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Perturbation finite element method of structural analysis under fuzzy environments

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Abstract

A preliminary analysis of the perturbation fuzzy finite element was provided by Yang et al. (Appl. Math. Mech. 20(7) (1999) 795). In this paper, we provide a detailed analysis of the perturbation fuzzy finite element method based on variational principle. Firstly, on the basis of the second-order perturbation principle of small parameter, the fuzzy functional of total potential energy and the definite perturbation expansions are proposed. Secondly, definite recursion equation of fuzzy variational principle is deduced and fuzzy finite element recursion functional is presented based on fuzzy variational principle. Thirdly, the proposed approach is compared with the conventional fuzzy finite element method. Finally, a numerical example is given to illustrate the method.

Keywords: Fuzzy finite element; Perturbation method; Variational principle; Structural analysis

1. Introduction

The finite element method is a very popular tool for both static and dynamic analysis of engineering systems. The ability to predict the behavior of a structure under static or dynamic loads is not only of great scientific value, it is also very useful from an economical point of view. A reliable finite element analysis could make prototype production and testing obsolete and therefore significantly reduce the associated design validation cost. Traditional finite element approaches require crisp or well-defined input parameters. For instance, given the geometry, material properties, load and boundary conditions as deterministic values, a crisp result can be calculated on an element by element basis. Unfortunately, it sometimes is very difficult to ensure the reliability of the result of a finite element analysis for realistic structures that are not precisely defined (Moens and Vandepitte, 2002). For instance, the geometric properties, as well as the effects of service conditions on the physical, mechanical and electromechanical properties of smart structures, are vaguely understood and therefore cannot be precisely defined. If the uncertainty is due to vaguely defined system characteristics, imprecision of data, insufficient information and/or subjectivity of opinion or judgement, then fuzzy set-based treatment is appropriate.

The fuzzy finite element methodology is a new area of finite element analysis that began in the early 1990s. Valliappan and Pham (1993, 1995) applied fuzzy set-based methods to geotechnical finite element analysis of soils and foundations, elasto-plastic finite element analysis. Pham et al. (1995) applied fuzzy set-based methods to modelling of damping in dynamic finite element analysis. Chao and Ayyub (1995, 1996) applied fuzzy set-based methods to static analysis of structures. Rao and Sawyer (1995) proposed a fuzzy finite element method for static analysis of engineering systems using an optimization-based scheme for the numerical

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solution of systems of fuzzy parameters, geometry and applied loads was considered and implemented in their approach. Chen and Rao (1997) developed a fuzzy finite element method for vibration analysis of imprecisely defined systems by using a search-based algorithm. Their approach enhances the computational efficiency in fuzzy operations for identifying the system dynamic responses. Muhanna and Mullen (1995, 1999), Mullen and Muhanna (1999) developed a finite element analysis procedure utilizing the concept of fuzzy sets through interval calculations and also computed the response of different structural systems due to geometric and loading uncertainties. Noor et al. applied fuzzy finite element method to analysis of space structures (Wasfy and Noor, 1998a), flexible multibody systems (Wasfy and Noor, 1998b), welding residual stress fields (Abdel-Tawab and Noor, 1999), composite structures (Noor et al., 2000) and tethered satellite system (Leamy et al., 2001). Lallemand et al. (1999) gave a Neumann expansion for fuzzy finite element analysis. Akpan et al. (2001) proposed a fuzzy finite element approach for modelling smart structures with vague or imprecise uncertainties. Hanss and Willner (2000) proposed a fuzzy arithmetical approach to the solution of finite element problems with uncertain parameters.

Perturbation method comes from celestial mechanics. The effect of astronomic gravitation that comes from outside the main star is called perturbation. In the 1980s, Lindstedt and Poincare had studied problems of celestial mechanics with perturbation method. Perturbation method, a general method to deal with non-linear problem, has been developed quickly since 1950s. The content of perturbation is universal and it has been widely applied on oscillation principle, hydromechanics, modern physics, autocontrol, marine engineering, biology, chemistry and economics, demography and so on (Wang, 1994).

The property of perturbation method is that it translates non-linear equation into multilevel linear equation and solves it, which can overcome the difficulty of directly solving non-linear equation. In structural analysis, someone has successfully solved a large deflection problem with perturbation method in 1947 (Xie et al., 1984). But the method needs to find an analytical solution of a linear equation at first as the basis of the solving process. If the structural geometry shape and boundary conditions are complicated, it is difficult to find an analytical solution. The advantage of finite element method is that it can solve the structural problems with more complicated geometry shape and boundary conditions. But when solving non-linear problem, the method costs lots of time. So people developed perturbation finite element method that combines finite element with perturbation approach. Thompson and Walker (1968) developed perturbation analysis concept of disperse structure and applied it on solving a large deflection problem. Yokoo and Nakamura (1976) used the incremental perturbation method to solve a large deformation problem of elastic–plastic structure in 1976. Xie et al. (1983) introduced the perturbation process into variational principle and finite element method was used to solve the perturbation equation. Perturbation finite element method is not only efficient in certain structural analysis but also in uncertain structural analysis. Stochastic finite element method that combines perturbation method with finite element method has been developed, and the method has shown great advantage and wide applications in solving stochastic problems (Chen and Liu, 1993). On the basis of the second order perturbation method of small parameter, Yang (1998) and Yang and Li (1999) provided the fuzzy functional of overall potential energy and the second order perturbation expansions, and deduced definite recursion equation of fuzzy variational principle. Fuzzy finite element recursion equation was established based on fuzzy variational principle.

