Optimal Replacement Policy for Multi-State System Under Imperfect Maintenance

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***Abstract*—A multi-state system (MSS) has more than two discrete states corresponding to different performance rates. Usually, MSS is viewed as in a failure state once its performance rate falls below user demand, and maintenance is carried out immediately. Generally, the repaired system cannot be regarded as good as new, and oftentimes the system restoration is stochastic. We introduce an optimal replacement policy for MSSs, called policy N. Under this policy, a MSS is replaced whenever its failure number reaches N. We assess the dynamic element state probabilities of each aging multi-state element (MSE) using a stochastic process model which is identified as a non-homogeneous continuous time Markov model (NHCTMM), and we evaluate the state distribution of the entire MSS via the combination of the stochastic process, and the universal generating function (UGF). To quantify the quality of imperfect maintenance, a quasi-renewal process is used to describe the stochastic behavior of each individual MSE after repair. Moreover, we derive an explicit expression of the long-run expected profit per unit time, and determine the optimal failure number N^* to replace the entire system. The proposed models are demonstrated via an illustrative case, followed by some comparative studies.**

***Index Terms*—Imperfect maintenance, maintenance policy, multi-state systems, non-homogeneous continuous time Markov model, quasi-renewal process, universal generating function.**

ACRONYMS

MSS	multi-state system
MSE	multi-state element
MTTF	mean time to failure
UGF	universal generating function
NHCTMM	non-homogeneous continuous time Markov model

NOTATION

M	Number of the s-independent element in the MSS
k_l	Number of possible states for element l
$g_{l,i}$	Performance rate of element l in state i
$\mathbf{g}_l$	Set of all possible performance rates of element l
$G_l(t)$	Random variable representing the performance rate of element l at time t
$p_{l,i}(t)$	Probability of element l in state i at time t in the first repair cycle
$\mathbf{p}_l(t)$	Set of probabilities associated with different states of element l
$\lambda^l_{i,j}(t)$	Intensity of element l transiting from state i to state j at time t in the first repair cycle
$\Lambda^l(t)$	Instantaneous transition intensity matrix of element l
K_s	Number of possible system states
g_i	Performance rate of MSS in its state i
$p_i(t)$	Probability of MSS in state i at time t in the first repair cycle
$p_i^n(t)$	Probability of MSS in state i at time t in the nth repair cycle
$G_s(t)$	Random variable representing the performance rate of MSS at time t
$\phi(\cdot)$	MSS structure function
w	User demand
$1(x)$	Unity function: $1(TRUE) = 1$, and $1(FALSE) = 0$
δ_R	Operator used in UGF to calculate MSS reliability
δ_E	Operator used in UGF to calculate the expected performance rate
$R(t,w)$	System reliability at time t under the user demand w
$f(t,w)$	System failure probability density function under user demand w
$\lambda(t,w)$	System failure rate under the user demand w
$f_n(t,w)$	System failure probability density function in the nth system repair cycle under user demand w
$R_n(t,w)$	System reliability function in the nth system repair cycle under user demand w
$\lambda_n(t,w)$	System failure rate in the nth system repair cycle under user demand w
$u_l^n(z,t)$	UGF of the element in the nth repair cycle
$U_s^n(z,t)$	UGF of the entire MSS in the nth repair cycle
$X^l_{n,i}$	Random time of the element l sojourning in its state i in the nth system repair cycle
α_l	Quasi-renewal parameter for the lifetime random variable of element l
β_l	Quasi-renewal parameter for the repair time random variable of element l

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c_w	Working reward of system per unit performance rate per unit time
c_f	Repair cost per unit time
c_r	Complete replacement cost for the entire system
c_{rw}	Replacement cost per unit time
$S_{n,i}$	Random time of system sojourning in state i in the n th repair cycle
X_n^s	Random lifetime of system in the nth repair cycle
Y_n^l	Random repair time of element l in the n th repair cycle
Y_n^s	Random repair time of system in the nth repair cycle
T_n	Time duration of the n th replacement cycle
N	Number of system failures before complete replacement
N^*	Optimal number of system failures before complete replacement
$C(N)$	Long run expected profit per unit time under policy N
$C(N^*)$	Maximum long run expected profit per unit time under the optimal policy N^*
$\mu_{X_n^s}$	Mean time to system failure in the nth repair cycle
$\mu_{Y_n^l}$	Mean time for element l repair in the nth repair cycle
$\mu_{Y_n^s}$	Mean time for system repair in the nth repair cycle
$\mu_{S_{n,i}}$	Mean time of system sojourning in state i during the nth repair cycle
$E(\cdot)$	Expectation operator

I. Introduction

VARIOUS kinds of systems suffer deterioration, and unexpected shock damages after being launched. A complex system consisting of multiple elements deteriorates gradually with the failure and degradation of its elements. For the non-repairable system, replacement is carried out once it fails; while most repairable systems, such as aircrafts, power generators, nuclear systems, and computing systems, can be restored to a functioning state through a specific maintenance activity. Therefore, selecting cost efficient, effective maintenance actions and strategies to improve the reliability and physical performance of complex systems is important work.

Generally speaking, maintenance activities can be classified into two major categories: corrective (un-planed), and preventive (planned) [1]. Corrective maintenance (CM) refers to any action that restores the failed system to its working state; preventive maintenance (PM) is defined as a maintenance action carried out when the system is still functioning, with the goal to restore the system to a specified better condition through systematic detection, correction of slight flaws, inspection, and other activities that will prolong system life.

To measure the quality of maintenance activities, Pham *et al.* [1] classify maintenance into five widely accepted categories according to its impact on the system or element condition: perfect, minimal, imperfect, worse, and worst. In most cases, maintenance does not make a system "as good as new" (perfect maintenance), or "as bad as old" (minimal maintenance). It is more realistic to consider that maintenance restores a system to somewhere between these two extremes, and that activity is called imperfect maintenance. A large amount of literature models imperfect maintenance for traditional binary state systems with either completely working or totally failed state, and the most significant among these models include the (p, q) model (see Nakagawa [2]), the $(p(t), q(t))$ model (see Block *et al.* [3]), the $(p(n,t), q(n,t), s(n,t))$ model (see Makis & Jardine [4]), the Kijima Type I and II models (see Kijima *et al.* [5] and Kijima [6]), the improvement factor method (see Malik [7]), the hybrid imperfect model (see Lin *et al.* [8]), the geometric process (see Lam [9]), and the quasi-renewal model (see Wang & Pham [10], [11]). Some applications of these imperfect maintenance models can be found in [11]–[16]. A comprehensive survey of maintenance policies in binary state systems is presented by Wang [17]. For systems with more than two states, there exist some papers discussing the optimal maintenance strategy for the system with discrete degraded states resulting from cumulative damage [18]–[23]; or state transition [24]–[26]. Most of these papers assume maintenance action is perfect, or minimal. Imperfect maintenance for a multi-failure-state system is developed in [12], [27]–[29]; and is extended to a multi-working and failure-state system in [30].

The multi-state system (MSS) concerned in the present paper is defined as a system that has a range of performance levels, from perfectly functioning to complete failure, resulting from the degradation or/and failure of some elements in the system [31]. Such a MSS is usually viewed as in a failure state once its performance rate falls below the user demand. With these properties, reliability assessment and optimizing methodology in the MSSs framework becomes more complicated than in traditional binary state systems; meanwhile, maintenance modeling in a binary state system context might be problematic when directly applied to such MSSs.

