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A Simulation Method to Estimate Two Types of Time-Varying Failure Rate of Dynamic Systems

The failure rate of dynamic systems with random parameters is time-varying even for linear systems excited by a stationary random input. In this paper, we propose a simulation-based method to estimate two types (type I and type II) of time-varying failure rate of dynamic systems. The input stochastic processes are discretized in time and the trajectories of the output stochastic process are calculated. The time of interest is partitioned into a series of time intervals and the saddlepoint approximation (SPA) is employed to estimate the probability of failure in each interval. Type I follows the commonly used definition of failure rate. It is estimated at discrete time intervals using SPA and the correlation information from a properly selected time-dependent copula function. Type II is a proposed new concept of time-varying failure rate. It provides a way to predict the failure rate considering a virtual "good-as-old" repair action of repairable dynamic systems. The effectiveness of the proposed method is illustrated with a vehicle vibration example.
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1 Introduction

The failure rate quantifies the probability of failure over a small time interval dt after time t under the condition that the system did not fail before time t. In reliability engineering, it is common to derive the failure rate from the cumulative distribution function (CDF) [1–4]. Sufficient data samples are needed to estimate the CDF accurately with statistical methods, especially for a small probability of failure. For example, about 234,000 data samples are needed to estimate a 0.00171 probability of failure with 10% error under 95% confidence. Estimation of the CDF using physics of failure (POF) has attracted more attention for structural systems by considering the time-dependent properties. The time horizon is partitioned into many time intervals for discrete CDF estimation and then a continuous CDF is fitted. Accordingly, discrete and continuous time-varying failure rates could be derived from the discrete and continuous CDFs, respectively.

To estimate a time-varying failure rate, time-dependent reliability analysis must be conducted over the planning horizon. Many methods have been proposed for time-dependent reliability analysis of structural systems. Because of the independent non-negative increments of the Gamma process, a combination of Gamma process and statistics of live load models is used in Ref. [5]. A distribution of extreme values approach is employed in Ref. [6] to transform the time-dependent reliability problem to a time-independent reliability problem. An equivalent time-invariant composite limit state is defined and used in Ref. [7] for time-dependent reliability analysis. The composite limit state method is further studied for time-dependent reliability analysis based on total probability theorem [8]. Also, a time-dependent reliability analysis is performed in Ref. [9] using a nested kriging based prediction model where time-independent reliability methods such as the first order reliability method (FORM) [10] and the saddlepoint approximation method (SPA) [11] can be used. The out-crossing rate method has been widely used for time-dependent reliability problems and many advanced methods have been proposed [12–15] under the assumption of independent out-crossings following the Poisson distribution, which may result in a poor accuracy. A parallel system reliability formulation, the so-called PHI2 method, using the out-crossing rate is proposed in Ref. [16]. A Monte Carlo based set theory method, similar with the PHI2 method, is reported in Ref. [17]. Studies reveal that the PHI2 based method shows relatively poor accuracy when dealing with nonmonotonic problems [6,18]. In order to avoid the calculation of out-crossing rate, a time-dependent reliability analysis method based on stochastic process discretization is presented in Ref. [19].

Most of time-dependent reliability analysis methods provide only the reliability over a specific time interval. Design under uncertainty considering maintenance and lifecycle cost requires however, the reliability at all times within the planning horizon. Singh et al. [20] proposes such a method by estimating the failure rate of random dynamic systems using importance sampling. Their method is suitable for low dimensional limit state functions. Wang et al. [21] also presents a method to estimate the failure rate at all times within the planning horizon using an improved subset simulation with splitting by partitioning the original high dimensional random process into a series of correlated, short duration, low dimensional random processes. However in estimating the time-varying failure rate, the high computational effort is still a challenge.

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Copula functions have been widely used for modeling the correlation of random variables and time series in economic field [22,23]. Because of their great capacity to represent correlation, Copulas have been recently introduced in the structural safety field. Noh et al. [24] uses Gaussian Copula to describe the correlation of input random variables for reliability-based design optimization and further identifies the joint CDFs from provided data using a copula function [25]. In order to express the multidimensional correlation, the vine-copula has been also introduced in structural reliability analysis [26]. Tang et al. [27] studies the effect of different copula models on the structural system reliability results.