An analysis method of structure, perturbation fuzzy finite element method based on variational principle, is introduced in this paper. On the basis of the second order perturbation principle of small parameter, the fuzzy functional of total potential energy and the definite perturbation expansions are proposed. Then definite recursion equation of fuzzy variational principle is deduced and fuzzy finite element recursion functional is presented based on fuzzy variational principle. The process that element stiffness matrix is assembled into total stiffness matrix is taken into account, and the fuzzy finite element method in papers (Yang, 1998; Yang and Li, 1999) is improved. Compared with the conventional fuzzy finite element method (Huang and Li, 2003; Li et al., 2003), this method can avoid calculation error caused by the expansion of interval number's operation (Moore, 1966). The method is illustrated through a numerical simulation example.

2. Perturbation fuzzy finite element method based on fuzzy variational principle

When finding the displacement of structure, the functional of overall potential energy of elastic object is

$$\Pi = \iiint_{\Omega} \left\{ \frac{1}{2} D_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - f_i u_i \right\} \mathrm{d}\Omega - \iint_{sp} u_i \bar{p}_i \, \mathrm{d}S, \tag{1}$$

where D_{ijkl} is the tensor of elastic moduli, ε_{ij} the strain tensor, f_i the body force per unit volume, u_i the displacement and $\bar{p}_i$ the boundary forces.

The functional of overall potential energy is a fuzzy functional with fuzzy parameters when the engineering system has fuzzy factors; the fuzzy functional is

$$\tilde{\Pi} = \iiint_{\Omega} \left\{ \frac{1}{2} \tilde{D}_{ijkl} \tilde{\varepsilon}_{ij} \tilde{\varepsilon}_{kl} - \tilde{f}_i \tilde{u}_i \right\} \mathrm{d}\Omega - \iint_{sp} \tilde{u}_i \tilde{\bar{p}}_i \, \mathrm{d}S, \tag{2}$$

where $\tilde{D}_{ijkl}$ is the tensor of fuzzy elastic moduli, $\tilde{\varepsilon}_{ij}$ the fuzzy strain tensor, $\tilde{f}_i$ the fuzzy body force per unit volume, $\tilde{u}_i$ the fuzzy displacement and $\tilde{\bar{p}}_i$ the fuzzy boundary forces.

All the fuzzy variables that affect load and deformation of structure are denoted with fuzzy vector field with n sub-vectors.

$\tilde{\mathbf{X}} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\}$.

In fuzzy vector field $\tilde{\mathbf{X}}$, there is little fuzzy perturbation $\tilde{\beta}$ near the real point $\mathbf{X}^0$, so $\tilde{\mathbf{X}}$ can be expressed as

$$\tilde{\mathbf{X}} = \mathbf{X}^0 + \tilde{\beta}, \tag{3}$$

where

$\mathbf{X}^0 = \{x_1^0, x_2^0, \ldots, x_n^0\}$,
$\tilde{\beta} = \{\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n\}$

$\tilde{\beta}_i$ $(i = 1, 2, \ldots, n)$ is a fuzzy number with a mean value of zero. $\tilde{u}$, $\tilde{\varepsilon}$, $\tilde{D}$, $\tilde{f}$, $\tilde{\bar{p}}$ and $\tilde{\Pi}_p$ are all dependent on fuzzy vector field $\tilde{\mathbf{X}}$, so they can be expanded according to the second order perturbation at real point in fuzzy vector field.

$$\tilde{u} = u^0 + \sum_{i=1}^{n} \tilde{\beta}_i u_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j u_{ij}'', \tag{4}$$

$$\tilde{\varepsilon} = \varepsilon^0 + \sum_{i=1}^{n} \tilde{\beta}_i \varepsilon_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \varepsilon_{ij}'', \tag{5}$$

$$\tilde{D} = D^0 + \sum_{i=1}^{n} \tilde{\beta}_i D_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j D_{ij}'', \tag{6}$$

$$\tilde{\bar{p}} = \bar{p}^0 + \sum_{i=1}^{n} \tilde{\beta}_i \bar{p}_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \bar{p}_{ij}'', \tag{7}$$

$$\tilde{f} = f^0 + \sum_{i=1}^{n} \tilde{\beta}_i f_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j f_{ij}'', \tag{8}$$

$$\tilde{\Pi} = \Pi^0 + \sum_{i=1}^{n} \tilde{\beta}_i \Pi_i' + \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \Pi_{ij}'', \tag{9}$$

where $u_i'(\varepsilon_i', D_i', f_i', \bar{p}_i', \Pi_i')$ denotes the first-order partial derivative of $\tilde{u}(\tilde{\varepsilon}, \tilde{D}, \tilde{f}, \tilde{\bar{p}}, \tilde{\Pi}_p)$ to fuzzy variable $\tilde{x}_i$ at the real point $\mathbf{X}^0$ in fuzzy vector field. $u_{ij}''(\varepsilon_{ij}'', D_{ij}'', f_{ij}'', \bar{p}_{ij}'', \Pi_{ij}'')$ denotes the second-order derivative to fuzzy variable $\tilde{x}_i$, $\tilde{x}_j$ at real point $\mathbf{X}^0$ in fuzzy vector field.