There are some efforts focusing on the maintenance problem for the aforementioned MSSs. For example, Nourelfath *et al.* [32], [33] discuss redundancy optimization under limited maintenance repairmen. To achieve the required system availability with minimal lifecycle cost, Levitin & Lisnianski [34] formulate a joint redundancy and replacement schedule optimization problem in which the element version, redundancy level, and replacement interval are optimized simultaneously. Liu & Huang [35] studied the optimal replacement policy for MSE under fuzzy uncertainty. However, these works only assume that the maintenance action is either perfect, or minimal. Taking imperfect maintenance into consideration, Levitin *et al.* [36], and Nahas *et al.* [37] generalize a maintenance optimization problem to a MSS with binary capacity elements intervened by imperfect PM. The imperfect maintenance model with age reduction characteristic was directly employed to reflect the improvement of element condition resulting from the PM separately performed on each element. The authors assume that repair or PM is executed immediately when any element fails, or the system reliability reaches the fixed threshold. With the aim to minimize the total maintenance cost while providing the desired system reliability, an optimization is therefore formulated to find the optimal sequence of PM actions chosen from a set of available actions. Nevertheless, the restriction of the proposed age reduction model is that the model is only feasible if every MSE is a binary capacity element. In reality, many MSEs cannot be categorized or simplified to binary capacity elements.

In [38], authors propose a practical model for MSSs predictive maintenance, and the "system perspective" maintenance strategy is first introduced. The "system perspective" means that the maintenance schedules are predicted or arranged based on the system state, or performance trend, rather than the states of individual elements. It is realistic that, in many situations, one may only be concerned with the whole system performance trend, and not consider restoring the failed element until system performance does not satisfy user demand. The main reason is that, usually, performing maintenance for individual failed elements requires turning off the whole system, which would incur too much product lost [39]. For example, oftentimes, repairing the failed element (the element could be a machine or subsystem) in the manufacturing line, with no buffer among subsystems, requires someone to turn off the whole system. Also, special facilities and/or repairmen usually have to be contracted to repair the failed element [46]. Instead of contracting for these resources every time when individual elements fail, it is more cost effective to contract only when system performance is unacceptable. Furthermore, for some systems, only total system performance can be monitored directly, and distinguishing the failed element becomes infeasible except under elaborate, expensive inspection involving disassembly. In all of these cases, maintenance actions are only executed whenever the total system performance falls into an unacceptable region, and then the whole system will have a comprehensive recovery. Based on the "system perspective" concept, [38] introduces a random restoration factor (RF) to describe the imperfect restoration of the whole system after repair, and the impact from the RF on the time to replacement (TTR) is demonstrated via the studied cases. Regarding some possible errors and misunderstandings in [38], we have published a critical paper to present some comments to their current methodology in [40], and the authors' reply is available in [41].

Motivated by [38], we propose a new approach to investigate the optimal replacement strategy from the "system perspective" for MSSs incorporating imperfect maintenance quality. Different from the method proposed in [38] which may cause definition conflict on MSSs if the RF is used [40], we assume that, after repair, each individual MSE would be restored to its best functional state, but not in "as good as new" condition. The transition intensities between element states will proportionally increase after repair, which means that MSEs degrade more rapidly to a lower performing state. Similar to the imperfect maintenance model in traditional binary state systems, the quasi-renewal process introduced in [10] is employed to describe the imperfect maintenance quality after repair through an element state probability function. Furthermore, considering the age effect, the non-homogeneous continuous time Markov model (NHCTMM), where the state transition intensity varies with time, is applied to model the aging MSE. An optimal failure number N^* can be obtained by minimizing the long-run expected system profit per unit time.

The remainder of this paper is organized as follows. Section II briefly introduces the definition of MSSs, aging MSEs, NHCTMM, and the universal generating function (UGF). Basic concepts of the quasi-renewal process in binary state systems is reviewed in Section III. Assumptions, the formulations of the quasi-renewal process for individual aging MSE, and the expression of the long-run expected profit per unit time under policy N are presented in Section IV. The proposed models are demonstrated via an illustrative case of a three-element series-parallel MSS in Section V, and followed by discussion, and conclusion in Section VI.

II. MSS Definition and Reliability Assessment

A. Definition of MSS, and MSE

A system that can have a finite number of discrete performance rates is called a MSS [31]. To analyse MSS behavior, one has to know the characteristics of its elements. Any system element l can have k_l different states corresponding to the performance rates, represented by the set

$$\mathbf{g}_l = \{g_{l,1}, g_{l,2}, \ldots, g_{l,k_l}\}. \tag{1}$$

The performance rate $G_l(t)$ of element l at any instant $t \geq 0$ is a random variable taking a value from $\mathbf{g}_l : G_l(t) \in \mathbf{g}_l$. Therefore, for the time interval $[0, T]$, where T is the MSS operation period, the performance rate of element l forms a stochastic process. The state distribution of element l at any instant t can be represented by the set

$$\mathbf{p}_l(t) = \{p_{l,1}(t), p_{l,2}(t), \ldots, p_{l,k_l}(t)\}, \tag{2}$$

where $p_{l,i}(t)$ represents the probability that $G_l(t) = g_{l,i}$.

B. NHCTMM for Aging MSE

To describe the degradation process of individual MSE, Lisnianski & Levitin [31], [42] employ the homogeneous continuous time Markov model. Basically, it assumes the time of transition between any two states follows a negative exponential distribution, and thus the deteriorating process of MSE has the memoryless property [43]. The hypothesis that the transition intensity to the next state only depends on the current state is, however, only applicable to elements having no age effect [44], [45]. In fact, it is more realistic to consider the case that an element's deterioration process is not only related to the current element state, but also to the age of the element [45], [46]. Taking this concept into account, the NHCTMM is utilized in this paper to derive the stochastic behavior of individual aging elements through considering the age-related increasing state transition intensity.

For a non-repairable aging MSE, the element transits from the states with greater performance rate to states with lower performance rate. A typical state-space diagram of an aging MSE is illustrated in Fig. 1.

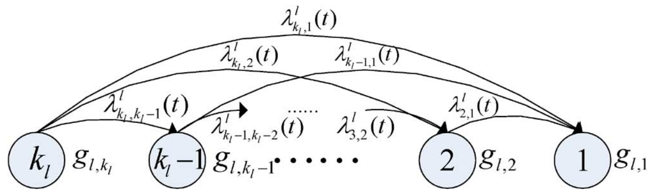

Fig. 1. A typical state-space diagram of an aging MSE l.

For element l, this performance degradation is characterized by the stochastic process $\{G_l(t)|t \geq 0\}$. The intensity $\lambda^l_{i,j}(t)$, $j \in \{i-1, i-2, \ldots, 1\}$ of any transition from state i to state j is a monotonically increasing function with respect to element age. The element state probability is written as

$$p_{l,i}(t) = \Pr\{G_l(t) = I\}, \quad i = 1, \ldots, k_l\ t \geq 0. \tag{3}$$

Similar to [44], and [45], the instantaneous transition intensity matrix $\Lambda^l(t)$ of element l is written as (4) at the bottom of the page, where

$$\lambda^l_{i,j}(t) = \lim_{\Delta t \to 0} \frac{\Pr\{G_l(t+\Delta t) = j | G_l(t) = I\}}{\Delta t}. \tag{5}$$

Distinct from the homogeneous continuous time Markov model proposed in [31], [42], solving the NHCTMM is a complicated challenge [44]–[47]. How to solve a NHCTMM in an efficient manner is outside the scope of this paper, and the formulations given in [44], and [45] are directly used here to obtain the dynamic state probability of individual elements. For example, with the initial conditions $p_{l,k_l}(0) = 1$, and $p_{l,i}(0) = 0$ for $1 \leq i < k_l$, the probability that element l is in its best state k_l is expressed asd

$$p_{l,k_l}(t) = \exp\left[-\int_0^t \sum_{i=1}^{k_l-1} \lambda^l_{k_l,i}(\tau) d\tau\right], \tag{6}$$

and the probability that element l is in its state $k_l - 1$ is expressed as

$$\begin{aligned} p_{l,k_l-1}(t) = \int_0^t \exp\left[-\int_0^{\tau_1} \sum_{i=1}^{k_l-1} \lambda^l_{k_l,i}(s) ds\right] \\ \times \exp\left[-\int_{\tau_1}^t \sum_{i=1}^{k_l-2} \lambda^l_{k_l-1,i}(s) ds\right] \lambda^l_{k_l,k_{l-1}}(\tau_1) d\tau_1. \end{aligned} \tag{7}$$

The element state distribution satisfies the condition $\sum_{i=1}^{k_l} p_{l,i}(t) = 1$ because, at any time instant t, the element can always be in one and only in one of k_l states, and all the states of individual elements compose the complete group of mutually exclusive events.

