optimal corrective maintenance contract planning for aging multi-state system. *Applied Stochastic Models in Business and Industry, 25*, 612–631.

El-Neweihi, E., Proschan, F., & Sethuraman, J. (1978). Multistate coherent systems. *Journal of Applied Probability*, 675–688.

Grall, A., Berenguer, C., & Dieulle, L. (2002). A condition-based maintenance policy for stochastically deteriorating systems. *Reliability Engineering and System Safety, 76*, 167–180.

Ivy, J. S., & Pollock, S. M. (2005). Marginally monotonic maintenance policies for a multi-state deteriorating machine with probabilistic monitoring, and silent failures. *IEEE Transactions on Reliability, 54*, 489–497.

Kao, E. P. C. (1973). Optimal replacement rules when changes of states are semi-Markovian. *Operations Research, 21*, 1231–1249.

Kim, M. J., & Makis, V. (2009). Optimal maintenance policy for a multi-state deteriorating system with two types of failures under general repair. *Computers & Industrial Engineering, 57*, 298–303.

Kwok, Y. K., & Barthez, D. (1989). An algorithm for the numerical inversion of Laplace transforms. *Inverse Problems, 5*, 1089–1095.

Levitin, G., & Lisnianski, A. (2000). Optimization of imperfect preventive maintenance for multi-state systems. *Reliability Engineering and System Safety, 67*, 193–203.

Maillart, L. M., & Zheltova, L. (2007). Structured maintenance policies on interior sample paths. *Naval Research Logistics, 54*, 645–655.

Moustafa, M. S., Abdel Maksoud, E. Y., & Sadek, S. (2004). Optimal major and minimal maintenance policies for deteriorating systems. *Reliability Engineering and System Safety, 83*, 363–368.

Nakagawa, T., & Mizutani, S. (2009). A summary of maintenance policies for a finite interval. *Reliability Engineering and System Safety, 94*, 89–96.

Pham, H., Suprasad, A., & Misra, R. B. (1997). Availability and mean life time prediction of multistage degraded system with partial repairs. *Reliability Engineering and System Safety, 56*, 169–173.

Ross, S. M. (1979). Multivalued state component systems. *The Annals of Probability, 1979*, 379–383.

Sim, S. H., & Endrenyi, J. (1993). A failure-repair model with minimal and major maintenance. *IEEE Transactions on Reliability, 42*, 134–140.

Soro, I. W., Nourelfath, M., & Ait-Kadi, D. (2010). Performance evaluation of multi-state degraded systems with minimal repairs and imperfect preventive maintenance. *Reliability Engineering and System Safety, 95*, 65–69.

Su, C. T., & Chang, C. C. (2000). Minimization of the life cycle cost for a multistate system under periodic maintenance. *International Journal of Systems Science, 31*, 217–227.

Tai, A. H., & Chan, L. Y. (2010). Maintenance models for a continuously degrading system. *Computers* & *Industrial Engineering*. doi:10.1016/j.cie.2009.12.005.

Tian, Z. G., Yam, R. C. M., Zuo, M. J., & Huang, H. Z. (2008a). Reliability bounds for multi-state k-out-of-n systems. *IEEE Transactions on Reliability, 57*, 53–58.

Tian, Z. G., Zuo, M. J., & Huang, H. Z. (2008b). Reliability-redundancy allocation for multi-state series-parallel systems. *IEEE Transactions on Reliability, 57*, 303–310.

Wu, S. (2005). Joint importance of multistate systems. *Computers & Industrial Engineering, 49*, 63–75.

Xue, J., & Yang, K. (1995). Dynamic reliability analysis of coherent multistate systems. *IEEE Transactions on Reliability, 44*, 683–688.

Yeh, R. H. (1996). Optimal inspection and replacement policies for multi-state deteriorating systems. *European Journal of Operational Research, 96*, 248–259.

Zuo, M. J., Liu, B., & Murthy, D. N. P. (2000). Replacement-repair policy for multi-state deteriorating products under warranty. *European Journal of Operational Research, 123*, 519–530.

Zuo, M. J., & Tian, Z. G. (2006). Performance evaluation of generalized multi-state k-out-of-n systems. *IEEE Transactions on Reliability, 55*, 319–327.