The above studies have played an important role in fostering applications of Copula functions for time-independent problems in the structural safety field. For dynamic systems, the correlation of performance between adjacent time intervals can be established accounting therefore, for time. In order to address the time-dependent probability of failure or failure rate, of dynamic systems, a simulation-based method is proposed in this article to estimate two types of time-varying failure rate.

A simulation approach is used because the failed samples still remain in the operating queue, resulting in correlated failures in time. The input stochastic processes are discretized using a small time step into a set of high-dimensional random variables and the trajectories of the output stochastic process are calculated accurately with simulation. The planning horizon is then partitioned into a series of time intervals and the saddlepoint approximations (SPA) method is used to calculate the probability of failure in each time interval. With the same trajectories used in SPA, a time-dependent copula is built to provide the correlation between the maximum response in each time interval and the maximum responses up to that time interval. A discrete mean time-varying failure rate in each time interval is calculated using an estimated probability of failure from the SPA and the correlation information from the estimated time-dependent copula. A continuous mean time-varying failure rate is also derived from the fitted CDF from the discrete cumulative probabilities of failure based on a validated Weibull distribution. This time-varying failure rate, called Type I failure rate, is calculated using the classical definition.

During simulation, we assume that after failure, performance is restored to the state just before failure occurred (good-as-old repair assumption). In order to compare the failure evolution of the repaired and nonrepaired samples, we propose the concept of time-varying failure rate for repaired samples, called type II failure rate, derive its expression and compare it with type I. Type II provides a new way to predict the failure rate by essentially building a relationship between the system failure rate and the capacity for maintenance of repairable dynamic systems.

The contributions of this article are: (1) a new time-varying failure rate (type I) estimation method is proposed for dynamic systems; (2) a time-dependent copula function approach is proposed to establish the failure correlation as time progresses without additional computational effort; and (3) a new concept of time-varying failure rate (type II) is presented for repaired samples and its expression is derived.

The reminder of the paper is organized as follows. Section 2 reviews the SPA method and the method to construct a time-dependent copula function. We use both methods in the proposed time-varying failure rate approach. Section 3 provides details on the proposed approach and Sec. 4 uses a vehicle vibration example to illustrate its effectiveness. Finally, Sec. 5 summarizes and concludes.

2 Saddlepoint Approximation and Copula Function

2.1 Saddlepoint Approximation. The SPA is an attractive method to perform structural reliability analysis because of its high computational accuracy in approximating the tails of a distribution and its comparable efficiency with FORM [11,28]. If Y is a random variable with probability density function (PDF) $f_Y(y)$, the moment generating function (MGF) of Y is given by

$$M(\xi) = E[e^{\xi Y}] = \int_{-\infty}^{\infty} e^{\xi y} f_Y(y)\mathrm{d}y \tag{1}$$

where $E[\cdot]$ denotes expectation. The cumulative generating function (CGF) $K(\xi)$ is

$$K(\xi) = \log[M(\xi)] \tag{2}$$

where $\log[\cdot]$ is the natural logarithm. The inverse Fourier transform is used to obtain $f_Y(y)$ from $K(\xi)$ as

$$f_Y(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp[K(\xi) - \xi y]\mathrm{d}\xi \tag{3}$$

Analytical forms of the CGF are only available for certain distributions [29]. If there is no analytical form, an approximation can be obtained using samples. It has been shown that an accurate approximation of the CGF can be obtained using the following four cumulants [30]

$$\begin{cases} k_1 = \dfrac{s_1}{N_s} \\ k_2 = \dfrac{N_s s_2 - s_1^2}{N_s(N_s-1)} \\ k_3 = \dfrac{2s_1^3 - 3N_s s_1 s_2 + N_s^2 s_3}{N_s(N_s-1)(N_s-2)} \\ k_4 = \dfrac{-6s_1^4 + 12N_s s_1^2 s_2 - 3N_s(N_s-1)s_2^2 - 4N_s(N_s+1)s_1 s_3 + N_s^2(N_s+1)s_4}{N_s(N_s-1)(N_s-2)(N_s-3)} \end{cases} \tag{4}$$

where $s_r = \sum_{j=1}^{N_s}(y^j)^r (r = 1,2,3,4)$ and $y^j (j = 1,\ldots,N_s)$ is the output of the performance function. $N_s (N_s \geq 4)$ is the sample size.