C. MSS Reliability Evaluation

Given the MSE state distribution $\mathbf{p}_l(t) = \{p_{l,1}(t), p_{l,2}(t), \ldots, p_{l,k_l}(t)\}$ solved from the NHCTMM, the element state distribution can be expressed via the universal generating function (UGF), which is written in the polynomial form [31], [48]

$$\begin{aligned} u_l(z,t) = \sum_{i=1}^{k_l} p_{l,i} \cdot z^{g_{l,i}} = p_{l,1}(t) \cdot z^{g_{l,1}} + p_{l,2}(t) \cdot z^{g_{l,2}} \\ + \cdots + p_{l,k_l}(t) \cdot z^{g_{l,k_l}}. \end{aligned} \tag{8}$$

The system performance distribution at any time instant can be determined based on the element state distribution, and it is written in the same UGF fashion as

$$\begin{aligned} U_s(z,t) = \sum_{i=1}^{K_s} p_i \cdot z^{g_i} = p_1(t) \cdot z^{g_1} + p_2(t) \cdot z^{g_2} \\ + \cdots + p_{N_s}(t) \cdot z^{g_{K_s}}. \end{aligned} \tag{9}$$

Suppose the MSS consists of M MSEs. To obtain the UGF of arbitrary system structure based on the element UGF, one needs to apply the composition operations $\otimes$ recursively as [31]

$$\begin{aligned} &U_s(z,t) \\ &= \otimes\{u_1(z,t), \ldots, u_M(z,t)\} \\ &= \otimes\left\{\sum_{i_1=1}^{k_1} p_{1,i_1}(t) \cdot z^{g_{1,i_1}}, \ldots, \sum_{i_M=1}^{k_M} p_{M,i_M}(t) \cdot z^{g_{M,i_M}}\right\} \\ &= \sum_{i_1=1}^{k_1} \cdots \sum_{i_M=1}^{k_M} \left(\prod_{j=1}^{M} p_{j,i_j}(t) \cdot z^{\phi(g_{1,i_1}, \ldots, g_{M,i_M})}\right) \\ &= \sum_{i=1}^{K_s} p_i(t) \cdot z^{g_i}. \end{aligned} \tag{10}$$

State K_s is the best state with the maximum performance rate, whereas state 1 is the worst one. This polynomial $U_s(z,t)$ represents all of the possible mutually exclusive combinations of realizations of the variables by relating the probabilities of each combination to the value of function $\phi(G_1(t), \ldots, G_M(t))$, which is determined by both the system structure, and performance rates' combination property. For example, in the case of

$$\Lambda^l(t) = \begin{bmatrix} \lambda^l_{k_l,k_l-1}(t) & \lambda^l_{k_l,k_l-2}(t) & \cdots & \lambda^l_{k_l,2}(t) & \lambda^l_{k_l,1}(t) \\ 0 & \lambda^l_{k_l-1,k_l-2}(t) & \cdots & \lambda^l_{k_l-1,2}(t) & \lambda^l_{k_l-1,1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda^l_{2,1}(t) \end{bmatrix}, \tag{4}$$

a *flow transmission* type system with two elements connected in *series*, one may have

$$G_s(t) = \min\{G_1(\mathrm{t}), G_2(t)\} \tag{11}$$

and for the case of a *series-connected time processing* system, one has

$$G_s(t) = \left(G_1^{-1}(t) + G_2^{-1}(t)\right)^{-1}. \tag{12}$$

Generally, the UGF approach is a general, efficient method to evaluate the state probability, and performance rate in many cases; and it can deal with different kinds of dependencies between system performance rate, and element performance rate [49]. With the assistance of the combination of the UGF approach with the stochastic process identified as a NHCTMM for individual aging MSE, the system state distribution can be computed in an efficient way, as stated in [42]. Due to the age effect on the individual elements, the deterioration process of the entire MSS is age-related also.

Furthermore, the MSS reliability that, at time instant t, the system has not reached any of the states with the performance rate less than the specified user demand w, is defined using the operator δ_R as

$$\begin{aligned} R(t,w) &= \delta_R\left(U_s(z,t),w\right) = \delta_R\left(\sum_{i=1}^{K_s} p_i(t)\cdot z^{g_i}, w\right) \\ &= \sum_{i=1}^{K_s} p_i(t) 1(g_i - w \geq 0), \end{aligned} \tag{13}$$

where the expected performance rate with respect to any instant t is equal to

$$E(t) = \delta_E\left(U_s(z,t)\right) = \delta_E\left(\sum_{i=1}^{K_s} p_i(t)\cdot z^{g_i}\right) = \sum_{i=1}^{K_s} p_i(t) g_i. \tag{14}$$

The MSS reliability is a summation of probabilities of all the acceptable states. All the unacceptable states can be regarded as failed states, and the failure probability is a sum of probabilities of all the unacceptable states. From this perspective, a MSS under the user demand context can also be regarded as a binary state system, because one groups all the acceptable states as a working state, and unacceptable states as a failed state. With the same manner as in the traditional binary state system, the failure rate of the MSS is formulated as

$$\begin{aligned} \lambda(t,w) &= \frac{f(t,w)}{R(t,w)} = \frac{\frac{d(1-R(t,w))}{dt}}{R(t,w)} \\ &= \frac{-\frac{d\left(\sum_{i=1}^{K_s} p_i(t)1(g_i(t)-w\geq 0)\right)}{dt}}{\sum_{i=1}^{K_s} p_i(t) 1\left(g_i(t) - w \geq 0\right)}, \end{aligned} \tag{15}$$

which represents the transition intensity of MSS from acceptable states to unacceptable states under the user demand w, and the *mean time to failure (MTTF)* can be written as

$$MTTF = \int_0^{+\infty} R(t,w)dt = \int_0^{+\infty}\left(\sum_{i=1}^{K_s} p_i(t) 1(g_i - w \geq 0)\right)dt. \tag{16}$$

Therefore, with known system failure rate and MTTF, one can clearly quantify the degradation trends after system repairs, and further gets more insights to make maintenance decision.

III. Quasi-Renewal Process Definition

Wang & Pham introduced quasi-renewal processes [11] to model the imperfect maintenance of binary state systems. Later on, a comprehensive application of this theory in maintenance and warranty issues is summarized in [10].

The original definition of a quasi-renewal process is elaborated as follows. Let $\{N(t), t > 0\}$ be a counting process, and X_n be a random variable denoting the time between the $(n-1)$th and nth event of the process, where $n \geq 1$. Observing the sequence of nonnegative random variables $\{X_1, X_2, X_3, \ldots\}$, the counting process $\{N(t), t \geq 0\}$ is said to be a quasi-renewal process with parameter a with the first inter-arrival time X_1 if $X_1 = Z_1$, $X_2 = aZ_2$, $X_3 = a^2 Z_3, \ldots$, where Z_i are i.i.d., and $a > 0$. For $a = 1$, the quasi-renewal process becomes the ordinary renewal process; when $0 < a < 1$, it is called a decreasing quasi-renewal process, which can be used to model the successive system lifetime after repair; $a > 1$ is known as an increasing quasi-renewal process that represents the successive increasing repair time, and cost in every maintenance activity.

