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Optimal Replacement Policy for Fuzzy Multi-State Element

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In this paper, a system consisting of single multi-state element (MSE) with performance rates and transition intensities presented as fuzzy values is introduced. Due to the lack, inaccuracy or fluctuation of collected data, it is often too difficult to evaluate the performance rates and transition intensities of multi-state element/system with precise value, especially in the continuous degradation element/system which is usually simplified to finite multi-state element/system to avoid "dimension damnation". To overcome this challenge, fuzzy set theory as a promising methodology to quantify the non-probabilistic uncertainty is employed here to facilitate the multi-state element/system performance assessment. Given the fuzzy transition intensities and performance rates, the state probabilities of multi-state element are fuzzy also. Meanwhile, when considering the replacement policy, fuzzy continuous-time Markov model with finite discrete states is proposed to assess the fuzzy mean time between replacement (MTBR) and the cumulative fuzzy performance reward in each replacement cycle. In order to obtain the membership functions of the fuzzy indices of interest, parametric programming technique is employed based on the Zadeh's extension principle. The expected fuzzy average profit per unit time is computed under different replacement policy, and then three fuzzy decision making (ordering) methods are adapted to determine the optimal replacement threshold state θ^* with aim to maximize the expected fuzzy average profit per unit time. The effectiveness of the proposed method is illustrated via an example of multi-state power generator.

Keywords: Fuzzy multi-state system (FMSS), fuzzy multi-state element (FMSE), FMSE replacement, maintenance policy, fuzzy Markov reward process, parametric programming.

1 INTRODUCTION

In the real world, many systems are able to perform their task with partial performance rate of their original design. This phenomenon is usually resulting from the degradation of components and parts in the system or/and the failures of some elements which decrease system performance. This type of system is called multi-state system (MSS) and was primarily introduced in the mid-1970s by Murchland [1].

After Barlow [2], Ross [3] and other researchers extended some basic concepts and definitions in the traditional binary-state system reliability theory to the MSS framework, many novel methods were developed to facilitate the MSS reliability and performance assessment, e.g. the extended decision diagram-based method [4], the stochastic process modeling [5], the universal generating function (UGF) [6], the Monte Carlo simulation [7], etc., and these methods have been widely applied to industrial engineering system [8], such as power generating system, computing system, transportation system and radio relay station, etc. On the other hand, some MSS with specific characteristics existing in particular industrial field were also introduced and studied in recent years, e.g. the multiple failure modes MSS [9], the dependent MSS [10], the multi-state weighted system [11], the generalized multi-state k-out-of-n:F system [12], and the acyclic multi-state-node networks [13], and so forth.

Beside the reliability assessment, the MSS maintenance decision making is more crucial and attracts much attentions in both academic and industrial field. Some of recent research on MSS maintenance optimization problem are worth mentioning here, for example, Zhang *et al.* [14] introduced a replacement policy for the repairable system with multiple failure modes, and the geometric process repair model was proposed. A bivariate replacement policy was proposed and discussed in their later work [15], where the number of failure times N and working age T were considered as the bivariate decision parameters. Moustafa *et al.* [16] presented a maintenance model for multi-state semi-Markov deteriorating system and three optional maintenance decisions (do-nothing, minimal maintenance or replacement) are optimally selected to determine what action should be taken at each system state. Chen *et al.* [17] established a semi-Markov decision process (SMDP) model for MSS under condition-based maintenance framework. Chiang *et al.* [18] proposed a state-dependent maintenance policy for MSS subject to aging and fatal shocks under the periodic inspection strategy. Levitin and Lisnianski [19] studied the imperfect preventive maintenance strategy for the MSS with binary-state element. Some other maintenance problems with the aforementioned methods can refer to [20, 21]. All of these works are based on the probabilistic uncertainty framework where the uncertainty can be fully described via probabilistic or stochastic model.

However, the conventional MSS reliability and performance assessment methods are usually based on the following two assumptions [22, 23]:

1. The state probabilistic distributions of MSE in the MSS are precisely known and measurable;
2. The performance rates of MSE are precisely determined.

Actually, there assumptions do not always hold when precisely evaluating the state probability distribution and performance rates is difficult. There are mainly two reasons [23]:

1. Due to the budget or time limitation, getting accurate and sufficient data is impossible or prohibitive. Therefore, the evaluation of element/system degradation behavior can be only expressed in terms like “a unit would fail in about 1 year” and “system performance degrades nearly 200 per unit time”. Thus, crisp values used to represent the probabilistic distributions and performance rates sometimes make no sense.
2. Many elements/systems deteriorate continuously or nearly continuously with time. To avoid the “dimension damnation” [24], the model is oftentimes simplified via state combination to reduce the computational burden. The continuously degrading element/system is finally simplified to one with several discrete states separated by the distinguishable performance rates, and the number of discrete states is usually not too large to make the computation tractable [25].

Because of these two reasons, the conventional approach of representing the performance distribution of MSE/MSS in crisp values involves more risks to describe the actual behavior of the element/system.

Fuzzy reliability theory which employs the fuzzy set theory introduced by Zadeh [26, 27] is becoming a new methodology to deal with the imprecision and uncertainty phenomena in reliability engineering [28]. With the assistance of fuzzy reliability theory, the reliability and risk of complex system can be assessed even with some imprecise information, such as linguistic variable, scarce data, etc. Therefore, it has since received increasing attention in recent years, for example, Cai *et al*. [29] introduced the fuzzy success/failure state and the reliability model to study a gradually degrading computing system. Huang [30] assessed the reliability of a system in the presence of fuzziness in operating time. Huang *et al*. [31] proposed to evaluate the failure possibility via posbist fault tree analysis when statistical data is scarce or failure probability is extremely small. A novel fuzzy Bayesian approach was developed by Wu [32] to create the fuzzy Bayes point estimator of reliability. Huang *et al*. [33] introduced a Bayesian method to assess system reliability when lifetime data is presented as a fuzzy value. Fuzzy dynamic reliability evaluation for a deteriorating system under imperfect repair action was addressed by Verma *et al*. [34]. Ke *et al*. [35] developed a procedure to construct the fuzzy steady-state availability when obtained data are subjective. Two-unit repairable systems suffering common-cause failure was discussed by Huang *et al*. [36], where the time to failure follows fuzzified exponential distribution. Pandey *et al*. [37] proposed a new method to assess the profust reliability indices. To consider the maintenance action involving fuzzy value, Popova *et al*. [38] discussed a T-age replacement policy with fuzzy reward. In their model, the fuzzy theory was applied to the renewal reward processes, and operational reward was also regard as a fuzzy random variable. The optimal T-age replacement policy was determined via nonlinear programming algorithm. However, up to present, most of the reported works mainly focus on binary-state system issues. As stated in Ref. [8], the MSS is already very popular in industry, so the fuzzy reliability and it related issue, such as maintenance decision, warranty analysis etc., under MSS context remains an emerging research paradigm. The concept of fuzzy multi-state system (FMSS) was first used by Ding and Lisnianski [22] in a modeling study where the state probabilities and performances of a component were presented as fuzzy values. In their work, fuzzy UGF (FUGF) method was proposed to assess reliability and availability of FMSS under the fuzzy demand. Afterwards, some general definitions involving relevancy, coherency, dominance and equivalence in FMSS were provided by Ding *et al*. [39], to extend the basic properties of MSS in crisp case to the fuzzy context. Liu *et al*. [23] did more extending work through considering the fuzzy transition intensity and performance rate for MSE, and fuzzy Markov model was proposed to compute the fuzzy state probability distribution and then applied to assess the fuzzy availability under fuzzy user demand. To make a further investigation on some related issues in FMSS, this paper does more efforts to study the maintenance decision problem in FMSE based on our previous work and tries to build up an FMSE replacement model to facilitate making a reasonable maintenance planning in practical applications.