Using the above four cumulants, the CGF is approximated as

$$K(\xi) = k_1\xi + \frac{1}{2}k_2\xi^2 + \frac{1}{3!}k_3\xi^3 + \frac{1}{4!}k_4\xi^4 \tag{5}$$

If the derivative of $K(\xi)$ with respect to ξ is set equal to y as

$$K'(\xi) = y \tag{6}$$

the solution to Eq. (6) is the saddlepoint ξ_{sp}, which is also the extreme point of $K(\xi) - \xi y$ in Eq. (3).

Daniels [31] uses a power series expansion to evaluate the integral in Eq. (3) resulting in the following expression of the PDF of Y

$$f_Y(y) = \left[\frac{1}{2\pi K''\left(\xi_{sp}\right)}\right]^{\frac{1}{2}} e^{\left[K\left(\xi_{sp}\right) - \xi_{sp} y\right]} \tag{7}$$

The saddlepoint approximation approach can also be used to approximate the CDF $F_Y(y)$. A reasonable and widely used formula is provided by Lugannini and Rice [32]

$$F_Y(y) = \Pr\{Y \le y\} = \Phi(w) + \phi(w)\left(\frac{1}{w} - \frac{1}{v}\right) \tag{8}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of a standard normal distribution, and w and v can be expressed by

$$w = \text{sgn}(\xi_{sp})\{2[\xi_{sp} y - K(\xi_{sp})]\}^{1/2} \tag{9}$$

and

$$v = \xi_{sp}[K''(\xi_{sp})]^{1/2} \tag{10}$$

where $\text{sgn} = 1, 0,$ or -1 depending on whether ξ_{sp} is positive, zero, or negative. $K''(\cdot)$ is the second derivative of $K(\cdot)$ and $K''(\cdot) \ge 0$ is always satisfied. More details on SPA are provided in Refs. [33,34].

2.2 Copula Function. The copula function is a general tool to represent statistical dependence for multivariate distributions [35]. According to Sklar's theorem [36], the n-dimensional copula function C uniquely defines the joint CDF $F(x_1, x_2, \ldots, x_n)$ as a function of the marginal CDFs $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$ as

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \tag{11}$$

if the marginal CDFs are continuous functions. Note that $F_i(x_i) = \Pr\{X_i \le x_i\}$, $i = 1, 2, \ldots, n$, represent probabilities. The notation $\Pr\{\cdot\}$ is used to denote probability. In this paper, the Clayton copula is used extensively. Its CDF is defined as

$$C(u, v|\theta) = (-1 + u^{-\theta} + v^{-\theta})^{-1/\theta} \tag{12}$$

where u and v are the marginal distribution functions, and θ is the Clayton parameter. For more information on different copula functions, such as Clayton, Gumbel, Frank, Gaussian, Ali-Mikhail-Haq (AMH), Farlie–Gumbel–Morgenstern (FGM), Arch 12 and Arch 14, please refer to Refs. [22–27,37].

First, a copula function is selected among existing copula functions to ensure we describe the statistical dependence of a multivariate distribution properly. Subsequently, the parameters of the copula functions are estimated to guarantee the accuracy of the statistical dependence representation. Many methods have been developed for parameter estimation, including the likelihood approach, the inference for the margins method, the semiparametric estimation method, the nonparametric estimation method, and the Bayesian copula approach [38,39].

The semiparametric estimation method uses an empirical CDF which is built from samples [40]. In this paper, we construct copula functions to describe the dependence of maximum response among different time intervals. As such, they are time-dependent copulas.

3 Proposed Method

3.1 Overview. We estimate the time-varying failure rate of a dynamic (rigid-body or vibratory) system whose equations of motion are discretized in time and expressed in a state-space form. The discretized equations are time integrated using for example, a Runge–Kutta method or Newmark-beta method [41] and are expressed as

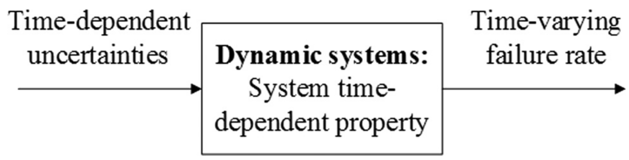

Fig. 1 Schematic of the time-varying failure rate estimation process

$$\mathbf{X}(t + \Delta t) = h(\mathbf{X}(t), \mathbf{Z}, \mathbf{M}(t), t) \tag{13}$$