Let's assume, in a binary state system, that the random variable X_j denotes the system lifetime between the $(j-1)$ th and j th repair, and $X_1, X_2, X_3, \ldots, X_n$ form a quasi-renewal process with parameter a. According to [10], one may have

$$F_n(t) = F_1(a^{1-n}t), \tag{17}$$
$$R_n(t) = 1 - F_n(t) = R_1(a^{1-n}t), \tag{18}$$
$$f_n(t) = \alpha^{1-n} f_1(a^{1-n}t), \text{ and} \tag{19}$$
$$\lambda_n(t) = a^{1-n}\lambda_1(a^{1-n}t) \tag{20}$$

where $f_1(t)$, $F_1(t)$, $R_1(t)$, and $\lambda_1(t)$ are the failure probability density function, failure probability function, reliability function, and failure rate of a new system respectively; $f_n(t)$, $F_n(t)$, $R_n(x)$, $\lambda_n(t)$ are the equivalent for the system in the n th repair cycle, respectively. Moreover, the expected value, and variance of the lifetime between each repair can be given as

$$E(X_n) = a^{n-1}E(X_1), \text{ and} \tag{21}$$
$$\mathrm{Var}(X_n) = a^{2n-2}\mathrm{Var}(X_1). \tag{22}$$

The quasi-renewal process is an effective method to model imperfect maintenance because it directly describes the decreasing trend of the lifetime after each repair through a sequence of monotone random variables. However, the original quasi-renewal process is only successfully adopted to model the imperfect maintenance for binary state systems. The details of using quasi-renewal processes in MSEs are described next.

IV. General Maintenance Model, and Policy

In this section, we describe the studied MSS with some assumptions, and propose models to determine the optimal replacement strategy.

Assumption 1: A new MSS is installed at time 0, and the whole system will eventually be replaced by a s-identical one at replacement time.

Assumption 2: The studied MSS might consist of an arbitrary number of aging MSE connected in an arbitrary structure. The state probability function $p_i(t)$ of the MSS at any time instant can be assessed through the NHCTMM of MSE, and the UGF method as presented in Section II.

Assumption 3: All the MSEs in the MSS, either failed or working, are restored to their best state after repair. Therefore, the MSS will return to its best state (state K_s) also. However, even if every MSE is restored to its best state, the repaired MSE cannot be viewed as in a completely new condition. It is more realistic to consider that repair activities restore the MSE to somewhere between "as good as new" and "as bad as old" conditions. This characterization is true because every MSE degrades with its usage, even if it is recovered to its best state after repair. The state transition intensities increase proportionally after repair. Hence, the pace of degradation from best state to lower state can become faster with repair times. We use the quasi-renewal process to describe the degradation trend at the element level for each repair cycle as follows.

Let us define a sequence of nonnegative random variables $\{X_{1,i}^l, X_{2,i}^l, \ldots, X_{n,i}^l, \ldots\}$, where $X_{n,i}^l$, $(i \in \{1,2,\ldots,k_l\}, n \in \{1,2,\ldots\})$ denotes the random time that the MSE l sojourns in the ith state in the nth repair cycle. If $X_{1,i}^l = Z_{1,i}$, $X_{2,i}^l = \alpha_l Z_{2,i}$, $X_{3,i}^l = \alpha_l^2 Z_{3,i}, \ldots$ where random variables $Z_{n,i}$, $n \in \{1,2,\ldots\}$ are i.i.d., and $0 < \alpha_l < 1$, the counting process $\{N^l(t), t \geq 0\}$, which represents the cumulative times that MSE l falls from the best state k_l into the worst state 1 after repair, is said to be a decreasing quasi-renewal process with the parameter α_l once the transition rates among any pair of states somehow increases proportionally after each repair activity. Thus, suppose the state probability of the MSE at the first repair cycle is denoted as $p_{l,i}(t)$, $i \in \{1,2,\ldots,k_l\}$, which can be obtained via the NHCTMM mentioned in Section II, or another stochastic process modeling method, such as a homogeneous Markov model, or Monte Carlo simulation. The state probability of this repaired MSE in the n th repair cycle is written as $p_{l,i}(\alpha_l^{1-n}t)$, $i \in \{1,2,\ldots,k_l\}$. We also assume that the repair activity has the identical recovery ability whichever elements state the MSE stays in. In other words, the state probability function of each repair cycle only depends on the number of repairs, but not the element state in which the repair activity is executed. The smaller α_l is, the lower the maintenance quality. Apparently, if $\alpha_l = 1$, the quasi-renewal process becomes the ordinary renewal process, and the repaired MSE can be regarded as a completely new element in the next repair cycle.

Assumption 4: From the "system perspective," the MSS is regarded as in a failure state whenever its performance rate does not satisfy the user demand. Repair is triggered once the MSS falls into one of the unacceptable performance states, where the performance rate is less than the user demand. Any element state transition or element failure will not trigger maintenance unless these element state transitions result in a system transition from an acceptable state to an unacceptable state under a user demand context. After repair, all the MSE in the MSS, either failed or working, are restored to their best state, but with accelerated state transition rates, which is described via imperfect maintenance as given in Assumption 3. The system state distribution in the nth repair cycle therefore can be computed via the UGF operations as follows.

$$\begin{aligned} U_s^n(z,t) &= \otimes\left\{u_1^n(z,t),\ldots,u_M^n(z,t)\right\} \\ &= \otimes\left\{\sum_{i_1=1}^{k_1} p_{1,i_1}\left(\alpha_1^{1-n}t\right)\right. \\ &\qquad \left.\cdot z^{g_{1,i_1}},\ldots,\sum_{i_M=1}^{k_M} p_{M,i_M}\left(\alpha_M^{1-n}t\right)\cdot z^{g_{M,i_M}}\right\} \\ &= \sum_{i_1=1}^{k_1}\cdots\sum_{i_M=1}^{k_M}\left(\prod_{j=1}^{M} p_{j,i_j}\left(\alpha_j^{1-n}t\right)\cdot z^{\phi(g_{1,i_1},\ldots,g_{M,i_M})}\right) \\ &= \sum_{i=1}^{K_s} p_i^n(t)\cdot z^{g_i}, \end{aligned} \tag{23}$$

If the performance rates of the states below state k (from state $k-1$ to state 1) cannot satisfy user demand, then the probability function (or reliability defined in (13)) that the MSS does not fall into any unacceptable state in the nth repair cycle is written as

$$R_n(t,w) = \sum_{i=1}^{K_s} p_i^n(t) 1(g_i - w \geq 0) = \sum_{i=k}^{K_s} p_i^n(t). \tag{24}$$

In accordance with (15), the failure rate in the nth repair cycle is given by

$$\begin{aligned} \lambda_n(t,w) &= \frac{f_n(t,w)}{R_n(t,w)} = \frac{\frac{d(1-R_n(t,w))}{dt}}{R_n(t,w)} \\ &= \frac{-\frac{d\left(\sum_{i=1}^{K_s} p_i^n(t)1(g_i(t)-w\geq 0)\right)}{dt}}{\sum_{i=1}^{K_s} p_i^n(t)1\left(g_i(t)-w\geq 0\right)}. \end{aligned} \tag{25}$$

Let a sequence of random variables $\{X_1^s, X_2^s, X_3^s, \ldots, X_n^s, \ldots\}$ represent the random lifetimes of the MSS in each repair cycle. Then the expected mean time to failure for the first repair cycle is formulated as

$$E\left(X_1^s\right) = MTTF = \int_0^{+\infty} R_1(t,w)dt = \mu_{X_1^s}. \tag{26}$$

Thus, according to (16) and (24), the expected mean time to failure in the nth failure cycle is

$$E\left(X_n^s\right) = \int_0^{+\infty} R_n(t,w)dt = \mu_{X_n^s}. \tag{27}$$

The expected mean time of the MSS sojourning in acceptable state i $(i \geq k)$ during the nth repair cycle is

$$E(S_{n,i}) = \int_0^{+\infty} p_i^n(t)dt = \mu_{S_{n,i}}. \tag{28}$$

Specifically, if the repair activity has identical impact on each of the MSE in the MSS, i.e. $\alpha_l = \alpha$, $l \in \{1, 2, \ldots, M\}$, then the UGF of the entire system in the nth repair cycle becomes

$$U_s^n(z,t) = \sum_{i=1}^{K_s} p_i(\alpha^{1-n}t) \cdot z^{g_i}. \tag{29}$$

Then, instead of (24) and (25), one has

$$R_n(t,w) = \sum_{i=k}^{K_s} p_i(\alpha^{1-n}t), \text{ and} \tag{30}$$

$$\lambda_n(t,w) = \frac{\alpha^{1-n} f(a^{1-n}t, w)}{R(a^{1-n}t, w)} = \alpha^{1-n}\lambda_1(\alpha^{1-n}t, w). \tag{31}$$

Moreover, $E(X_n^s) = \alpha^{n-1}E(X_1^s) = \alpha^{n-1}\mu_{X_1^s}$. In the same manner, one can also have $E(S_{n,i}) = \alpha^{n-1}E(S_{1,i}) = \alpha^{n-1}\mu_{S_{1,i}}$. And the sequence of random variables $\{X_1^s, X_2^s, X_3^s, \ldots, X_n^s, \ldots\}$, which represents the random lifetimes of MSS in each repair cycle, forms a decreasing quasi-renewal process. However, these properties don't hold once the α_l, $l \in \{1, 2, \ldots, M\}$ are not identical.