In this paper, FMSE is introduced to overcome the deficiencies of the conventional MSS theory. The state performance rates and transition intensities among each state are treated as fuzzy values. Fuzzy mean time between replacement (MTBR) and fuzzy performance reward is computed via the proposed fuzzy Markov reward model, and the expected fuzzy average profit per unit time is also formulated. To calculate the membership functions of the quantities of interest, the parametric programming algorithm is executed. The replacement policy for the FMSE is introduced and the definition of the threshold state under the replacement policy is also given. Finally, the optimal policy among the possible candidates is elected based on the three fuzzy decision making methods proposed in the past literatures.

The remainder of this paper is organized as follows: In Section 2, fuzzy set, fuzzy number and extension principle are briefly reviewed. The definition of FMSE and FMSS are given in Section 3 and fuzzy Markov model and fuzzy Markov reward model are discussed also. Replacement policy for single FMSE is introduced in Section 4, and three types of fuzzy decision making methods proposed in the literatures are briefly reviewed in Section 5. The proposed model and approach are illustrated in Section 6 via a power generator. Conclusion is given in Section 7.

2 FUZZY SET THEORY

2.1 Fuzzy set and fuzzy number

A fuzzy subset $\tilde{X}$ of a universal set U is defined by its membership (or characteristic) function $\mu_{\tilde{X}} : U \to [0, 1]$. The values of $\mu_{\tilde{X}}(x)$ extends from zero to one which can be interpreted as the membership degree at which x belongs to $\tilde{X}$.

Let $\Re$ be an universal set of real numbers and $\tilde{X}$ be a fuzzy subset of $\Re$. $\tilde{X}_\alpha = \{x|\mu_{\tilde{X}}(x) \geq \alpha\}$ denotes the α-cut level set of $\tilde{X}$ where $\alpha \in [0, 1]$. The interval of this set is written as $\tilde{X}_\alpha = [\tilde{X}^L_\alpha, \tilde{X}^U_\alpha]$, and $\tilde{X}_0$ is the closure of the set $\tilde{X}_0 = \{x|\mu_{\tilde{X}}(x) \geq 0\}$.

$\tilde{X}$ is called a fuzzy real number if: (1) it is a normal and convex fuzzy set; (2) its membership function is upper semi-continuous; (3) the 0-cut level set $\tilde{X}_0$ is bounded in $\Re$; (4) the 1-cut level set $\tilde{X}_1$ is a singleton set, and $\tilde{X}^L_1 = \tilde{X}^U_1$; (5) the boundary functions $L(\alpha) = \tilde{X}^L_\alpha$ and $U(\alpha) = \tilde{X}^U_\alpha$ of membership function are continuous with respect to $\alpha \in [0, 1]$.

The membership function of a typical triangle fuzzy number (TFN) $\tilde{X}$, parameterized by the triplet (a, b, c), is formulated as:

$$\mu_{\tilde{X}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x < b \\ 1, & x = b \\ \dfrac{x-c}{b-c}, & b < x \leq c \\ 0, & otherwise \end{cases}, \tag{1}$$

and is plotted in Fig. 1.

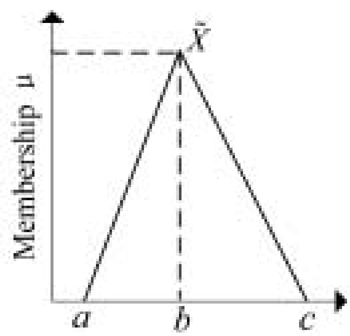

FIGURE 1
The membership function of a triangle fuzzy number.

In this paper, both the state transition intensity and performance rate of each element are treated as TFN; because it is straightforward to manipulate TFN in mathematical calculation, and it has been widely used in many practical situations and reliability engineering [22, 23, 30, 34, 35, 39–41].

2.2 Extension principle and parametric programming algorithm

Zadeh [26, 27] introduced the extension principle to obtain the membership function of a function with n fuzzy numbers as inputs:

$$\mu_{\tilde{p}(\tilde{\mathbf{x}})}(z) = \sup_{\substack{\mathbf{x}\in\mathbf{R}^n \\ z=p(\mathbf{x})}} \min\{\mu_{\tilde{\mathbf{x}}}(\mathbf{x})\} = \sup_{\substack{x_1\in\Re_1,\ldots,x_n\in\Re_n \\ z=p(x_1,\ldots,x_n)}} \min\{\mu_{\tilde{X}_1}(x_1), \ldots, \mu_{\tilde{X}_n}(x_n)\}, \tag{2}$$

where $\tilde{\mathbf{X}}$ represents a set of input fuzzy numbers $\{\tilde{X}_1, \ldots, \tilde{X}_n\}$, $\mathbf{x}$ is a set of inputs variables $\{x_1, \ldots, x_n\}$. $\mathbf{R^n}$ is a set $\{\Re_1, \ldots, \Re_n\}$ representing the universal sets of real numbers, and $p(\cdot)$ is a function mapping inputs $\mathbf{x}$ to a output variable z. According to the extension principle, the interval of α-cut level set of fuzzy number $\tilde{p}(\tilde{\mathbf{x}})$ is given by:

$$\tilde{p}_\alpha(\tilde{\mathbf{x}}) = [\min p(\mathbf{x}; \mu_{\tilde{\mathbf{x}}}(\mathbf{x}) \geq \alpha), \max p(\mathbf{x}; \mu_{\tilde{\mathbf{x}}}(\mathbf{x}) \geq \alpha)] = [\tilde{p}_\alpha^L, \tilde{p}_\alpha^U]. \tag{3}$$

Thus, the lower and upper bounds of $\tilde{p}(\tilde{\mathbf{x}})$ at α-cut level could be obtained by a pair of parametric programming as follows:

$$\begin{array}{lll} \tilde{p}_\alpha^L : & \min & p(x_1, \ldots, x_n) \\ & s.t. & \tilde{x}_{1\alpha}^L \leq x_1 \leq \tilde{x}_{1\alpha}^U \\ & & \vdots \\ & & \tilde{x}_{n\alpha}^L \leq x_n \leq \tilde{x}_{n\alpha}^U \end{array}, \tag{4}$$

$$\begin{array}{lll} \tilde{p}_\alpha^U : & \max & p(x_1, \ldots, x_n) \\ & s.t. & \tilde{x}_{1\alpha}^L \leq x_1 \leq \tilde{x}_{1\alpha}^U \\ & & \vdots \\ & & \tilde{x}_{n\alpha}^L \leq x_n \leq \tilde{x}_{n\alpha}^U \end{array}. \tag{5}$$

This parametric programming problem can be realized by computer program, for which a couple of extreme values subjected to different intervals of input variables $\mathbf{x}$ at α-cut level can be easily found.