where $\mathbf{X}(t + \Delta t) \in \Re^c$ is the vector of uncertain states $x_s(t + \Delta t)$, $s = 1, 2, \ldots, c$ at time $t + \Delta t$, Δt is the integration time step and $h(\cdot)$ indicates a function of the arguments. $\mathbf{Z} \in \Re^q$ is the time-independent vector of random variables (e.g., system parameters such as vehicle suspension stiffness $\mathbf{Z}_s$, and excitation parameters $\mathbf{Z}_e$), and $\mathbf{M}(t) = \mathbf{M}(\mathbf{Z}, t)$ is the time-dependent vector of excitation random processes (e.g., road elevation at a vehicle tire location through time). The trajectories $\mathbf{X} = \{\mathbf{X}(t),\ t \in [0, T]\}$ of all states are calculated at discrete times $t_i = i\Delta t$, $i = 0, 1, 2, \ldots, N_1;\ t_0 = 0,\ t_{N_1} = T$ over the planning horizon $[0, T]$. Figure 1 shows a schematic of the time-varying failure rate estimation process.

The computational effort to calculate the system states $\mathbf{X}(t + \Delta t)$ at time $t + \Delta t$ as a function of the known system states $\mathbf{X}(t)$ at time t, is considered one function evaluation. The states $\mathbf{X}(t)$ are in turn a function of the states $\mathbf{X}(t - \Delta t)$. In order to calculate $\mathbf{X}(t)$, all previous states over the time interval $[0, t]$ should be obtained. Thus, treating the number of sample functions (or trajectories) as the computational efficiency index, instead of the number of function evaluations, is preferable for dynamic systems.

To address the time-varying failure rate estimation for dynamic systems, we propose a simulation-based method using the SPA and a copula function (SPA/Copula). The planning horizon $[0, T]$ is partitioned into a series of N time intervals $[0, T_1]$, $[T_1, T_2], \ldots, [T_{n-1}, T_n], \ldots, [T_{N-1}, T_N]$ using a time step $\Delta T = T_n - T_{n-1}$ which is a multiple of Δt so that $\Delta T = k\Delta t$ where the integer k can be between 20 and 250. The SPA estimates the CDF of maximum response in each time interval and copula functions estimate the dependence of maximum response among all time intervals.

Without loss of generality, we define the time-dependent performance function as $L(t) = g(\mathbf{X}(t), t)$. A series of N failure regions are then formed and defined by the following events:

$$E_n^{\text{int}} = \left\{\max_{t \in [T_{n-1}, T_n]} g(\mathbf{X}(t), t) \ge S_t\right\}\ (n = 1, \ldots, N; T_0 = 0) \tag{14}$$

where $g : \Re^p \to \Re$ is a function mapping $\mathbf{X}(t)$ to a response g and S_t is a predetermined threshold. If $g(\mathbf{X}(t), t) \ge S_t, \exists t \in [T_{n-1}, T_n]$, the system is considered failed while if $g(\mathbf{X}(t), t) < S_t$, $\forall t \in [T_{n-1}, T_n]$, the system operates safely. The superscript "int" (first three letters of interval) in Eq. (14) indicates time interval. The equation $\max_{t \in [T_{n-1}, T_n]} g(\mathbf{X}(t), t) - S_t = 0$ defines the time-dependent limit state surface. Since the event E_n^{int} is time-dependent, time-dependent reliability methods should be used to calculate the probability $\Pr\{E_n^{\text{int}}\}$. Time series is first used to characterize the input random processes, and the SPA using the first four cumulants, is subsequently employed to estimate the CDF of maximum response within each time interval in order to calculate $\Pr\{E_n^{\text{int}}\}$.

Because the event E_n^{int} depends on events $E_{n_i}^{\text{int}} (n_i = 1, \ldots, n-1)$, we define N conditional failure events

$$E_{cn}^{\text{int}} = \left\{ \max_{t \in [T_{n-1}, T_n]} g(\mathbf{X}(t), t) \geq S_t \,|\, \bar{E}_{n-1} \right\} (n = 1, \ldots, N; T_0 = 0) \quad (15)$$

where

$$\bar{E}_{n-1} = \left\{ \max_{t \in [0, T_{n-1}]} g(\mathbf{X}(t), t) < S_t \right\} (n = 1, \ldots, N; T_0 = 0) \quad (16)$$

is the safe event over the time interval $[0, T_{n-1}]$. The subscript "c" in E_{cn}^{int} (Eq. (15)) stands for "conditional," while the subscript "n" is the time interval counter.