Assumption 5: The element repair time doesn't depend on the state in which the element stays currently, but only depends on the number of the repair cycle. We assume that the repair time of the MSE l follows an increasing quasi-renewal process with parameter $\beta_l > 1$, and the repair time is represented by a sequences of nonnegative random variables $\{Y_1^l, Y_2^l, Y_3^l, \ldots\}$. The expected repair time for the MSE l in the nth repair cycle is given by

$$E\left(Y_n^l\right) = \beta_l^{n-1} E\left(Y_1^l\right) = \beta_l^{n-1}\mu_{Y_1^l}. \tag{32}$$

If there is only one maintenance facility and team, then the total repair time required for the entire system in the nth repair cycle is given by

$$E\left(Y_n^s\right) = \sum_{l=1}^{M} \beta_l^{n-1}\mu_{Y_1^l}. \tag{33}$$

On the other hand, if there are multiple facilities and teams working simultaneously, then the total repair time required is given by

$$E\left(Y_n^s\right) = \frac{\sum_{l=1}^{M} \beta_l^{n-1}\mu_{Y_1^l}}{H_F}, \tag{34}$$

where H_F is the number of facilities/team. Or the total repair time required is equal to the maximum time among all of the repair time for individual MSE,

$$E\left(Y_n^s\right) = \max\left\{\beta_1^{n-1}\mu_{Y_1^1}, \beta_2^{n-1}\mu_{Y_1^2}, \ldots, \beta_M^{n-1}\mu_{Y_1^M}\right\}. \tag{35}$$

The time duration for eventual replacement of the entire system is denoted by the random variable Z, where $E(Z) = \mu_Z > 0$.

Assumption 6: Random variables $X_{n,i}^l$, Y_n^l, and Z, for $n = 1, 2, 3, \ldots$, $l = 1, 2, \ldots, M$, and $i = 1, 2, \ldots, k_l$, are s-independent random variables.

Assumption 7: We use c_w, c_f, c_{rw}, and c_r to represent the working reward of MSS per unit performance rate at per unit time, the cost of per unit time under repair when failed, the replacement cost per unit time, and the fixed replacement cost of the system, respectively. Without loss of generality, let $c_f \ll c_r$ [12].

The replacement policy being considered is called policy N, in which whether we replace the whole system or not is based on the system failure number. The MSS is regarded as failed when its performance falls below the demand w, and the system is replaced by a s-identical, new one when the total number of failure reaches N. Our objective is to determine the optimal value N^* to maximize the long run expected profit per unit time.

Let T_1 be the first replacement time of the MSS under policy N, and $T_i(i \geq 2)$ be the time duration between the $(i-1)$th replacement and ith replacement of the system. Then $\{T_1, T_2, T_3, \ldots, T_i, \ldots\}$ forms a renewal process, and the end time of the replacement forms a renewal point between each replacement cycle. We use $C(N)$ to denote the long run expected profit per unit time of the MSS under policy N. Thus, we have

$$C(N) = \frac{\text{the expected profit in a renewal cycle}}{\text{the expected length of a renewal cycle}} = \frac{c_w E\left(S(N)\right) - c_f E\left(Y(N-1)\right) - \left(c_{rw}E(Z) + c_r\right)}{E\left(Y(N-1)\right) + E\left(X(N)\right) + E(Z)}, \tag{36}$$

where

$$E(Z) = \mu_Z, \tag{37}$$

$$E\left(X(N)\right) = E\left(\sum_{n=1}^{N} X_n^s\right) = \sum_{n=1}^{N} E(X_n^s) = \sum_{n=1}^{N} \mu_{X_n^s}, \tag{38}$$

$$E\left(Y(N-1)\right) = E\left(\sum_{n=1}^{N-1} Y_n^s\right) = \sum_{n=1}^{N-1} E\left(Y_n^s\right), \tag{39}$$

and

$$\begin{aligned} E\left(S(N)\right) &= E\left(\sum_{n=1}^{N}\sum_{i=1}^{K_s} S_{n,i} \cdot g_i \cdot 1(g_i - w \geq 0)\right) \\ &= \sum_{i=1}^{K_s}\sum_{n=1}^{N} E(S_{n,i}) \cdot g_i \cdot 1(g_i - w \geq 0) \\ &= \sum_{\substack{i=k \\ g_k \geq w}}^{K_s} \sum_{n=1}^{N} \mu_{S_{n,i}} \cdot g_i. \end{aligned} \tag{40}$$

Especially, if $\alpha_l = \alpha$, and $\beta_l = \beta$ for $l \in \{1, 2, \ldots, M\}$, instead of (38)–(40), we have

$$E\left(X(N)\right) = \sum_{n=1}^{N} \alpha^{n-1}\mu_{X_1^s} = \frac{\mu_{X_1^s}(1-\alpha^N)}{1-\alpha}, \tag{41}$$

$$E\left(Y(N-1)\right) = \sum_{n=1}^{N-1} \beta^{n-1}\mu_{Y_1^s} = \frac{\mu_{Y_1^s}\left(1-\beta^{N-1}\right)}{1-\beta}, \tag{42}$$

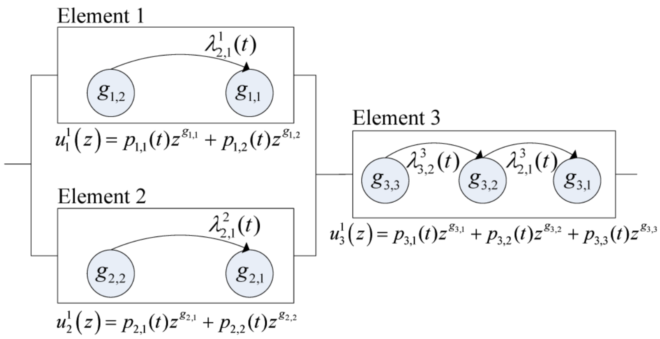

Fig. 2. MSS structure, and elements' state-space diagrams.

TABLE I
PARAMETERS FOR EACH ELEMENT

Element(#)	Performance rate (tons/min)	Initial condition	Transition intensities (year^{-1})	α_l	β_l
1	$g_{12} = 1.5$	$p_{12}(0) = 1$	$\lambda^1_{2,1}(t) = 0.8 + 0.2t$	0.7	1.1
	$g_{11} = 0.0$	$p_{11}(0) = 0$			
2	$g_{22} = 2.0$	$p_{22}(0) = 1$	$\lambda^2_{2,1}(t) = 1.5 + 0.1t^2$	0.8	1.2
	$g_{22} = 0.0$	$p_{22}(0) = 0$			
3	$g_{33} = 4.0$	$p_{33}(0) = 1$	$\lambda^3_{3,2}(t) = 1.2 + 0.15t$	0.9	1.3
	$g_{32} = 1.8$	$p_{32}(0) = 0$	$\lambda^3_{2,1}(t) = 2.0 + 0.2t$		
	$g_{31} = 0.0$	$p_{31}(0) = 0$			

and

$$E\left(S(N)\right) = \sum_{\substack{i=k \\ g_k \geq w}}^{K_s} \left(\frac{\mu_{S_{1,i}}(1-\alpha^N)}{1-\alpha} \right) \cdot g_i. \tag{43}$$

Thus, the optimal policy N^* can be solved via

$$N^* = \arg_N \left\{ C(N) \rightarrow \max \right\}. \tag{44}$$

V. AN ILLUSTRATIVE CASE

We consider a type of *flow transmission* MSSs with flow dispersion, say a water pipe system [38]. As shown in Fig. 2, the MSS consists of three aging MSEs, where element 1 and element 2 are connected in parallel with each other, while the element 3 is connected in series with the others.