3 FMSE FMSS AND PERFORMANCE ASSESSMENT

3.1 Definition of FMSE and FMSS

As defined by Ding and Lisnianski [22] and Liu *et al.* [23], FMSE is the MSE in which the element state performance rate, the associated state probabilities or the transition intensities between any pair of states are treated as fuzzy values. In such case, any element j having k_j different states is characterized by fuzzy performance rates $\tilde{\mathbf{g}}_j = \{\tilde{g}_{j,1}, \ldots, \tilde{g}_{j,k_j}\}$ and the associated state probabilities are represented by fuzzy values $\tilde{\mathbf{p}}_j(t) = \{\tilde{p}_{j,1}(t), \ldots, \tilde{p}_{j,k_j}(t)\}$. The fuzzy UGF (FUGF) proposed by Ding and Lisnianski [22] can be applied to describe the behavior of FMSE in a polynomial form as:

$$\begin{aligned}\tilde{u}_j(z,t) &= \sum_{i_j=1}^{k_j} \tilde{p}_{j,i_j}(t) \cdot z^{\tilde{g}_{j,i_j}} \\ &= \tilde{p}_{j,1}(t) \cdot z^{\tilde{g}_{j,1}} + \tilde{p}_{j,2}(t) \cdot z^{\tilde{g}_{j,2}} + \cdots + \tilde{p}_{j,k_j}(t) \cdot z^{\tilde{g}_{j,k_j}}. \qquad (6)\end{aligned}$$

Since an MSS is consisting of more than one FMSE, this kind of MSS is called FMSS for its state probability and performance rate inherit the fuzzy property from FMSE. Once the dynamic fuzzy state probability of each individual FMSE is available, the dynamic behavior of the FMSS can be expressed through some combination rules based on the system structure function and the property of its performance rate under the fuzzy context.

3.2 Fuzzy Markov model and reward model

Based on the definition of FMSE, the state-space diagram of a non-repairable FMSE is shown in Fig. 2, where state k is the best state with highest performance rate while state 1 is worst state with zero performance rate and often considered as failure state. The transition intensities between states i and j are presented by the fuzzy values $\tilde{\lambda}_{i,j}$ and the associated performance rate in each state i is viewed as the fuzzy value $\tilde{g}_i$.

With the assumption of fuzzy transition intensities, the state probability of elements at time t must also be a fuzzy value denoted as $\tilde{p}_i(t)$. In order to obtain the fuzzy dynamic probability $\tilde{p}_i(t)$ of FMSE, the fuzzy Markov model is introduced as follows [23].

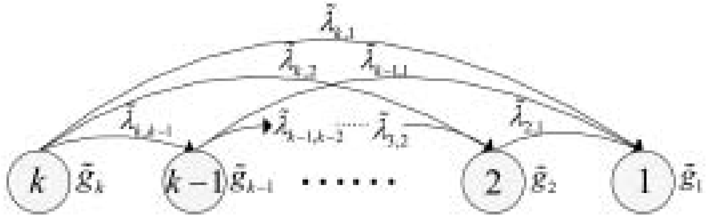

FIGURE 2
The state-space diagram of non-repairable FMSE.

The matrix of FMSE fuzzy transition intensities is given by:

$$\tilde{\boldsymbol{\lambda}} = |\tilde{\lambda}_{i,j}| = \begin{array}{c|ccc} \text{state} & 1 & \cdots & k \\ \hline 1 & \tilde{\lambda}_{1,1} & \ldots & \tilde{\lambda}_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ k & \tilde{\lambda}_{k,1} & \cdots & \tilde{\lambda}_{k,k} \end{array}, \tag{7}$$

where $\tilde{\lambda}_{i,i} = -\sum_{j=1, j\neq i}^{k} \tilde{\lambda}_{i,j}$. For non-repairable FMSE $\tilde{\lambda}_{i,j} = 0$ $(j > i)$. Then, the Kolmogorov's equation with fuzzy transition intensities (fuzzy Kolmogorov's equations) takes the form [42]:

$$\begin{cases} \dfrac{d\tilde{p}_k(t)}{dt} = -\tilde{p}_k(t) \displaystyle\sum_{j=1}^{k-1} \tilde{\lambda}_{k,j} \\ \dfrac{d\tilde{p}_i(t)}{dt} = \displaystyle\sum_{j=i+1}^{k} \tilde{\lambda}_{j,i}\tilde{p}_j(t) - \tilde{p}_i(t) \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j}, \quad 1 < i < k,\ t \geq 0 \\ \dfrac{d\tilde{p}_1(t)}{dt} = \displaystyle\sum_{j=2}^{k} \tilde{\lambda}_{j,1}\tilde{p}_j(t) \end{cases}, \tag{8}$$

with initial conditions: $\tilde{p}_k(0) = 1$, $\tilde{p}_i(0) = 0 (i \neq k)$.

To evaluate the performance reward produced by the system operation, the fuzzy Markov reward model is proposed to handle the case with fuzzy performance rates at each element state. Because the performance reward associated with element remaining at each state per unit time should equal to the unit performance reward multiplying the performance rate, therefore, it is also a fuzzy value denoting as $\tilde{r}_{i,i}$ if the element is staying in state i. Sometime, the state transition involves reward or loss, e.g. the expenditure of repair activities, thus, the fuzzy transition reward due to the element state transition from state i to state j is denoted as $\tilde{r}_{i,j}$. The fuzzy reward matrix takes the form as:

$$\tilde{\mathbf{r}} = |\tilde{r}_{i,j}| = \begin{array}{c|ccc} \text{state} & 1 & \cdots & k \\ \hline 1 & \tilde{r}_{1,1} & \ldots & \tilde{r}_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ k & \tilde{r}_{k,1} & \cdots & \tilde{r}_{k,k} \end{array}. \tag{9}$$