The mean time-varying failure rate over the time interval $[T_{n-1}, T_n]$ is provided by

$$\lambda(T_{n-1}, T_n) = \frac{\Pr\{E_{cn}^{\text{int}}\}}{T_n - T_{n-1}} \quad (17)$$

If $T_n - T_{n-1}$ is sufficiently small, $\lambda(T_{n-1}, T_n)$ is the instantaneous failure rate at time instant T_{n-1}.

Using the conditional probability definition, $\Pr\{E_{cn}^{\text{int}}\}$ is equal to $\Pr\{E_n^{\text{int}}\bar{E}_{n-1}\}/\Pr\{\bar{E}_{n-1}\}$ and Eq. (17) can be rewritten as

$$\lambda(T_{n-1}, T_n) = \frac{\Pr\{E_n^{\text{int}}\bar{E}_{n-1}\}}{\Pr\{\bar{E}_{n-1}\}(T_n - T_{n-1})} \quad (18)$$

where $\Pr\{\bar{E}_{n-1}\}$ is calculated recursively using the copula function as

$$\Pr\{\bar{E}_{n-1}\} = 1 - \Pr\{E_{n-1}\}(n = 2, \ldots, N; Pr\{E_0\} = 0) \quad (19)$$

where

$$\Pr\{E_{n-1}\} = [\Pr\{E_{n-2}\} + \Pr\{E_{n-1}^{\text{int}}\} - C(\Pr\{E_{n-1}^{\text{int}}\}, \Pr\{E_{n-2}\})] \quad (20)$$

Note that $\Pr\{E_n^{\text{int}}\bar{E}_{n-1}\}$ is expressed in terms of $\Pr(E_n^{\text{int}})$ and $\Pr\{\bar{E}_{n-1}\}$ as

$$\Pr\{E_n^{\text{int}}\bar{E}_{n-1}\} = C(\Pr(E_n^{\text{int}}), \Pr(\bar{E}_{n-1})) \quad (21)$$

Equations (18)–(21) are used sequentially for all time intervals. For each cycle (new time interval), the SPA estimates the CDF of maximum response to calculate $\Pr\{E_n^{\text{int}}\}$, and a copula function provides the dependence between E_n^{int} and $\bar{E}_{n-1}$. We can then calculate a discrete mean time-varying failure rate for each time interval.

Based on the statistics of maximum response, we use a validated Weibull distribution to fit the discrete cumulative probabilities of failure with a continuous CDF and thus obtain a continuous time-varying failure rate. This failure rate is calculated using the common definition as the conditional probability of failure over a small time interval dt after time t under the condition that the system did not fail before time t. We call this definition type I failure rate.

The copula $C(\Pr\{E_{n-1}^{\text{int}}\}, \Pr\{E_{n-2}\})$ in Eq. (20) represents the probability that samples failed in both time intervals $[0, T_{n-2}]$ and $[T_{n-2}, T_{n-1}]$. It appears because the failed samples still remain in the operating queue during the simulation. The copula can be viewed as an index describing the failure correlation as time progresses. This is similar to the failed items still remaining in the operating queue after engineering maintenance. During simulation, we assume that after failure, performance is restored to the state just before failure occurred (good-as-old repair assumption). In order to compare the failure evolution of the repaired and nonrepaired samples, we propose the concept of time-varying failure rate for repaired samples. A discrete mean time-varying failure rate over the time interval $[T_{n-1}, T_n]$ is defined as

$$\lambda^r(T_{n-1}, T_n) = \frac{C(\Pr\{E_n^{\text{int}}\}, \Pr\{E_{n-1}\})}{\Pr\{E_{n-1}\}(T_n - T_{n-1})} \quad (22)$$

which we call type II failure rate. It provides a new way to predict the failure rate by essentially building a relationship between the system failure rate and the capacity for maintenance of repairable dynamic systems. Also, the calculation of type II failure rate does not require additional computational effort.

Section 3.2 provides step-by-step details of the proposed approach.

3.2 Procedure.

The proposed method consists of two stages. In the first stage, we calculate the probabilities of failure over a series of time intervals. The second stage builds the time-dependent copula function and estimates the type I and type II failure rates. Figure 2 summarizes all steps of the proposed method. Details are provided below.

Stage 1: Estimation of probabilities of failure $\Pr\{E_n^{\text{int}}\}$ over the N time intervals $[T_{n-1}, T_n](n = 1, \ldots, N; T_0 = 0)$.