The age-related transition intensities in the first repair cycle, which are linear or quadratic functions in terms of time instant t, are in Table I. The associated performance rate at each element state, and the parameters for imperfect repair are also given.

A. Elements and System Performance Rates, and State Distributions

According to the NHCTMM presented in Section II-B, the element state distribution can be formulated as follows.

For element 1,

$$\begin{cases} p_{1,2}(t) = \exp\left[-\int_0^t \lambda^1_{2,1}(\tau)d\tau\right] \\ p_{1,1}(t) = 1 - \exp\left[-\int_0^t \lambda^1_{2,1}(\tau)d\tau\right]. \end{cases}$$

For element 2,

$$\begin{cases} p_{2,2}(t) = \exp\left[-\int_0^t \lambda^2_{2,1}(\tau)d\tau\right] \\ p_{2,1}(t) = 1 - \exp\left[-\int_0^t \lambda^2_{2,1}(\tau)d\tau\right]. \end{cases}$$

For element 3,

$$\begin{cases} p_{3,3}(t) = \exp\left[-\int_0^t \lambda^3_{3,2}(\tau)d\tau\right] \\ p_{3,2}(t) = \int_0^t \exp\left[-\int_0^{\tau_1} \lambda^3_{3,2}(s)ds\right] \\ \qquad\qquad \times \exp\left[-\int_{\tau_1}^t \lambda^3_{2,1}(s)ds\right] \lambda^3_{3,2}(\tau_1)d\tau_1 \\ p_{3,1}(t) = 1 - p_{3,2}(t) - p_{3,3}(t). \end{cases}$$

Following Section II-C, the UGFs for the individual elements in the first repair cycle are expressed as

$$\begin{aligned} u^1_1(z,t) &= p_{1,1}(t)z^{g_{1,1}} + p_{1,2}(t)z^{g_{1,2}} \\ &= p_{1,1}(t)z^0 + p_{1,2}(t)z^{1.5}, \\ u^1_2(z,t) &= p_{2,1}(t)z^{g_{2,1}} + p_{2,2}(t)z^{g_{2,2}} \\ &= p_{2,1}(t)z^0 + p_{2,2}(t)z^{2.0}, \text{ and} \\ u^1_3(z,t) &= p_{3,1}(t)z^{g_{3,1}} + p_{3,2}(t)z^{g_{3,2}} + p_{3,3}(t)z^{g_{3,3}} \\ &= p_{3,1}(t)z^0 + p_{3,2}(t)z^{1.8} + p_{3,3}(t)z^{4.0}. \end{aligned}$$

According to the system structure function and performance rate combination property, the output performance rate of the entire MSS is defined as

$$G(t) = \min\left\{G_1(t) + G_2(t), G_3(t)\right\}.$$

Using the composition operations $\underset{Ser}{\otimes}$, and $\underset{Par}{\otimes}$, the UGF for the MSS in the first repair cycle is given by

$$\begin{aligned} U^1_s(z,t) &= \underset{Ser}{\otimes}\left(\underset{Par}{\otimes}\left(u^1_1(z,t), u^1_2(z,t)\right), u^1_3(z,t)\right) \\ &= \underset{Ser}{\otimes}\left(\underset{Par}{\otimes}\left(\sum_{i_1=1}^{2} p_{1,i_1}(t)\cdot z^{g_{1,i_1}}, \right.\right. \\ &\qquad \left.\left. \sum_{i_2=1}^{2} p_{2,i_2}(t)\cdot z^{g_{2,i_2}}\right), \sum_{i_3=1}^{3} p_{3,i_3}(t)\cdot z^{g_{3,i_3}}\right) \\ &= \sum_{i_3=1}^{3}\sum_{i_2=1}^{2}\sum_{i_1=1}^{2}\left(\prod_{j=1}^{3} p_{j,i_j}(t)\cdot z^{\min(g_{1,i_1}+g_{2,i_2},g_{3,i_3})}\right). \end{aligned}$$

One can obtain the UGF of the entire MSS with the output performance distributions in the first repair cycle following

$$U^1_s(z,t) = \sum_{i=1}^{5} p^1_i(t)z^{g_i},$$

TABLE II
PERFORMANCE RATES AND STATE PROBABILITIES OF MSS

MSS performance rate (tons/min)	State probabilities
$g_1 = 0$	$p_1^1(t) = p_{1,1}(t)p_{2,1}(t) + p_{3,1}(t)p_{1,2}(t) + p_{3,1}(t)p_{1,1}(t)p_{2,2}(t)$
$g_2 = 1.5$	$p_2^1(t) = p_{1,2}(t)p_{2,1}(t)\left[p_{3,2}(t) + p_{3,3}(t)\right]$
$g_3 = 1.8$	$p_3^1(t) = p_{3,2}(t)p_{2,2}(t)$
$g_4 = 2.0$	$p_4^1(t) = p_{3,3}(t)p_{1,1}(t)p_{2,2}(t)$
$g_5 = 3.5$	$p_5^1(t) = p_{3,3}(t)p_{1,2}(t)p_{2,2}(t)$

TABLE III
REPAIR AND REPLACEMENT PARAMETERS

$\mu_{Y_1^1}$	$\mu_{Y_1^2}$	$\mu_{Y_1^3}$	μ_Z	c_f ($)	c_r ($)	c_w($ unit time)	c_{rw}($ unit time)
0.02	0.02	0.03	0.1	5.0	80	40	10

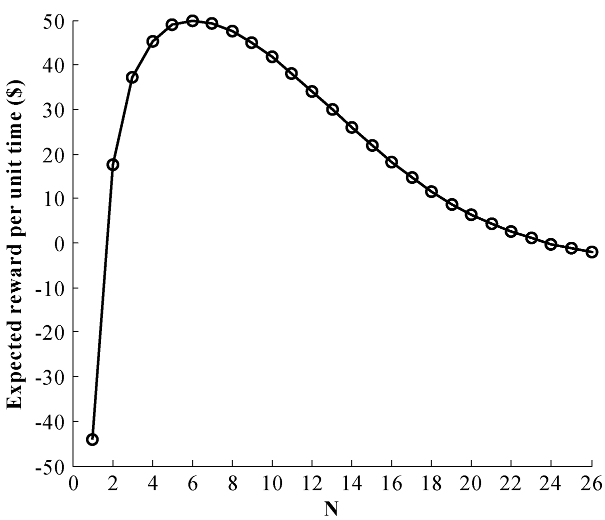

Fig. 3. Expected profit per unit time, corresponding to N.

where g_i, and $p_i^1(t)$ in terms of element state distributions are in Table II.

Assuming the user demand $w = 1.8$, the reliability of the MSS in the first repair cycle is given by

$$R_1(t,w) = \delta_R\left(U_s^1(z,t),w\right) = \delta_R\left(\sum_{i=1}^{5} p_i^1(t)z^{g_i}, 1.8\right) = \sum_{i=1}^{5} p_i^1(t)1((g_i - 1.8) \geq 0) = p_3^1(t) + p_4^1(t) + p_5^1(t).$$

Instead of using the NHTCMM to obtain the element state probability for the new elements, other stochastic process modeling methods, such as Monte Carlo simulation [50], can also facilitate the evaluation of the element state probability. With the combination of the stochastic process model for individual MSE with the UGF method, the reliability of the entire MSS can be assessed efficiently to avoid the dimensionality explosion issue [42].