Let $\tilde{V}_i(t)$ be the total expected fuzzy reward accumulated up to time t, given the initial element state i at time instant $t = 0$. In order to obtain the total expected fuzzy reward, one must sum up all of the performance rate reward with associated state distribution. To this end, we first examine the reward variation for a small time interval Δt, and it can be written as:

$$\tilde{V}_i(t+\Delta t) = \left(1 - \sum_{\substack{j=1 \\ j\neq i}}^{k} \tilde{\lambda}_{i,j}\Delta t\right)(\tilde{r}_{i,i}\Delta t + \tilde{V}_i(t)) + \sum_{\substack{j=1 \\ j\neq i}}^{k} \tilde{\lambda}_{i,j}\Delta t(\tilde{r}_{i,j} + \tilde{V}_j(t)), \quad i = 1, \dots, k, \tag{10}$$

which can be transformed into the following form as:

$$\frac{\tilde{V}_i(t+\Delta t) - \tilde{V}_i(t)}{\Delta t} = \tilde{r}_{i,i} + \sum_{\substack{j=1 \\ j\neq i}}^{k} \tilde{\lambda}_{i,j}\tilde{r}_{i,j} + \sum_{j=1}^{k} \tilde{\lambda}_{i,j}\tilde{V}_j(t) + \tilde{\lambda}_{i,i}\tilde{r}_{i,i}\Delta t, \quad i = 1, \dots, k. \tag{11}$$

Therefore, the first order derivative of $\tilde{V}_i(t)$, with respect to t, is written as:

$$\frac{d\tilde{V}_i(t)}{dt} = \lim_{\Delta t\to 0} \frac{\tilde{V}_i(t+\Delta t) - \tilde{V}_i(t)}{\Delta t} = \tilde{r}_{i,i} + \sum_{\substack{j=1 \\ j\neq i}}^{k} \tilde{\lambda}_{i,j}\tilde{r}_{i,j} + \sum_{j=1}^{k} \tilde{\lambda}_{i,j}\tilde{V}_j(t), \quad i = 1, \dots, k. \tag{12}$$

If we assume the reward from the transition between any two states in the non-repairable FMSE are equal to 0 ($\tilde{r}_{i,j} = 0$ *for* $i \neq j$). Thus, one can write the fuzzy differential equations corresponding to different initial states as follows:

$$\begin{cases} \dfrac{d\tilde{V}_k(t)}{dt} = \tilde{r}_{k,k} + \displaystyle\sum_{j=1}^{k} \tilde{\lambda}_{k,j}\tilde{V}_j(t) \\ \dfrac{d\tilde{V}_i(t)}{dt} = \tilde{r}_{i,i} + \displaystyle\sum_{j=1}^{i} \tilde{\lambda}_{i,j}\tilde{V}_j(t), \quad 1 < i < k, \\ \dfrac{d\tilde{V}_1(t)}{dt} = \tilde{r}_{1,1} + \tilde{\lambda}_{1,1}\tilde{V}_1(t) \end{cases} \tag{13}$$

with initial condition $\tilde{V}_i(0) = 0$, and the equations for the long-run (stationary) fuzzy reward are written as:

$$\begin{cases} 0 = \tilde{r}_{k,k} + \sum_{j=1}^{k} \tilde{\lambda}_{k,j} \tilde{V}_j \\ 0 = \tilde{r}_{i,i} + \sum_{j=1}^{i} \tilde{\lambda}_{i,j} \tilde{V}_j, \quad 1 < i < k, \\ 0 = \tilde{r}_{1,1} + \tilde{\lambda}_{1,1} \tilde{V}_1 \end{cases} \tag{14}$$

where all of the time-derivative terms $\frac{d\tilde{V}_i(t)}{t}$ are equal to zero representing stationary reward.

Solving the fuzzy differential in Eqs. (8) and (13) can be resorted to the Laplace-Stieltjes transform and Laplace-Stieltjes inverse transform [23][42]. The instantaneous state probability and performance reward are expressed in terms of fuzzy parameters, and their membership functions can be derived through the proposed parametric programming straightforwardly. In the same manner, the long-run reward given by Eqs. (14) can also be computed via solving the linear equations and the parametric programming proposed in Section 2.2.

4 REPLACEMENT POLICY FOR FMSE

It always troubles the decision maker that when MSE degrades to the lower performance rate states, whether it is cost efficient to replace the element. In fact, it is necessary to find a proper threshold state, and based on this threshold state, the element will be replaced right away once it falls into the threshold state or below. This is realistic in many situations where the condition of the system/element is monitored all the time, and whenever the condition becomes lower than threshold value, the system/element needs replacing to ensure the cost efficiency or system safety.

The proposed replacement policy in the present work is to select an optimal threshold state θ^* with the aim to maximize the expected average profit per unit time, and the MSE replacement is carried out immediately once the element falls into the state θ^*or below. It assumes that the replacement restores the element to "as good as new" condition. A simple policy where $\theta = 2$ is demonstrated in Fig. 3. Under this policy, the performance reward for state 2 and 1 is zero for the element is replaced as soon as it falls into these states, and the associated cost of replacement is denoted by fuzzy value $\tilde{M}$.

The MTBR and total expected performance reward of the FMSE in each replacement cycle are fuzzy values, because they inherit the fuzzy character

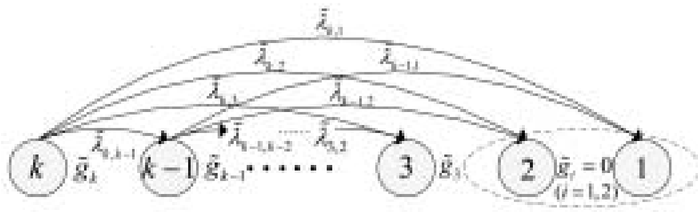

FIGURE 3
The state-space diagram for the case with replacement policy $\theta = 2$.

from the fuzzy transition intensities and performance rates. In order to compute the fuzzy MTBR and the fuzzy total expected performance reward in per replacement cycle, the proposed fuzzy Markov reward model is employed.

According the replacement policy, the states $\theta, \theta - 1, \theta - 2, \ldots, 0$ are absorbing states simply denoted by a single state θ. The Markov reward matrix $\mathbf{r}_T$ to obtain the MTBR is given by:

$$\mathbf{r}_T = |r_{i,j}| = \begin{array}{c c} & \begin{array}{cccc} \theta & \theta+1 & \cdots & k \end{array} \\ \begin{array}{c} \theta \\ \theta+1 \\ \vdots \\ k \end{array} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \end{array}, \tag{15}$$

where $r_{i,j} = 0 (i \neq j)$ representing the reward associated with the transitions is zeroed, and the reward associated with remaining in absorbing state should be zeroed also, i.e. $r_{\theta,\theta} = 0$. $r_{i,i}(i \neq \theta)$ is assigned equal to 1 representing the reward from sojourning in state i per unit time. Thus, the fuzzy MTBR is solved via following long-run stationary algebraic equations mentioned in Eq. (14):

$$\begin{cases} 0 = r_{k,k} + \tilde{T}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{k,j} + \tilde{\lambda}_{k,\theta+1} \tilde{T}_{\theta+1} + \cdots - \tilde{T}_k \sum_{j=1}^{k-1} \tilde{\lambda}_{k,j} \\ 0 = r_{i,i} + \tilde{T}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{i,j} + \tilde{\lambda}_{i,\theta+1} \tilde{T}_{\theta+1} + \cdots - \tilde{T}_i \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j}, \qquad \theta + 1 < i < k, \\ 0 = r_{\theta+1,\theta+1} + \tilde{T}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{\theta+1,j} - \tilde{T}_{\theta+1} \sum_{j=1}^{\theta-1} \tilde{\lambda}_{\theta+1,j} \end{cases} \tag{16}$$

where $\tilde{T}_i$ denote the long-run reward with initial state i. When the FMSE starts from the absorbing state θ, it will never leave this state and no reward can be accumulated, therefore $\tilde{T}_\theta = 0$.