In this stage, we calculate the probability $\Pr\{E_n^{\text{int}}\}$ over the time interval $[T_{n-1}, T_n](n = 1, \ldots, N; T_0 = 0)$ using the SPA. $\Pr\{E_n^{\text{int}}\}$ is used in Eqs. (20) and (21). The process consists of the following six steps.

Step 1: Construct the time-dependent performance function $L(t) = g(\mathbf{X}(t), t)$.

Step 2: Characterize each input stochastic process.

We use time series model to characterize a stochastic process. Many time series models are available including autoregressive (AR), moving average, autoregressive moving average, and autoregressive integrated moving average models [42–44]. In this work, we use AR models.

An input stochastic process $X(t)$ is represented as a collection of random variables $X(t_i)$ at the discrete times $t_i = i\Delta t,\ i = 0, 1, 2, \ldots, N_1;\ t_0 = 0,\ t_{N_1} = T$. The time step Δt is much smaller than $\Delta T = T_n - T_{n-1}$ resulting in $N_1 \gg N$. For an AR(p) model of order p, the discretized sample function (trajectory) is represented as

$$x(t_i) - \mu = \varphi_1(x(t_{i-1}) - \mu) + \varphi_2(x(t_{i-2}) - \mu) + \cdots + \varphi_p(x(t_{i-p}) - \mu) + \varepsilon_i \quad (23)$$

where $x(t_i)$ is the value of the sample function at time t_i, μ is the temporal mean of the stochastic process $X(t)$, $\varepsilon_i \sim N(0, \sigma_e^2)$ is Gaussian white noise, and $\varphi_1, \varphi_2, \ldots, \varphi_p$ are feedback parameters to be estimated. For the AR(2) model, the variance σ_e^2 can be determined by

$$\sigma_e^2 = \gamma(0)(1 - \varphi_1\rho_1 - \varphi_2\rho_2 - \cdots - \varphi_p\rho_p) \quad (24)$$

where $\gamma(0)$ is the variance of the stochastic process and ρ_i $(i = 1, 2, \ldots, p)$ is the value of the autocorrelation function at time lag $\tau = i\Delta t$. More details are provided in Ref. [43].

Step 3: Partition time of interest $[0, T]$.

The time of interest $[0, T]$ is partitioned into a series of N time intervals $[T_{n-1}, T_n],\ n = 1, 2, \ldots, N$ using a time step $\Delta T = T_n - T_{n-1}$ which is a multiple of Δt.

Step 4: Calculate K trajectories of the limit state function over each time interval $[T_{n-1}, T_n]$.

We generate K sample trajectories of the limit state function $L(t) = g(\mathbf{X}(t), t)$ by solving the discretized dynamic equations of motion $\mathbf{X}(t + \Delta t) = h(\mathbf{X}(t), \mathbf{Z}, \mathbf{M}(t), t),\ T_{n-1} \leq t \leq T_n,\ K$ times using the Δt time step. Each trajectory $l^j(t),\ j = 1, \ldots, K$ is represented by $NP = (T_n - T_{n-1}/\Delta t) + 1$ values $l^j(t_i)$ at time instants $t_i = T_{n-1} + (i-1)\Delta t,\ i = 1, \ldots, NP$.

Step 5: Calculate extreme values over each time interval $[T_{n-1}, T_n]$.

For the nth time interval, the extreme value r_n^j of the jth trajectory from step 4, is