B. Maintenance Policy

With the assumption that the user demand is $w = 1.8$ tons/min, states 2 and 1 are regarded as failure states, and the repair activity is triggered whenever the MSS falls into these two states. Thus, according to (26), the $MTTF$ of the MSS in the first repair cycle is equal to

$$E\left(X_1^s\right) = \int_0^{+\infty} R_1(t, 1.8)dt = \int_0^{+\infty} \left[p_3^1(t) + p_4^1(t) + p_5^1(t)\right]dt = \mu_{X_1^s} \approx 0.4858\ year.$$

Furthermore, the expected time that the MSS sojourns in state 5 within the first repair cycle is equal to

$$E(S_{1,5}) = \int_0^{\infty} p_5^1(t)dt = \int_0^{\infty} p_{3,3}(t)p_{1,2}(t)p_{2,2}(t)dt = \mu_{S_{1,5}} \approx 0.2660\ year,$$

where the numerical integral with interval $\Delta t = 0.006\ year$, and rectangle integral function are adopted. Further, one has

$$E(S_{1,4}) = \int_0^{\infty} p_4^1(t)dt = \int\_0^{\infty} p_{3,3}(t)p_{1,1}(t)p_{2,2}(t)dt = \mu_{S_{1,4}} \approx 0.0972\ year, \text{ and}$$

$$E(S_{1,3}) = \int_0^{\infty} p_3^1(t)dt = \int_0^{\infty} p_{3,2}(t)p_{2,2}(t)dt = \mu_{S_{1,3}} \approx 0.1226\ year.$$

According to Assumption 3, the system state probability function in the nth repair cycle is

$$p_5^n(t) = p_{3,3}\left(\alpha_3^{1-n}t\right)p_{1,2}\left(\alpha_1^{1-n}t\right)p_{2,2}\left(\alpha_2^{1-n}t\right),$$
$$p_4^n(t) = p_{3,3}\left(\alpha_3^{1-n}t\right)p_{1,1}\left(\alpha_1^{1-n}t\right)p_{2,2}\left(\alpha_2^{1-n}t\right), \text{ and}$$
$$p_3^n(t) = p_{3,2}\left(\alpha_3^{1-n}t\right)p_{2,2}\left(\alpha_2^{1-n}t\right).$$

The integral in (27) and (28) can be approximately solved using the numerical integral method. Suppose there is only one repair facility available, and the associated parameters, e.g. the expected repair time, the repair fee per unit time, as well as the complete replacement cost, are tabulated in Table III. The total repair time required per repair cycle can be calculated via (33). Thus, the expected profit per unit time corresponding to N is shown in Fig. 3.

As shown in Fig. 3, the optimal number of failures before replacement is $N^* = 6$, with the maximum expected profit per unit time $C(N^*) = \$48.35$.

The reliability, and failure rate in each repair cycle are plotted in Figs. 4 and 5, respectively. At any fixed point in time in each repair cycle, the reliability becomes lower with the index of the repair cycle increasing, and the failure rate is also increasing with the index of the repair cycle. The results illustrate that the MSS in each repair cycle has a worse condition than the previous cycle, and the repair action can not completely restore the MSS to the "as good as new" condition. It can only recover the failed MSS to its best state N_s after repair, but with a faster deterioration process in the next cycle.

The expected total performance reward and repair cost are plotted in Fig. 6 with respect to policy N to show the trend.

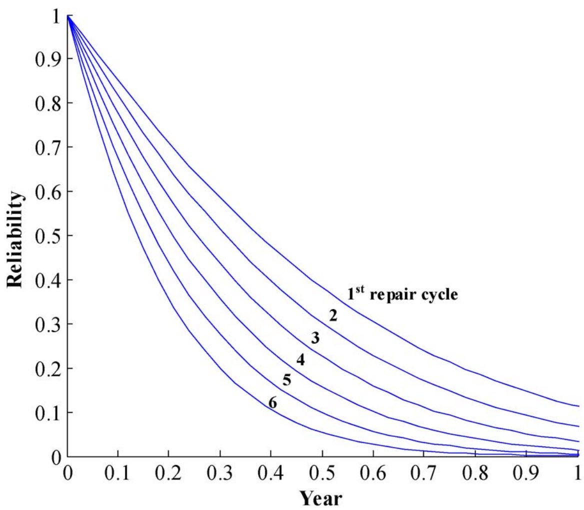

Fig. 4. System reliability in each repair cycle.

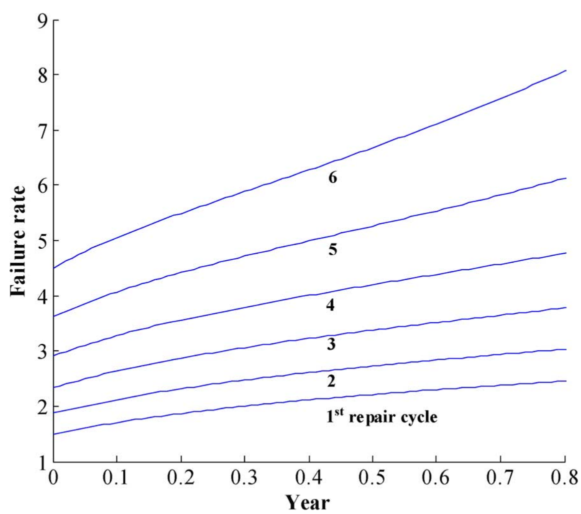

Fig. 5. System failure rate in each repair cycle.

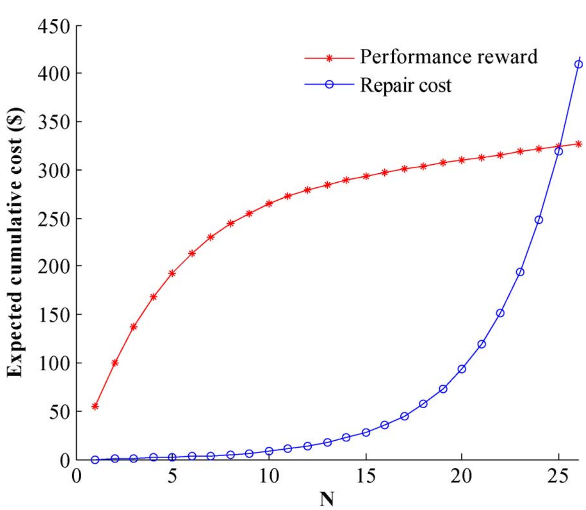

Fig. 6. Tendency of cost, corresponding to policy N.

Obviously, the repair cost is increasing exponentially with N. The increasing rate of performance reward decreases because the $MTTF$ shortens as N increases.

TABLE IV
OPTIMAL N^*, AND $C(N^*)$ VS. DEMAND w

Demand w	N^*	$C(N^*)$
1.5	6	51.20
1.8	6	48.35
2.0	6	43.40
3.5	6	31.66

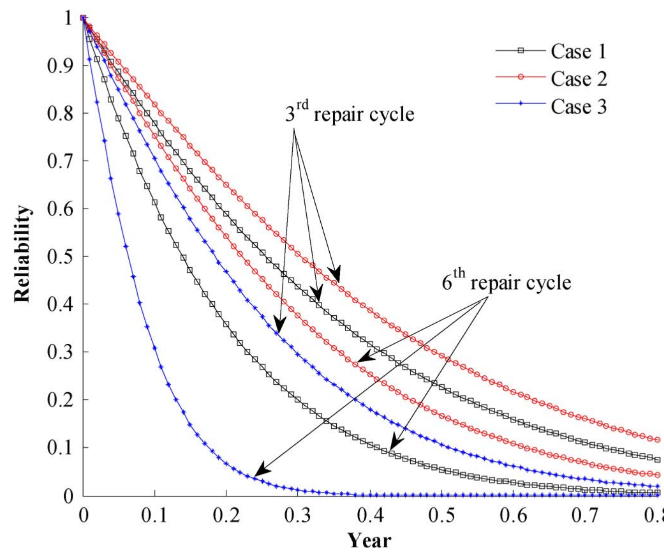

Fig. 7. System reliability comparison for the 3rd and 6th repair cycle.