$\tilde{T}_i$ can be solved via Eq. (16), and MTBR of FMSE is interpreted as the total expected fuzzy reward accumulated during the unrestricted time interval $[0, \infty]$. Thus, $\tilde{T}_i$ is the fuzzy MTBR where the FMSE starts operating from its state i.

In the same fashion, the total expected performance reward of the FMSE can be calculated through the fuzzy reward matrix representing its performance reward per unit time when FMSE is sojourning in each state.

The fuzzy reward matrix for the performance reward is written as:

$$\tilde{\mathbf{r}}_R = |\tilde{r}_{i,j}| = \begin{array}{c} \text{state} \\ \theta \\ \theta+1 \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{cccc} \theta & \theta+1 & \cdots & k \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \tilde{r}_{\theta+1,\theta+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{r}_{k,k} \end{pmatrix} \end{array}, \tag{17}$$

where $\tilde{r}_{i,i}$ is fuzzy number which is different from the previous reward matrix for MTBR, because the performance rate is of fuzzy value.

Similarly, the equations for the long-run expected fuzzy performance reward are formulated as:

$$\begin{cases} 0 = \tilde{r}_{k,k} + \tilde{R}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{k,j} + \tilde{\lambda}_{k,\theta+1}\tilde{R}_{\theta+1} + \cdots - \tilde{R}_k \sum_{j=1}^{k-1} \tilde{\lambda}_{k,j} \\ 0 = \tilde{r}_{i,i} + \tilde{R}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{i,j} + \tilde{\lambda}_{i,\theta+1}\tilde{R}_{\theta+1} + \cdots - \tilde{R}_i \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j}, \qquad \theta+1 < i < k \\ 0 = \tilde{r}_{\theta+1,\theta+1} + \tilde{R}_\theta \sum_{j=1}^{\theta} \tilde{\lambda}_{\theta+1,j} - \tilde{R}_{\theta+1} \sum_{j=1}^{\theta-1} \tilde{\lambda}_{\theta+1,j} \end{cases} \tag{18}$$

where $\tilde{R}_i$ represents the long-run fuzzy performance reward if the FMSE initially starts operating from state i, and one also has $\tilde{R}_\theta = 0$.

Because the replacement cycle forms a renewal process, the expected fuzzy average profit per unit time can be calculated as follows:

$$\begin{aligned} \tilde{C}_i &= \frac{\textit{expected fuzzy reward in a cycle } - \textit{fuzzy replacement cost}}{\textit{fuzzy MTBR}} \\ &= \left(\frac{\tilde{R}_k - \tilde{M}}{\tilde{T}_k}\right)_{\theta=i}, \end{aligned} \tag{19}$$

where fuzzy value $\tilde{C}_i$ denote the expected fuzzy average profit per unit time in the case where the replacement threshold state θ is equal to i, and the

replacement time is ignorable. The boundaries of $\tilde{C}_i$ at arbitrary α-cut level set can be calculated via a pair of parametric programming as follows:
Lower boundary:

$$\begin{array}{lll} \tilde{C}^L_{i\alpha}: & \min & h_i(\boldsymbol{\lambda}, \mathbf{r}_R, M) \\ & s.t. & \tilde{\lambda}^L_{(i,j)\alpha} \leq \lambda_{i,j} \leq \tilde{\lambda}^U_{(i,j)\alpha} \\ & & \tilde{r}^L_{(i,i)\alpha} \leq r_{i,i} \leq \tilde{r}^U_{(i,i)\alpha} \\ & & \tilde{M}^L_\alpha \leq M \leq \tilde{M}^U_\alpha \end{array}, \tag{20}$$

Upper boundary:

$$\begin{array}{lll} \tilde{C}^U_{i\alpha}: & \max & h_i(\boldsymbol{\lambda}, \mathbf{r}_R, M) \\ & s.t. & \tilde{\lambda}^L_{(i,j)\alpha} \leq \lambda_{i,j} \leq \tilde{\lambda}^U_{(i,j)\alpha} \\ & & \tilde{r}^L_{(i,i)\alpha} \leq r_{i,i} \leq \tilde{r}^U_{(i,i)\alpha} \\ & & \tilde{M}^L_\alpha \leq M \leq \tilde{M}^U_\alpha \end{array}, \tag{21}$$

where $h_i(\boldsymbol{\lambda}, \mathbf{r}_R, M)$ denotes the expected average profit per unit time with threshold state equal to i, which is a function in terms of state transition intensity, performance reward and replacement cost.

Under the different replacement policy, the fuzzy value $\tilde{C}_i(1 \leq i < k)$ is also different. In the next section, we will briefly introduce three fuzzy decision making methods to help determine the optimal solution under the fuzzy uncertainty context.

Solving the non-linear parametric programming in Eqs. (20) and (21), is straightforward, and standard optimization routines, such as the steepest descent method and the Newton-Raphson method, can be directly used. The command “fmincon” in the Matlab optimization toolbox is adopted to solve the constrained nonlinear problems.

5 FUZZY DECISION MAKING

There are three fuzzy decision making methods introduced separately in this section to select the best solution from all of the candidates under the fuzzy context.

5.1 Method I: fuzzy max order

This method is an extension of the interval comparison. According to the interval ordering method, the definition of a partial order for interval $[a, b]$ and $[c, d]$ is given as follows:

Definition 1.

$$[a, b] \le [c, d], \text{ if } a \le c \text{ and } b \le d;$$
$$[a, b] < [c, d], \text{ if } [a, b] \le [c, d] \text{ and } [a, b] \ne [c, d].$$

By extending the partial order criterion of interval value to fuzzy number, we suppose that $[\tilde{A}^L_\alpha, \tilde{A}^U_\alpha]$ and $[\tilde{B}^L_\alpha, \tilde{B}^U_\alpha]$ are the interval of α-cut level set of the two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively. The partial order criterion for the fuzzy number is stated in Definition 2.