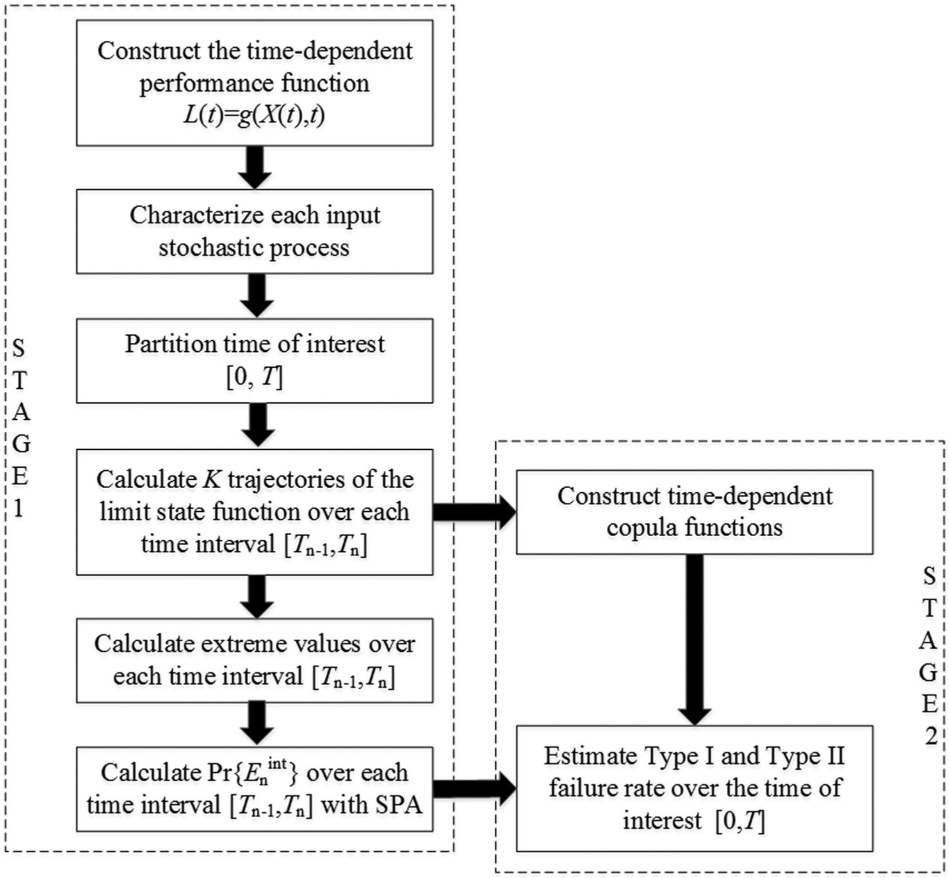

Fig. 2 Flowchart of the proposed method

$$r_n^j = \max_{i=1,\ldots,NP}(l^j(t_i)),\ j=1,\ldots,K,\ n=1,\ldots,N \tag{25}$$

The set of the K extreme values is represented as

$$r_n = \{r_n^1 \quad \cdots \quad r_n^K\} \tag{26}$$

Step 6: Calculate $\Pr\{E_n^{\text{int}}\}$ over each time interval $[T_{n-1}, T_n]$ with SPA.

The failure threshold S_t replaces y in Eq. (6) resulting in

$$K'(\xi) = S_t \tag{27}$$

Equations (8)–(10) are then used to calculate $\Pr\{E_n^{\text{int}}\}$.

Stage 2: Estimation of type I and type II failure rates over the planning horizon $[0, T]$.

In this stage, we build time-dependent copula functions using the same K trajectories from step 4 of stage 1, and use them sequentially to estimate type I and type II failure rates throughout the planning horizon using Eqs. (18)–(22). The process consists of the following two steps.

Step 1: Construct time-dependent copula functions.

In this step, we characterize the dependence of extreme values between time intervals $[T_{n-1}, T_n]$ and $[0, T_{n-1}]$. Note that in stage 1, we do not account for the dependence of extreme values of the performance function $L(t)$ among different time intervals.

We only consider the correlation of the extreme values between time intervals $[0, T_{n-1}]$ and $[T_{n-1}, T_n]$. This captures the correlation between the two time intervals by simply changing the number n. A suitable copula is selected among commonly used bivariate copulas such as Clayton, Gumbel, Frank, Gaussian, Ali-Mikhail-Haq (AMH), Farlie–Gumbel–Morgenstern (FGM), Arch 12, or Arch 14 and a Bayesian approach where the normalized weights are calculated as in Ref. [37]. The Clayton copula, one of the bivariate Archimedean copulas, is selected for the vehicle vibration example (Fig. 5 in Sec. 4).

In order to distinguish time intervals $[0, T_{n-1}]$ and $[T_{n-1}, T_n]$, we use subscripts $\Sigma(n-1)$ and n to represent the time intervals $[0, T_{n-1}]$ and $[T_{n-1}, T_n]$. Therefore, $r^j_{\Sigma(n-1)}$ and $r_{\Sigma(n-1)}$ represent the jth extreme value and the set of K extreme values over the time interval $[0, T_{n-1}]$. Similarly for the time interval $[T_{n-1}, T_n]$, r_n^j is the jth extreme value and r_n is the set of K extreme values.