Substituting Eqs. (4)–(9) into Eq. (2) and through comparing the coefficient of $\tilde{\beta}$, we can obtain

$$\Pi^0 = \iiint_{\Omega} \left\{ \frac{1}{2} \varepsilon^{0^{\mathrm{T}}} D^0 \varepsilon^0 - u^{0^{\mathrm{T}}} f^0 \right\} \mathrm{d}\Omega - \iint_{sp} u^{0^{\mathrm{T}}} \bar{p}^0 \, \mathrm{d}S, \tag{10}$$

$$\Pi_i' = \iiint_{\Omega} \left\{ \frac{1}{2} \varepsilon^{0^{\mathrm{T}}} D_i' \varepsilon^0 + \varepsilon^{0^{\mathrm{T}}} D^0 \varepsilon_i' - u^{0^{\mathrm{T}}} f_i' - u_i'^{\mathrm{T}} f^0 \right\} \mathrm{d}\Omega - \iint_{sp} u^{0^{\mathrm{T}}} \bar{p}_i' + u_i'^{\mathrm{T}} \bar{p}^0 \, \mathrm{d}S \quad i = 1, 2, \ldots, n, \tag{11}$$

$$\begin{aligned} \Pi_{ij}'' = & \iiint_{\Omega} \left\{ \frac{1}{2} \varepsilon^{0^{\mathrm{T}}} D_i'' \varepsilon^0 + \varepsilon_i'^{\mathrm{T}} D^0 \varepsilon_j' + \varepsilon^{0^{\mathrm{T}}} D_i' \varepsilon_j' + \varepsilon^{0^{\mathrm{T}}} D_j' \varepsilon_i' + \varepsilon^{0^{\mathrm{T}}} D^0 \varepsilon_{ij}'' \right\} \mathrm{d}\Omega \\ & - \iint_{sp} u^{0^{\mathrm{T}}} \bar{p}_{ij}'' + u_{ij}''^{\mathrm{T}} \bar{p}^0 + u_i'^{\mathrm{T}} \bar{p}_j' + u_j'^{\mathrm{T}} \bar{p}_i' \, \mathrm{d}S \\ & - \iiint_{\Omega} \left\{ u^{0^{\mathrm{T}}} f_{ij}'' - u_{ij}''^{\mathrm{T}} f^0 + u_i'^{\mathrm{T}} f_j' + u_i'^{\mathrm{T}} f_i' \right\} \mathrm{d}\Omega \quad i(j) = 1, 2, \ldots, n. \end{aligned} \tag{12}$$

Real displacement u^0, u', u'' must meet conditions that the first order variations of above three equations is zero (Yang, 1998; Yang and Li, 1999), i.e.

$$\delta \Pi^0 = 0, \tag{13}$$

$$\delta \Pi_i' = 0, \quad i = 1, 2, \ldots, n, \tag{14}$$

$$\delta \Pi_{ij}'' = 0, \quad i(j) = 1, 2, \ldots, n. \tag{15}$$

The following is the deducing process of finite element equation. Firstly, the object is dispersed, then through interpolation of element displacement field, we can obtain

$$\tilde{u}^e = \sum_{i=1}^{m} N_i(x) \tilde{a}_i^e = \mathbf{N} \tilde{\mathbf{a}}^e, \tag{16}$$

where

$\mathbf{N} = \{N_1(x, y, z),\ N_2(x, y, z), \ldots,\ N_m(x, y, z)\}$,
$\tilde{\mathbf{a}}^e = \{\tilde{a}_1^e, \tilde{a}_2^e, \ldots, \tilde{a}_m^e\}$,

where m denotes the node numbers of each element, $\tilde{a}_i^e$ is displacement of the ith node of element e and $\mathbf{N}$ is the shape function matrix.

For $\tilde{\mathbf{a}}^e$ is affected by fuzzy vector field $\tilde{\mathbf{X}}$ as $\tilde{u}^e$ is; $\tilde{\mathbf{a}}^e$ can be expanded according to the second-order perturbation near the real point in fuzzy vector field.

$$\tilde{\mathbf{a}}^e = \mathbf{a}^{0^e} + \sum_{i=1}^{n} \tilde{\beta}_i \mathbf{a}_i^{'e} + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \, \mathbf{a}_{ij}^{''e}, \tag{17}$$

where $\mathbf{a}^{0^e}$ is the value of $\tilde{\mathbf{a}}^e$ at the real point $\mathbf{X}^0$ in fuzzy vector field, $\mathbf{a}_i^{'e}$ is the value at real point $\mathbf{X}^0$ of the first-order partial derivative of $\tilde{\mathbf{a}}^e$ to fuzzy variable $\tilde{x}_i$ in fuzzy vector field, $\mathbf{a}_{ij}^{''e}$ is the value at real point $\mathbf{X}^0$ of the second-order partial derivative of $\tilde{\mathbf{a}}^e$ to fuzzy variable $\tilde{x}_i$ and $\tilde{x}_j$ in the fuzzy vector field.

Substituting Eqs. (4) and (9) into Eq. (16) and through comparing the coefficient of $\tilde{\beta}$, we can obtain

$$u^{0^e} = \mathbf{N}\mathbf{a}^{0^e}, \tag{18a}$$

$$u_i^{'e} = \mathbf{N}\mathbf{a}_i^{'e}, \tag{18b}$$

$$u_{ij}^{''e} = \mathbf{N}\mathbf{a}_{ij}^{''e}. \tag{18c}$$

Substituting Eq. (18) into Eq. (9), we can obtain

$$\tilde{u}^e = \mathbf{N}\mathbf{a}^{0^e} + \sum_{i=1}^{n} \tilde{\beta}_i \mathbf{N}\mathbf{a}_i^{'e} + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \mathbf{N}\mathbf{a}_{ij}^{''e}. \tag{19}$$