Definition 2.

$$\tilde{A} \le \tilde{B}, \text{ if } \left[\tilde{A}^L_\alpha, \tilde{A}^U_\alpha\right] \le \left[\tilde{B}^L_\alpha, \tilde{B}^U_\alpha\right] \text{ for all } \alpha \in [0, 1];$$
$$\tilde{A} < \tilde{B}, \text{ if } \tilde{A} \le \tilde{B} \text{ and } \tilde{A} \ne \tilde{B}.$$

Nevertheless, there are some special scenarios, for example:

(1) $\tilde{A}^L_\alpha \le \tilde{B}^L_\alpha$, but $\tilde{B}^U_\alpha \le \tilde{A}^U_\alpha$ for all $\alpha \in [0, 1]$;

(2) $\tilde{A}^L_\alpha \ge \tilde{B}^L_\alpha(\tilde{B}^U_\alpha \le \tilde{A}^U_\alpha)$ and $\tilde{A}^L_\alpha \le \tilde{B}^L_\alpha(\tilde{B}^U_\alpha \ge \tilde{A}^U_\alpha)$ for some of $\alpha \in [0, 1]$.

In these cases, one cannot decide which one is better than the other through the partial order method.

5.2 Method II: fuzzy number distance

This method was firstly introduced by Murakami *et al.* [43], and it was updated by Cheng [44] later on. The basic idea of this method is to rank the fuzzy numbers according to the distance from the centroid point of fuzzy alternative $\tilde{A}$ to the original point.

Let $\bar{x}$ denote the horizontal axis value corresponding to the centroid of $\tilde{A}$, and $\bar{y}$ denotes the vertical axis value. For a simple fuzzy number $\tilde{A} = [a, b, c, d]$ with its membership function $\mu_{\tilde{A}}$ given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x < b; \\ 1, & b \le x \le c; \\ \dfrac{x-d}{c-d}, & \text{c} < x \le d; \\ 0, & \text{otherwise} \end{cases}, \tag{22}$$

and let $\mu^L_{\tilde{A}}(x) = \mu_{\tilde{A}}(x)$ $(x \in [a, b])$ and $\mu^R_{\tilde{A}}(x) = \mu_{\tilde{A}}(x)(x \in [c, d])$. The inverse function of $\mu^L_{\tilde{A}}(x)$ and $\mu^R_{\tilde{A}}(x)$ are written as $g^L_{\tilde{A}} = a + (b-a)\mu_{\tilde{A}}$ and $g^R_{\tilde{A}} = d + (c-d)\mu_{\tilde{A}}$. Thus the coordinate of the centroid point $(\bar{x}, \bar{y})$ of $\tilde{A}$

can be computed as:

$$\bar{x} = \frac{\int_a^b x\mu_{\tilde{A}}^L dx + \int_b^c x dx + \int_c^d x\mu_{\tilde{A}}^R dx}{\int_a^b \mu_{\tilde{A}}^L dx + \int_b^c dx + \int_c^d \mu_{\tilde{A}}^R dx}, \tag{23}$$

$$\bar{y} = \frac{\int_0^1 \left(\mu_{\tilde{A}} g_{\tilde{A}}^L\right) d\mu_{\tilde{A}} + \int_0^1 \left(\mu_{\tilde{A}} g_{\tilde{A}}^R\right) d\mu_{\tilde{A}}}{\int_0^1 g_{\tilde{A}}^L d\mu_{\tilde{A}} + \int_0^1 g_{\tilde{A}}^R d\mu_{\tilde{A}}}. \tag{24}$$

The distance index between the centroid point $(\bar{x}, \bar{y})$ and original point is defined as:

$$R(\tilde{A}) = \sqrt{(\bar{x})^2 + (\bar{y})^2}, \tag{25}$$

and the ranking of the fuzzy alternatives can be ordered according the distance index as below.

Definition 3.

$$\text{If } R(\tilde{A}) < R(\tilde{B}), \text{ then } \tilde{A} < \tilde{B};$$

$$\text{If } R(\tilde{A}) = R(\tilde{B}), \text{ then } \tilde{A} = \tilde{B};$$

$$\text{If } R(\tilde{A}) > R(\tilde{B}), \text{ then } \tilde{A} > \tilde{B}.$$

Based on this definition, order ranking of multiply fuzzy values becomes straightforward, and one of merits of this method is that it is able to deal with several fuzzy alternatives simultaneously.

5.3 Method III: Liou and Wang's method [45]

The advantages of this method are that it can not only handle several alternatives simultaneous as fuzzy number distance method does, but also it combines with the optimistic attitude of the decision maker. Therefore, an optimism index β is adopted in this method.

Based on this method, if the interval of α-cut level set of fuzzy value $\tilde{A}$ is denoted by $\tilde{A}_\alpha = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^U]$, then the left integral value of $\tilde{A}$ is defined as:

$$I_L(\tilde{A}) = \int_0^1 \tilde{A}_\alpha^L d\alpha, \tag{26}$$

while the right integral value is defined as:

$$I_R(\tilde{A}) = \int_0^1 \tilde{A}_\alpha^U d\alpha. \tag{27}$$

Thus, the total integral value of the fuzzy number $\tilde{A}$ is defined as:

$$I^\beta(\tilde{A}) = \beta I_R(\tilde{A}) + (1 - \beta) I_L(\tilde{A}), \tag{28}$$

where $\beta (\beta \in [0, 1])$ is an optimism parameter determined by decision maker, for example, $\beta = 0.5$ represents a neutral attitude.

According to the total integral value $I^{\beta}(\tilde{A})$, the fuzzy value can be ordered as following Definition 4.

Definition 4.

$$\begin{aligned}
&\text{If } I^{\beta}(\tilde{A}) < I^{\beta}(\tilde{B}),\ \text{then } A < B;\\
&\text{If } I^{\beta}(\tilde{A}) = I^{\beta}(\tilde{B}),\ \text{then } A = B;\\
&\text{If } I^{\beta}(\tilde{A}) > I^{\beta}(\tilde{B}),\ \text{then } A > B.
\end{aligned}$$

Based on these ranking methods, the optimal replacement policy can be clearly found by ranking the expected fuzzy average profit per unit time under each possible policy.

6 ILLUSTRATIVE CASE

The illustrative case is a non-repairable power generator. Generally, the generator is divided into 5 performance levels named perfect (state 5), partial perfect (state 4), medium (state 3), partial failed (state 2) and completed failed (state 1) by domain engineers according to ranges of its performance rates. The traditional statistical estimation method becomes problematic due to the lack of data, especially for new product. Thus, a crisp value is not suitable to assess the transition intensities. In addition, the performance rate may fluctuate around its expected value in each state, for example the capacity of solar generator is affected by power of sun and the power of water-turbine generator fluctuates with flow. It is reasonable to use fuzzy number to value transition intensities and performance rates according to expert's knowledge [34], and thus the generator can be regarded as an FMSE with five states as shown in Fig. 4.