C. Some Comparisons

We change the user demand w from 3.5 to 1.5 tons/min, and fix the other parameters. The optimal policy N^*, and $C(N^*)$ are tabulated in Table IV.

From Table IV, one observes that, although the optimal failure number N^* is the same for different user demands, the expected profit per unit time is increasing with the demand decreasing. Under the lower demand, more system states can be accepted, and the $MTTF$ is thus longer, while more performance rewards can be obtained during each repair cycle. Actually, a lower demand does not mean a higher expected profit per unit time. Although a lower demand would lead to a longer average length of repair cycle, the expected profit per unit time might become even lower if the system stays at states with lower performance rates for most of a repair cycle.

To demonstrate the impact from the imperfect repair parameters α_l, and β_l, we examine some special cases where $\alpha_l = \alpha$, and $\beta_l = \beta$ for $l = 1, 2, 3$. Suppose $\alpha = 0.7$ (Case 2), $\alpha = 0.9$ (Case 3), and $w = 1.8$ tons/min. We plot the reliability, and failure rate of the 3rd, and 6th repair cycles respectively in Figs. 7 and 8, and compare to the case in the previous subsection where $\alpha_1 = 0.7$, $\alpha_2 = 0.8$, and $\alpha_3 = 0.9$ (Case 1).

From Figs. 7 and 8, one can find that as the index of repair cycles is increasing, the reliability is decreasing, while the failure rate is increasing. Further, because the repair quality of case 1 is between that of case 2 and case 3, it can be observed that, for the 3rd and 6th repair cycles, both the reliability, and the failure

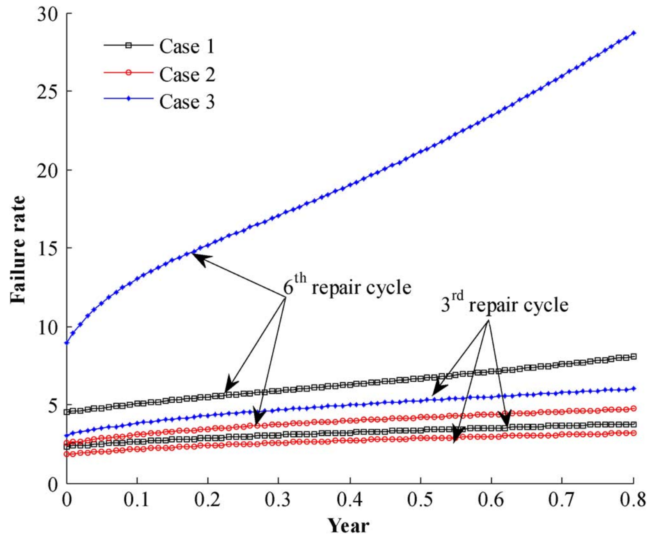

Fig. 8. System failure rate comparison for the 3rd and 6th repair cycle.

TABLE V
OPTIMAL N^*, AND $C(N^*)$ VS. α, AND β_i WHERE $\alpha_l = \alpha$, $\beta_l = \beta$, AND $w = 1.5\ \mathrm{tons/min}$

ID	α	β	N^*	$C(N^*)$
1	0.9	1.1	8	63.08
2		1.5	5	54.74
3		2.0	4	50.10
4		3.0	3	44.92
5	0.6	1.1	5	38.23
6		1.5	4	35.03
7		2.0	3	32.59
8		3.0	3	30.59
9	0.4	1.1	5	22.77
10		1.5	4	20.96
11		2.0	3	19.92
12		3.0	3	18.81

rate of case 1 are somewhere between those of case 2 and case 3.

More detailed comparisons about effect from the imperfect repair parameters α and β are tabulated in Table V when $w = 1.5\ \mathrm{tons/min}$. With a fixed β, both N^*, and $C(N^*)$ decrease as the value of α decreases. This relation is true because a smaller α will lower the $MTTF$ of the entire MSS in each repair cycle, and the expected reward from the performance rate simultaneously decreases. Therefore, a smaller N^* will maximize the expected profit per unit time. If one fixes α, and increases β, the expected repair time, and cost both increase monotonously. The expected profit per unit time, and the optimal policy N^* decrease because more repair time is required in each repair cycle when β is large. In other words, instead of performing imperfect repair, which would be much more cost expensive after several repair cycles, replacing the entire MSS is more cost efficient.

To illustrate the impact from changing the imperfect repair parameter of individual elements, we only change one of the parameters in Case 1, and the corresponding optimal results

TABLE VI
OPTIMAL N^*, AND $C(N^*)$ VS. α_l, AND β_l

ID	α_1	β_1	α_2	β_2	α_3	β_3	N^*	$C(N^*)$
1	0.4	1.1	0.8	1.2	0.9	1.3	6	42.02
2	0.7	2.5	0.8	1.2	0.9	1.3	4	42.16
3	0.7	1.1	0.4	1.2	0.9	1.3	5	17.89
4	0.7	1.1	0.8	2.5	0.9	1.3	4	42.31
5	0.7	1.1	0.8	1.2	0.4	1.3	5	23.10
6	0.7	1.1	0.8	1.2	0.9	2.5	4	41.15

are shown in Table VI. One observes that items #3, and #5 in Table VI have a lower expected profit per unit time compared with Case 1, which is $48.35. The imperfect repair parameters α_2, and α_3 have a large impact on the maximum expected profit per unit time, and it is worth allocating more effort and maintenance resources to improve the repair quality with the aim to restore elements 2 and 3 to a better condition. On the other hand, we can see from items #2, and # 4 that imperfect repair parameters β_1, and β_2 have the least impact, and less effort and resources are necessary to reduce their values.

VI. Discussion, and Conclusion

In this paper, we introduce an optimal replacement policy for MSSs. Maintenance activities are taken into account based on the overall performance degradation trend. This maintenance strategy would be efficient in the case where the strategy of separately performing inspection and maintenance for MSEs is expensive, or involves too much product lost. Through the combined NHCTMM and UGF method, the state performance rate, and associated state probabilities of the whole MSS are easily obtained. The MSS is regarded as failed when its total performance rate is lower than the user demand. A comprehensive repair is triggered immediately whenever the MSS fails, and it restores all the elements to their best state. However, this kind of repair restores the element to somewhere between the "as good as new" condition and the "as bad as old" condition. An imperfect maintenance model using a quasi-renewal process is proposed to model the degradation trend of the aging MSE after each repair action. The $MTTF$, and failure rate of the entire MSS under certain demand are formulated, and their trends in each imperfect repair cycle are demonstrated in the illustrative case. With the aim to maximize the long-run expected profit per unit time of the entire MSS, an explicit expression involving the maintenance cost and performance reward is formulated to obtain the optimal replacement policy N^*. The user demand, and quasi-renewal parameters impact the optimal policy significantly. These parameters need to be carefully determined according to field data in practical problems before decision making.

However, the proposed method has some restrictions.

1) Only one kind of repair action is considered in this work. In reality, preventive maintenance may intervene at pre-set time intervals to restore MSEs to a certain state to avoid a sudden failure. Incorporating the PM into the current replacement policy is worth investigating.

2) The user demand may not be a fixed value all the time. In some cases, such as in a power station or a water supply station, demands are stochastic, and the optimal maintenance policy needs to be further researched with considering the random uncertainty from user demand.
3) The imperfect repair parameters are pre-determined in our studied case. It would be more reasonable to regard these parameters as decision variables which can vary depending on how many maintenance resource are allocated. This approach will provide more flexibility in maintenance decision making.

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