For

$$\tilde{\varepsilon}^e = \mathbf{B}\tilde{\mathbf{a}}^e, \tag{20}$$

where $\mathbf{B}$ is the strain matrix. Substituting Eqs. (5) and (17) into Eq. (20) and through comparing the coefficient of $\tilde{\beta}$, we can obtain

$$\varepsilon^{0^e} = \mathbf{B}\mathbf{a}^{0^e}, \tag{21a}$$

$$\varepsilon_i^{'e} = \mathbf{B}\mathbf{a}_i^{'e}, \tag{21b}$$

$$\varepsilon_{ij}^{''} = \mathbf{B}\mathbf{a}_{ij}^{''}. \tag{21c}$$

Substituting Eqs. (18a) and (21a) into Eq. (10), we can obtain

$$\Pi^0 = \sum_{e=1}^{N} \left(\frac{1}{2} \mathbf{a}^{0^{e\mathrm{T}}} K^{0^e} \mathbf{a}^{0^e} - \mathbf{a}^{0^{e\mathrm{T}}} P^{0^e} \right), \tag{22}$$

where

$$K^{0^e} = \iiint_{\Omega e} B^{\mathrm{T}} D^0 B \, \mathrm{d}\Omega,$$

$$P^{0^e} = \iint_{Spe} \mathbf{N}^{\mathrm{T}} \bar{p}^0 \, \mathrm{d}S + \iiint_{\Omega e} \mathbf{N}^{\mathrm{T}} f^0 \, \mathrm{d}\Omega.$$

If

$$\mathbf{a}_S^0 = \begin{bmatrix} \mathbf{a}^{0^{1\mathrm{T}}} & \mathbf{a}^{0^{2\mathrm{T}}} & \cdots & \mathbf{a}^{0^{N\mathrm{T}}} \end{bmatrix}^{\mathrm{T}}, \tag{23}$$

$$\mathbf{K}_S^0 = \begin{bmatrix} K^{0^1} & 0 & \cdots & 0 \\ 0 & K^{0^2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & K^{0^N} \end{bmatrix}, \tag{24}$$

$$\mathbf{P}_S^0 = \begin{bmatrix} P^{0^{1\mathrm{T}}} & P^{0^{2\mathrm{T}}} & \cdots & P^{0^{N\mathrm{T}}} \end{bmatrix}^{\mathrm{T}} \tag{25}$$

then build the following equations:

$$\mathbf{a}^0 = J\mathbf{a}_S^0 \tag{26}$$

J is called transmit matrix, N is the number of elements.

Substituting Eq. (26) into Eq. (21), letc

$$\mathbf{K}^0 = J^{\mathrm{T}} \mathbf{K}_S^0 J,$$
$$\mathbf{P}^0 = J^{\mathrm{T}} \mathbf{P}_S^0 J,$$

then, Eq. (22) can be expressed as

$$\Pi^0 = \frac{1}{2} \mathbf{a}^{0^{\mathrm{T}}} \mathbf{K}^0 \mathbf{a}^0 - \mathbf{a}^{0^{\mathrm{T}}} \mathbf{P}^0,$$

When Eq. (13) is taken into account, we can obtain

$$\delta\Pi^0 = \frac{\partial \Pi^0}{\partial \mathbf{a}^0} \, \delta\mathbf{a}^0 = 0.$$

So

$$\begin{aligned} \frac{\partial \Pi^0}{\partial \mathbf{a}^0} &= \iiint_{\Omega} B^{\mathrm{T}} D^0 B \, \mathrm{d}\Omega \mathbf{a}^0 - \iint_{Sp} \mathbf{N}\bar{p}^0 \, \mathrm{d}S \\ &\quad - \iiint_{\Omega} \mathbf{N} f^0 \mathrm{d}\Omega \\ &= \mathbf{K}^0 \mathbf{a}^0 - \mathbf{P}^0 = \mathbf{0}. \end{aligned} \tag{27}$$

In fact, the above equation is equilibrium equation of finite element when fuzziness is not considered.

Similarly, substitute Eqs. (18a, b) and (21a, b) into Eq. (11), after elements are assembled and Eq. (14) is taken into account, then

$$\delta\Pi_i^{'} = \frac{\partial \Pi^0}{\partial \mathbf{a}^0} \, \delta\mathbf{a}^0 + \frac{\partial \Pi_i^{'}}{\partial \mathbf{a}_i^{'}} \, \delta\mathbf{a}_i^{'} = 0.$$

Taking the variation of $\delta\mathbf{a}^0$ and $\delta\mathbf{a}_i^{'}$, we can obtain

$$\begin{aligned} \frac{\partial \Pi_i^{'}}{\partial \mathbf{a}_i^{'}} &= \iiint_{\Omega} B^{\mathrm{T}} D^0 B \, \mathrm{d}\Omega \mathbf{a}^0 - \iint_{Sp} \mathbf{N}^{\mathrm{T}} \bar{p}^0 \, \mathrm{d}S \\ &\quad - \iiint_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{f}^0 \, \mathrm{d}\Omega \\ &= \mathbf{K}^0 \mathbf{a}^0 - \mathbf{P}^0 = \mathbf{0}, \end{aligned} \tag{28a}$$

$$\frac{\partial \Pi_i'}{\partial \mathbf{a}^0} = \iiint_{\Omega} B^{\mathrm{T}} D_i' B \mathbf{a}^0 + B^{\mathrm{T}} D^0 B \mathbf{a}_i' \,\mathrm{d}\Omega - \iint_{Sp} \mathbf{N}^{\mathrm{T}} \bar{p}_i' \,\mathrm{d}S - \iiint_{\Omega} \mathbf{N}^{\mathrm{T}} f_i' \,\mathrm{d}\Omega = \mathbf{K}_i' \mathbf{a}^0 + \mathbf{K}^0 \mathbf{a}_i' - \mathbf{P}_i' = \mathbf{0}, \quad i = 1, 2, \ldots, n, \tag{28b}$$

where

$$\mathbf{K}_i' = J^{\mathrm{T}} \begin{bmatrix} K_i'^1 & 0 & \cdots & 0 \\ 0 & K_i'^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & K_i'^N \end{bmatrix} J,$$

$$K_i'^l = \iiint_{\Omega l} B^{\mathrm{T}} D_i' B \,\mathrm{d}\Omega,$$

$$\mathbf{P}_i' = J^{\mathrm{T}} \left[P_i'^1 \;\; P_i'^2 \;\; \cdots \;\; P_i'^N \right]^{\mathrm{T}},$$

$$P_i'^l = \iint_{Spl} \mathbf{N}^{\mathrm{T}} \bar{p}_i' \,\mathrm{d}S - \iiint_{\Omega l} \mathbf{N}^{\mathrm{T}} f_i' \,\mathrm{d}\Omega.$$