The possible replacement policies ($\theta = 1, 2, 3, 4$) are circled by dash line in Fig. 4. That means the FMSE may replace immediately whenever it enters the circled states. The fuzzy transition intensity and fuzzy performance reward per unit time are treated as triangle fuzzy numbers and tabulated in Tables 1 and 2, respectively. The initial state is state 5, and the fuzzy replacement cost is $\tilde{M} = (50, 60, 70) \times 10^2\$$.

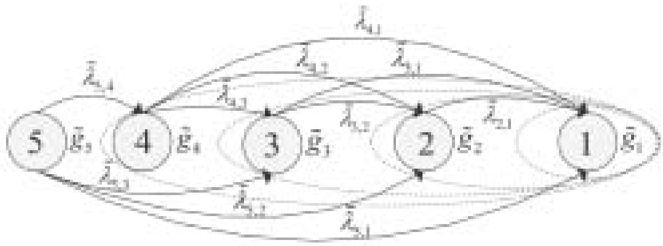

FIGURE 4
The state-space diagram of FMSE under different replacement policy.

State	1	2	3	4	5
1	/	0	0	0	0
2	(0.6,0.7,0.8)	/	0	0	0
3	(0.2,0.3,0.35)	(0.5,0.6,0.7)	/	0	0
4	(0.1,0.2,0.25)	(0.25,0.3,0.35)	(0.7,0.9,1.0)	/	0
5	(0.15,0.2,0.25)	(0.22,0.25,0.3)	(0.3,0.35,0.4)	(1.0,1.1,1.2)	/

TABLE 1
The fuzzy transition intensities between any pair of state (year^{-1})

State	1	2	3	4	5
Performance reward	0	(30,45,60)	(65,75,85)	(90,100,120)	(130,150,170)

TABLE 2
The fuzzy performance reward per unit time in each state ($\times 10^2$\$)

Under the policy $\theta = 2$, it means the generator is replaced once it transits into state 2 or below. In accordance to Eq. (16), the equations for the MTBR are written as:

$$\begin{cases} 0 = 1 + \sum_{j=3}^{4} \tilde{\lambda}_{5,j}\tilde{T}_j + (\tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1})\tilde{T}_2 - \tilde{T}_5 \sum_{j=1}^{4} \tilde{\lambda}_{5,j} \\ 0 = 1 + \sum_{j=3}^{i-1} \tilde{\lambda}_{i,j}\tilde{T}_j + \tilde{T}_2 \sum_{j=1}^{2} \tilde{\lambda}_{i,j} - \tilde{T}_i \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j}, \qquad 2 < i < 5. \\ 0 = \tilde{T}_2 \end{cases}$$

By solving the linear equations, the stationary results are:

$$\tilde{T}_3 = \frac{1}{\tilde{\lambda}_{3,1} + \tilde{\lambda}_{3,2}},$$

$$\tilde{T}_4 = \frac{\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1} + \tilde{\lambda}_{4,3}}{\left(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\right)\left(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\right)},$$

$$\begin{aligned} \tilde{T}_5 = \Big(\Big(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\Big)\Big(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\Big) + \tilde{\lambda}_{5,4}\Big(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\Big) \\ + \tilde{\lambda}_{5,4}\tilde{\lambda}_{4,3} + \tilde{\lambda}_{5,3}\Big(\tilde{\lambda}_{5,4} + \tilde{\lambda}_{5,3} + \tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1}\Big) \\ \Big/\Big(\Big(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\Big)\Big(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\Big)\Big(\tilde{\lambda}_{5,4} + \tilde{\lambda}_{5,3} + \tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1}\Big)\Big). \end{aligned}$$

Thus, the fuzzy MTBR is equivalent to $\tilde{T}_5$.

In the same manner, the equations for the stationary fuzzy performance reward are given by:

$$\begin{cases} 0 = \tilde{r}_{5,5} + \sum_{j=3}^{4} \tilde{\lambda}_{5,j} \tilde{R}_j + \left(\tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1}\right) \tilde{R}_2 - \tilde{R}_5 \sum_{j=1}^{4} \tilde{\lambda}_{5,j} \\ 0 = \tilde{r}_{i,i} + \sum_{j=3}^{i-1} \tilde{\lambda}_{i,j} \tilde{R}_j + \tilde{R}_2 \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j} - \tilde{R}_i \sum_{j=1}^{i-1} \tilde{\lambda}_{i,j}, & 2 < i < 5, \\ 0 = \tilde{R}_2 \end{cases}$$

and the total fuzzy performance reward $\tilde{R}_i$ is formulated as:

$$\tilde{R}_3 = \frac{\tilde{r}_{3,3}}{\tilde{\lambda}_{3,1} + \tilde{\lambda}_{3,2}},$$

$$\tilde{R}_4 = \frac{\tilde{r}_{4,4}(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}) + \tilde{r}_{3,3}\tilde{\lambda}_{4,3}}{\left(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\right)\left(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\right)},$$

$$\begin{aligned} \tilde{R}_5 = &\left(\tilde{r}_{5,5}\left(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\right)\left(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\right) + \tilde{r}_{4,4}\tilde{\lambda}_{5,4}\left(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\right)\right. \\ &\left. + \tilde{r}_{3,3}\tilde{\lambda}_{5,4}\tilde{\lambda}_{4,3} + \tilde{r}_{3,3}\tilde{\lambda}_{5,3}\left(\tilde{\lambda}_{5,4} + \tilde{\lambda}_{5,3} + \tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1}\right)\right) \\ &\bigg/ \left(\left(\tilde{\lambda}_{3,2} + \tilde{\lambda}_{3,1}\right)\left(\tilde{\lambda}_{4,3} + \tilde{\lambda}_{4,2} + \tilde{\lambda}_{4,1}\right)\left(\tilde{\lambda}_{5,4} + \tilde{\lambda}_{5,3} + \tilde{\lambda}_{5,2} + \tilde{\lambda}_{5,1}\right)\right). \end{aligned}$$

The $\tilde{R}_5$ is the total fuzzy performance reward when state 5 is initial state.

Refer to Eq. (19), the expected fuzzy average profit per unit time under the policy $\theta = 2$ is formulated as:

$$\tilde{C}_2 = \left(\frac{\tilde{R}_5 - \tilde{M}}{\tilde{T}_5}\right)_{\theta=2},$$

where the membership function at any α-cut level can be computed based on the couple of parametric programming given in Eqs. (20) and (21).

The other policies under different replacement threshold can be calculated in the same fashion, and the membership functions of the fuzzy MTBR and the total fuzzy performance reward obtained via parametrical programming are plotted in Figs. 5 and 6, respectively. The membership functions of the associated expected fuzzy average profit per unit time under each policy are shown in Fig. 7.

It is necessary to resort to aforementioned fuzzy decision methods to make correct judgment among the possible policies in order to maximize the expected fuzzy average profit per unit time.

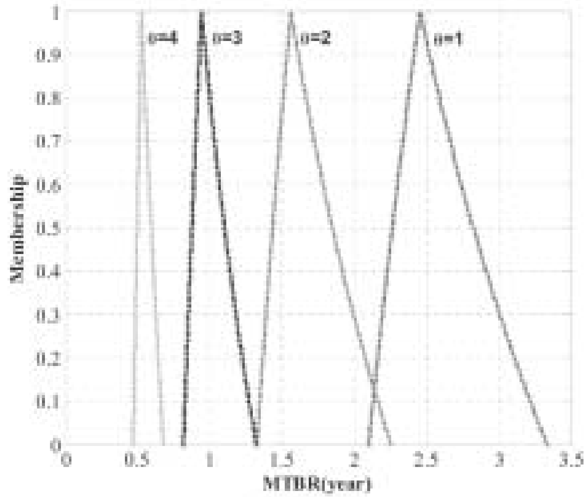

FIGURE 5
The fuzzy MTBR under different replacement policy.

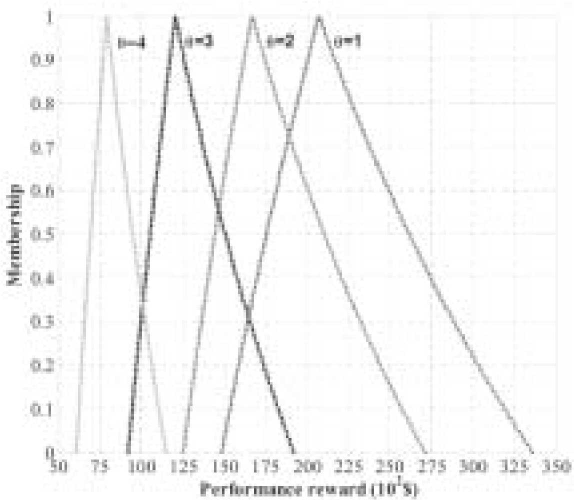

FIGURE 6
The total fuzzy performance reward under different replacement policy.

(1) **Method I:**

$\theta = 1$ vs. $\theta = 2$, $\tilde{C}^L_{1\alpha} < \tilde{C}^L_{2\alpha}$, $\tilde{C}^U_{1\alpha} < \tilde{C}^U_{2\alpha}$ apparently exist, thus $\tilde{C}_1 < \tilde{C}_2$;

$\theta = 2$ vs. $\theta = 3$, $\tilde{C}^L_{3\alpha} < \tilde{C}^L_{2\alpha}$ exists, but $\tilde{C}^U_{3\alpha} < \tilde{C}^U_{2\alpha}$ comes existence when $\alpha > 0.7$, thus one can not determine the better policy among $\tilde{C}_2$ and $\tilde{C}_3$;

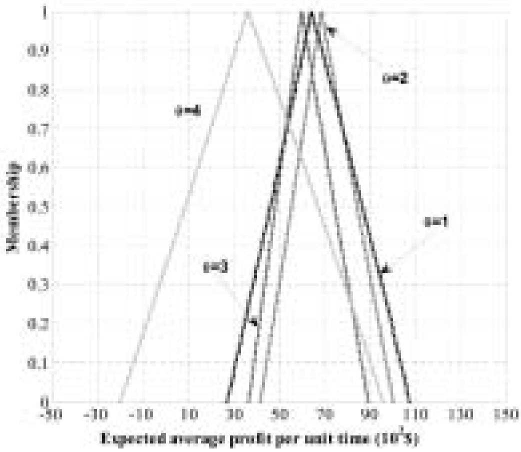

FIGURE 7
The expected fuzzy average profit per unit time under different replacement policy.

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$
$\tilde{C}_1$	/	$<$	–	–
$\tilde{C}_2$	$>$	/	–	$>$
$\tilde{C}_3$	–	–	/	$>$
$\tilde{C}_4$	–	$<$	$<$	/

TABLE 3
Fuzzy decisions comparison by method I

$\theta = 3$ vs. $\theta = 4$, there exists $\tilde{C}_4 < \tilde{C}_3$ for $\tilde{C}^L_{4\alpha} < \tilde{C}^L_{3\alpha}$, $\tilde{C}^U_{4\alpha} < \tilde{C}^U_{3\alpha}$ hold at any α-cut level. The comparison results are listed in Table 3, where "–" denotes cannot find out a better one between candidates.

This method can not make decision completely among all of the candidate policies, for one can only conclude that $\tilde{C}_1 < \tilde{C}_2$, $\tilde{C}_4 < \tilde{C}_3$, $\tilde{C}_4 < \tilde{C}_2$ but cannot determine the order between $\tilde{C}_3$ and $\tilde{C}_2$. The other two methods will be adopted to make further decision.

(2) **Method II:**

According to Eqs. (23), (24) and (25), the centroid point coordinates and distance are tabulated in Table 4. The rank of the fuzzy alternatives is $\tilde{C}_4 < \tilde{C}_1 < \tilde{C}_3 < \tilde{C}_2$, and the optimal replacement policy is $\theta^* = 2$.

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$
$\bar{x}_{\tilde{C}_i}$	60.8267	69.4005	65.6428	37.1175
$\bar{y}_{\tilde{C}_i}$	0.4958	0.4969	0.4960	0.4960
$R(\tilde{C}_i)$	60.8287	69.4023	65.6447	37.1208

TABLE 4
Fuzzy decisions comparison by method II

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$
$I_L(\tilde{C}_i)$	4867.1	5577.3	4625.2	821.77
$I_R(\tilde{C}_i)$	7524.2	8527.9	8741.9	6736
$I^{0.5}(\tilde{C}_i)$	6195.6	7052.6	6683.6	3778.9

TABLE 5
Fuzzy decisions comparison by method III

(3) **Method III:**

According to Eqs. (26), (27) and (28), the left, right and total integrals of alternative are tabulated in Table 5 with optimism parameter $\beta = 0.5$. The final rank of the fuzzy alternatives is $\tilde{C}_4 < \tilde{C}_1 < \tilde{C}_3 < \tilde{C}_2$ which is identical with the method II, and the optimal replacement state threshold is also state 2. Finally, it can conclude that the best replacement threshold is state 2 with the maximum expected fuzzy average profit per unit time under fuzzy uncertainty context.

7 CONCLUSION

In this paper, the definitions of FMSS and FMSE are first reviewed which extend the MSS model to the case where the transition rates and performance rates of MSE are imprecisely evaluated due to the lack of sufficient data. Fuzzy set theory is adopted to describe this kind of uncertainty. The fuzzy Markov model and fuzzy Markov reward model are proposed to assess the dynamic state probability, MTBR and stationary performance reward of the FMSE under the fuzzy uncertainty context. The parametric programming is also presented to obtain the membership functions of these indices of interest at different α-cut levels. To determine the optimal replacement policy of FMSE, three decision making methods under fuzzy uncertainty context are implemented to select the best policy among the candidates. Future work will be focused on extending this methodology to FMSS maintenance issue involving multiple FMSE.

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