Similarly, substitute Eqs. (18a, b, c) and (21a, b, c) into Eq. (12), after elements are assembled and Eq. (15) is taken into account, then

$$\delta \Pi_{ij}'' = \frac{\partial \Pi_{ij}''}{\partial \mathbf{a}^0} \delta \mathbf{a}^0 + \frac{\partial \Pi_{ij}''}{\partial \mathbf{a}_i'} \delta \mathbf{a}_i' + \frac{\partial \Pi_{ji}''}{\partial \mathbf{a}_j'} \delta \mathbf{a}_j' + \frac{\partial \Pi_{ij}''}{\partial \mathbf{a}_{ij}''} \delta \mathbf{a}_{ij}'' = 0.$$

Taking the variation of $\delta \mathbf{a}^0$, $\delta \mathbf{a}_i'$ and $\delta \mathbf{a}_{ij}''$, we can obtain

$$\frac{\partial \Pi_{ij}''}{\partial \mathbf{a}_{ij}''} = \iiint_{\Omega} B^{\mathrm{T}} D^0 B \,\mathrm{d}\Omega \mathbf{a}^0 - \iint_{Sp} \mathbf{N}^{\mathrm{T}} \bar{p}^0 \,\mathrm{d}S - \iiint_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{f}^0 \,\mathrm{d}\Omega = K^0 \mathbf{a}^0 - \mathbf{P}^0 = \mathbf{0}, \tag{29a}$$

$$\frac{\partial \Pi_{ij}''}{\partial \mathbf{a}_i'} = \iiint_{\Omega} B^{\mathrm{T}} D_j' B \mathbf{a}^0 + B^{\mathrm{T}} D^0 B \mathbf{a}_j' \,\mathrm{d}\Omega - \iint_{Sp} \mathbf{N}^{\mathrm{T}} \bar{p}_j' \,\mathrm{d}S - \iiint_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{f}_j' \,\mathrm{d}\Omega = \mathbf{K}_i' \mathbf{a}^0 + \mathbf{K}^0 \mathbf{a}_i' - \mathbf{P}_i' = \mathbf{0}, \quad i = 1, 2, \ldots, n, \tag{29b}$$

$$\frac{\partial \Pi_{ij}''}{\partial \mathbf{a}_i'} = \iiint_{\Omega} B^{\mathrm{T}} D_{ij}'' B \mathbf{a}^0 + D_i' B \mathbf{a}_j' + D_j' B \mathbf{a}_i' + D^0 B \mathbf{a}_{ij}'' \,\mathrm{d}\Omega - \iint_{Sp} \mathbf{N}^{\mathrm{T}} \bar{p}_{ij}'' \,\mathrm{d}S - \iiint_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{f}''_{ij} \mathrm{d}\Omega = \mathbf{K}^0 \mathbf{a}_{ij}'' + \mathbf{K}_i' \mathbf{a}_j' + \mathbf{K}_j' \mathbf{a}' + \mathbf{K}_{ij}'' \mathbf{a}^0 - \mathbf{P}_{ij}'' = \mathbf{0}, \quad i(j) = 1, 2, \ldots, n, \tag{29c}$$

where

$$\mathbf{K}_{ij}'' = J^{\mathrm{T}} \begin{bmatrix} K_{ij}''^1 & 0 & \cdots & 0 \\ 0 & K_{ij}''^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & K_{ij}''^N \end{bmatrix} J,$$

$$K_{ij}''^l = \iiint_{\Omega l} B^{\mathrm{T}} D_{ij}'' B \,\mathrm{d}\Omega,$$

$$\mathbf{P}_{ij}'' = J^{\mathrm{T}} \left[P_{ij}''^1 \;\; P_{ij}''^2 \;\; \cdots \;\; P_{ij}''^N \right]^{\mathrm{T}},$$

$$P_{ij}''^l = \iint_{Spl} \mathbf{N}^{\mathrm{T}} \bar{p}_{ij}'' \,\mathrm{d}S - \iiint_{\Omega l} \mathbf{N}^{\mathrm{T}} f_{ij}'' \,\mathrm{d}\Omega.$$

Substitute $\mathbf{a}^0$, $\mathbf{a}_i'$ and $\mathbf{a}_{ij}''$ into the following equation:

$$\tilde{\mathbf{a}} = \mathbf{a}^0 + \sum_{i=1}^{n} \tilde{\beta}_i \mathbf{a}_i' + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \tilde{\beta}_i \tilde{\beta}_j \mathbf{a}_{ij}''. \tag{30}$$

The fuzzy displacement of each element node can be obtained.

Variation Eq. (28) of the first-order perturbation expansion Π_i' includes variation Eq. (27) of functional Π^0, and variation Eq. (29) of the second-order perturbation expansion Π_{ij}'' includes variation Eq. (28) of functional Π_i'. In the deduction of fuzzy finite element equation based on the second-order perturbation method, we can obtain all the control function through taking the variation of the second-order perturbation expansion equation Π_{ij}''.

According to Eq. (29a), (29b) and (29c), we can obtain

$$\mathbf{K}^0 \mathbf{a}^0 = \mathbf{P}^0, \tag{31}$$

$$\mathbf{K}^0 \mathbf{a}_i' = \mathbf{P}_i' - \mathbf{K}_i' \mathbf{a}^0, \quad i = 1, 2, \ldots, n, \tag{32}$$

$$\mathbf{K}^0 \mathbf{a}_{ij}'' = \mathbf{P}_{ij}'' - \mathbf{K}_i' \mathbf{a}_j' + \mathbf{K}_j' \mathbf{a}' + \mathbf{K}_{ij}'' \mathbf{a}^0, \quad i(j) = 1, 2, \ldots, n, \tag{33}$$

From Eq. (31) to Eq. (33) we can see that, as compared with the finite element calculation of certain structure, we need to calculate n equations in Eq. (32) and n^2 equations in Eq. (33) besides calculating equations in Eq. (31) in the fuzzy finite element-based on perturbation method. The calculation of Eq. (31) is the same as that of conventional method. In the calculation of $n+n^2$ equations in Eqs. (32) and (33), $\mathbf{K}_i'$, $\mathbf{K}_{ij}''$, $\mathbf{P}_i'$ and $\mathbf{P}_{ij}''$ must be obtained first. From the expression of $\mathbf{K}_i'$, $\mathbf{K}_{ij}''$, $\mathbf{P}_i'$ and $\mathbf{P}_{ij}''$ we can see D_i', D_{ij}'', f_j', f_{ij}'', $\bar{p}_i'$ and $\bar{p}_{ij}''$ must be obtained before solving $\mathbf{K}_i'$, $\mathbf{K}_{ij}''$, $\mathbf{P}_i$ and $\mathbf{P}_{ij}''$. It is explained with plane triangular element.

To plane stress the problem in an isotropic material, the constitutive matrix D can be expressed as

$$D = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}.$$

The calculation of elastic modulus E and Poisson ratio v will produce fuzzy perturbation near the real value due to scarcity of experimental data and the limitations of measurement technology, which can be expressed as

$$\tilde{E} = E^0 + \tilde{e},$$

$$\tilde{v} = v^0 + \tilde{v}.$$

Above equations can be transformed as

$$\tilde{D} = \frac{E^0+\tilde{e}}{1-(v^0+\tilde{v})^2}\begin{bmatrix} 1 & v^0+\tilde{v} & 0 \\ v^0+\tilde{v} & 1 & 0 \\ 0 & 0 & \frac{1-(v^0+\tilde{v})}{2} \end{bmatrix}.$$

Let body force f and plane force $\bar{p}$ be fuzzy numbers. The calculation value will produce a fuzzy perturbation near the real value. The fuzzy field $\tilde{\mathbf{X}}$ of structure is denoted by a four-dimensional fuzzy vector, i.e.

$$\tilde{\mathbf{X}} = \mathbf{X}^0 + \tilde{\beta},$$

where

$$\mathbf{X}^0 = \{E^0, v^0, f^0, \bar{p}^0\},$$
$$\tilde{\beta} = \{\tilde{E}, \tilde{v}, \tilde{f}, \tilde{\bar{p}}\}.$$

According to the definition of D_i' and D_{ij}'', we can obtain

$$D_1' = \frac{1}{1-v^{0^2}}\begin{bmatrix} 1 & v^0 & 0 \\ v^0 & 1 & 0 \\ 0 & 0 & \frac{1-v^0}{2} \end{bmatrix},$$

$$D_2' = \frac{E^0}{1-v^{0^2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} + \frac{2E^0v^0}{(1-v^{0^2})^2}\begin{bmatrix} 1 & v^0 & 0 \\ v^0 & 1 & 0 \\ 0 & 0 & \frac{1-v^0}{2} \end{bmatrix},$$

$$D_{11}''=0 \quad D_{12}'' = D_{21}'' = \frac{1}{1-v^{0^2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} - \frac{2v^0}{(1-v^{0^2})^2}\begin{bmatrix} 1 & v^0 & 0 \\ v^0 & 1 & 0 \\ 0 & 0 & \frac{1-v^0}{2} \end{bmatrix},$$

$$D_{22}'' = \frac{2E^0v^0}{(1-v^{0^2})^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} + \frac{8E^0v^{0^2}}{(1-v^{0^2})^3} \times \begin{bmatrix} 1 & v^0 & 0 \\ v^0 & 1 & 0 \\ 0 & 0 & \frac{1-v^0}{2} \end{bmatrix} + \frac{2E^0v^0}{(1-v^{0^2})^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix},$$

$$\begin{cases} D_i' = 0 \\ D_{ij}'' = 0 \end{cases} \quad i(\text{or } j) = 3, 4,$$

$$f_i' = \begin{cases} 1 & i=3 \\ 0 & i=1,2,4 \end{cases} \quad f_{ij}'' = 0$$
$$p_i' = \begin{cases} 1 & i=4 \\ 0 & i=1,2,3 \end{cases} \quad p_{ij}'' = 0.$$

For the calculation of fuzzy perturbation finite element with four-dimensional vector field, we have obtained above all the coefficient of Eqs. (31)–(33). Dimension of fuzzy vector can be determined according to actual conditions. When calculating a fuzzy system with n fuzzy perturbation sources by the second-order perturbation fuzzy finite element method, it requires calculating $1+n+n^2$ finite element equations, which will cost lots of time. Because solving finite element equations equals to minimizing a quadratic energy function, we can construct a neural network according to the energy function. The solving finite element equation equals to the process of dynamitic adjustment of neural network. We can obtain the real-time calculation of finite element equations. Therefore all the neurocomputing methods (Huang and Li, 2003) can be applied to the calculation of perturbation fuzzy finite element. It is worthy of note that each equation's left side has the same coefficient matrix $\mathbf{K}^0$ in Eqs. (31)–(33). It does not require to change the neural network structure every time. We can obtain the solution through changing input electric current, which simplifies the neurocomputing process.

From Eq. (31) to Eq. (33) we can see that $\mathbf{a}^0$, $\mathbf{a}_i'$ and $\mathbf{a}_{ij}''$ must be obtained first. But in the process any fuzzy information should not be used, and fuzzy solution can be obtained through introducing fuzzy information $\tilde{\beta}$ into Eq. (30) after obtaining $\mathbf{a}^0$, $\mathbf{a}_i^{'}$ and $\mathbf{a}_{ij}^{''}$, which simplifies the solving process.

3. Numerical example

As shown in Fig. 1, the thickness of a uniform rectangle sheet is $h = 1$ mm. One of its ends is fixed, and the other is the applied uniform force $\tilde{q}$. $\tilde{q}$ is a triangular fuzzy number (0.98, 1, 1.02) kN/m, its length is 2 m, and width 1 m; elastic modulus $\tilde{E}$ is a triangular fuzzy number (204, 206, 208) $10^9\,\mathrm{N/m^2}$, and Poisson ratio is $\mu(=\frac{1}{3})$. Try to find each element displacement with perturbation finite element method when gravitation is neglected.

According to solving finite element equations, the arrays and columns corresponding to zero displacement constraints are deleted in $\tilde{\mathbf{K}}$ and corresponding sub-vector in $\tilde{\mathbf{F}}$ are deleted. We can obtain the equilibrium equation

$$\tilde{\tilde{\mathbf{K}}}\boldsymbol{\delta} = \tilde{\tilde{\mathbf{F}}},$$

$$\tilde{\tilde{\mathbf{K}}} = \begin{bmatrix} \tilde{K}_{22} & \tilde{K}_{23} \\ \tilde{K}_{32} & \tilde{K}_{33} \end{bmatrix} = \frac{3\tilde{E}h}{32}\begin{bmatrix} 7 & -4 & -4 & 2 \\ -4 & 13 & 2 & -12 \\ -4 & 2 & 7 & 0 \\ 2 & -12 & 0 & 13 \end{bmatrix}$$

$$d = \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} \quad \tilde{\tilde{\mathbf{F}}} = \begin{bmatrix} \tilde{F}_2 \\ \tilde{F}_3 \end{bmatrix} = \begin{bmatrix} \frac{\tilde{q}}{2} \\ 0 \\ \frac{\tilde{q}}{2} \\ 0 \end{bmatrix}.$$

Fuzzy field is a two-dimensional vector in the example, i.e. $\tilde{\beta} = \{\tilde{E}, \tilde{p}\}$.

According to the definition, calculating results are

$$\mathbf{K}_1^{'} = \mathbf{K}^0/E^0 \quad \mathbf{K}_2^{'} = 0 \quad \mathbf{P}_1^{'} = 0 \quad \mathbf{P}_2^{'} = \mathbf{1}$$

Substituting the above results into Eq. (32), we can obtain

$$\mathbf{a}_1^{'} = -\mathbf{a}^0/E^0, \tag{34}$$

$$\mathbf{a}_2^{'} = \mathbf{K}^{0^{-1}} \cdot \mathbf{1}. \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (30), we can obtain

$$\tilde{\mathbf{a}} = \mathbf{a}^0 + \left(-\mathbf{a}^0\frac{\tilde{E}}{E^0} + \mathbf{a}^0\frac{\tilde{q}}{q^0}\right), \tag{36}$$

where

$$\mathbf{a}^0 = \mathbf{K}^{0^{-1}}\mathbf{F} = (9.6295\ \ 1.7298\ \ 8.7069\ \ 0.1153)^{\mathrm{T}}$$

Substituting all parameters into Eq. (35), we can obtain

$$\tilde{\mathbf{a}} = \mathbf{a}^0\left(1 - \left(\frac{1}{103}\ \ 0\ \ \frac{-1}{103}\right) + (-0.02\ \ 0\ \ 0.02)\right)$$

i.e.

$$a_1(X_2) = (9.3434\ \ 9.6295\ \ 9.9156),$$
$$a_2(Y_2) = (1.6784\ \ 1.7298\ \ 1.7812),$$
$$a_3(X_3) = (8.4482\ \ 8.7069\ \ 8.9656),$$
$$a_4(Y_3) = (0.1119\ \ 0.1153\ \ 0.1187).$$

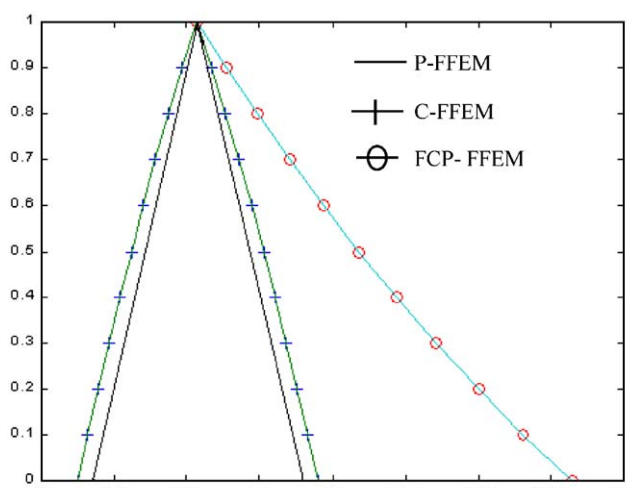

Fig. 2. Fuzzy solution to X_2.

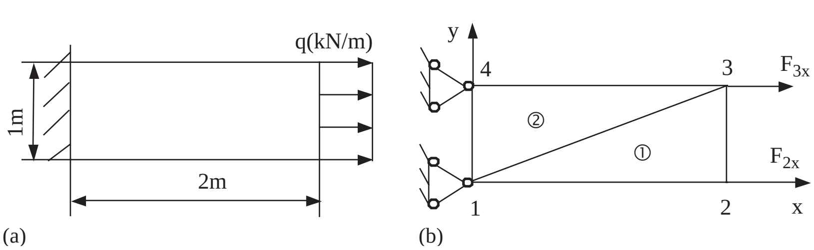

Fig. 1. (a), (b) The applied force and finite elements of the rectangle sheet.

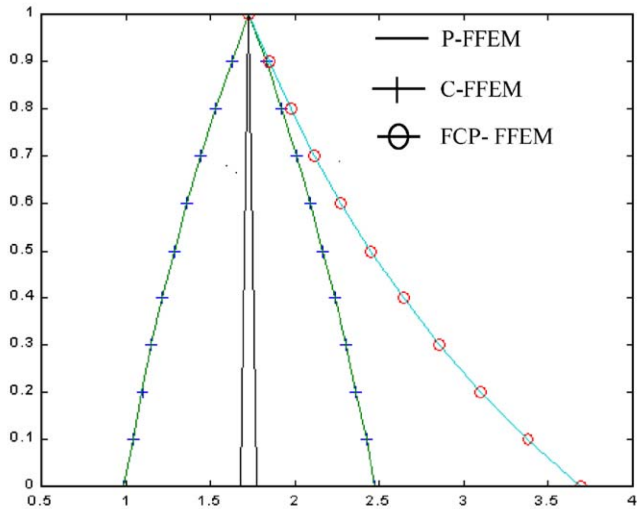

Fig. 3. Fuzzy solution to Y_2.

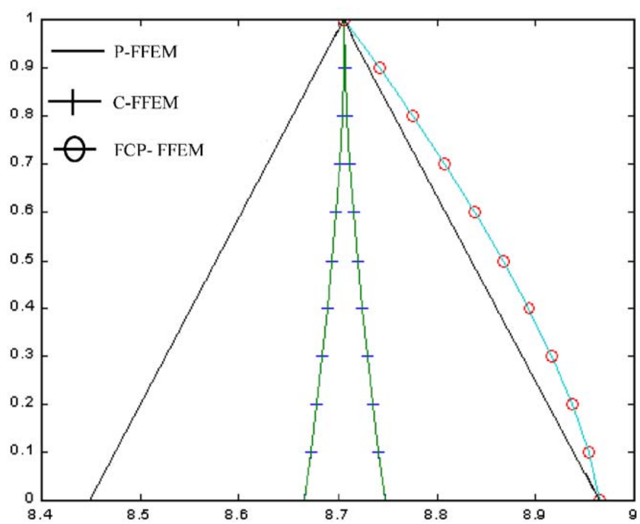

Fig. 4. Fuzzy solution to X_3.

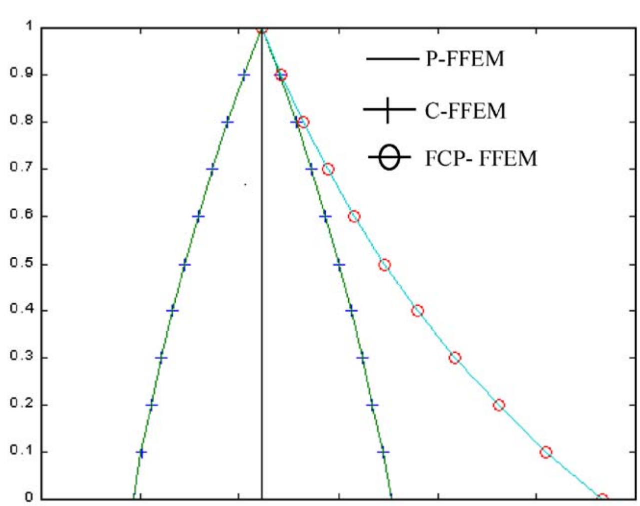

Fig. 5. Fuzzy solution to Y_3.

The results obtained from the above method (P-FFEM) and that obtained from conventional fuzzy finite element method (C-FFEM) and that obtained from the fuzzy finite element method based on fuzzy coefficient programming (FCP-FFEM) (Li et al., 2003) are shown in Figs. 2–5.

4. Conclusions

Perturbation fuzzy finite element method is introduced in this paper. The method is a fuzzy finite element method based on fuzzy variational principle. Its physical meaning is definite. Compared with the conventional fuzzy finite element methods, it avoids calculation error caused by the expansion of interval number's operation (Moore, 1966) . If the simulation results obtained from different methods are compared, when $\lambda = 1$, the result obtained from perturbation fuzzy finite element method based on fuzzy variational principle is the same as that obtained from conventional fuzzy finite element method and fuzzy finite element method based on fuzzy coefficient programming. With the decrease of λ, the difference between the interval's upper and lower boundary that was obtained from the conventional fuzzy finite element method is bigger. From Fig. 5 we can see that the upper and lower point of Y_3 occurs in opposite sign. The upper and lower boundaries of interval value obtained from perturbation fuzzy finite element method based on fuzzy variational principle have the same sign, which accords with reality. Besides, the absolute result obtained from fuzzy finite element method based on fuzzy coefficient programming is always the maximum. So it illustrates that fuzzy finite element method based on fuzzy coefficient programming is a reliable calculating method. It needs to be mentioned that fuzzy finite element method based on perturbation method is an approximated solution under small perturbation. When $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{F}}$ cannot be expanded to limited order expression of fuzzy resource, and fuzzy perturbation is big, the truncated error cannot be accepted (Chen and Liu, 1993). Under this condition, structural analysis will fail if perturbation fuzzy finite element method based on variational principle is used.

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