A semiparametric estimation method is used to determine the copula-based model parameters based on $r_{\Sigma(n-1)}$ and r_n in two steps. The first step estimates the marginal distribution functions using a nonparametric estimation method. The marginal distribution function of extreme values $U_{\Sigma(n-1)}$ over the time interval $[0, T_{n-1}]$ is empirically estimated as [45]

$$\widehat{U}_{\Sigma(n-1)}\left(r^j_{\Sigma(n-1)}\right) = \frac{1}{K+1}\sum_i^K I\left(r^i_{\Sigma(n-1)} \le r^j_{\Sigma(n-1)}\right) \tag{28}$$

where

$$\begin{cases} I(r^i_{\Sigma(n-1)} \le r^j_{\Sigma(n-1)}) = 1 \text{ if } r^i_{\Sigma(n-1)} \le r^j_{\Sigma(n-1)} \\ I(r^i_{\Sigma(n-1)} \le r^j_{\Sigma(n-1)}) = 0 \text{ if } r^i_{\Sigma(n-1)} > r^j_{\Sigma(n-1)} \end{cases} \tag{29}$$

Similarly, the marginal distribution function of extreme values U_n over the time interval $[T_{n-1}, T_n]$ is estimated as

$$\widehat{U}_n\left(r_n^j\right) = \frac{1}{K+1}\sum_i^K I\left(r_n^i \le r_n^j\right) \tag{30}$$

where

$$\begin{cases} I(r_n^i \le r_n^j) = 1 \text{ if } r_n^i \le r_n^j \\ I(r_n^i \le r_n^j) = 0 \text{ if } r_n^i > r_n^j \end{cases} \quad (31)$$

The second step estimates the copula parameter $\hat{\theta}_{n-1}$ using the maximum likelihood method. Given the nonparametric estimators $\widehat{U}_{\Sigma(n-1)}$ and $\widehat{U}_n$, the copula parameter $\hat{\theta}_{n-1}$ is estimated as [46]

$$\hat{\theta}_{n-1} = \arg\max_{\theta_{n-1}} \sum_{j=1}^{K} \log c(U_{\Sigma(n-1)}^{(j)}, U_n^{(j)}; \theta_{n-1}) \quad (32)$$

For the vehicle vibration example of Sec. 4, the Clayton copula is employed to describe the correlation as [25]

$$C\left(U_{\sum(n-1)}, U_n|\theta_{n-1}\right) = \left(\left(U_{\sum(n-1)}\right)^{-\theta_{n-1}} + (U_n)^{-\theta_{n-1}} - 1\right)^{-1/\theta_{n-1}} \quad (33)$$

where $\theta_{n-1} > 0$. Its corresponding density is provided by

$$\begin{aligned} &c\left(U_{\sum(n-1)}, U_n; \theta_{n-1}\right) \\ &= (\theta_{n-1}+1)\left(U_{\sum(n-1)}U_n\right)^{-(\theta_{n-1}+1)} \\ &\cdot\left(\left(U_{\sum(n-1)}\right)^{-\theta_{n-1}} + (U_n)^{-\theta_{n-1}} - 1\right)^{-1/\theta_{n-1}-2} \end{aligned} \quad (34)$$

Each copula parameter $\hat{\theta}_{n-1} (n = 2, \ldots, N)$ is estimated by repeating this procedure for time intervals $[0, T_{n-1}]$ and $[T_{n-1}, T_n] (n = 2, \ldots, N)$.

Step 2: Estimate type I and type II time-varying failure rates over the planning horizon $[0, T]$.

The discretized time-varying failure rate over the planning horizon $[0, T]$ is determined from type I failure rates $\lambda(T_{n-1}, T_n)$, $n = 1, 2, \ldots, N$. The failure rate $\lambda(T_{n-1}, T_n)$ over each time interval $[T_{n-1}, T_n]$ is calculated using Eqs. (18)–(21). $\Pr\{E_{cn}^{\text{int}}\}$ is calculated from the joint probability $\Pr\{E_n^{\text{int}}\bar{E}_{n-1}\}$ using the established copula function from step 1 of stage 2, the $\Pr\{E_n^{\text{int}}\}$ from stage 1, and the safe event probability $\Pr\{\bar{E}_{n-1}\}$ calculated using Eq. (19). Finally, Eq. (17) provides the failure rate $\lambda(T_{n-1}, T_n)$.

For dynamic systems, the extreme values of performance are usually Weibull distributed. The parameters of Weibull distribution are estimated by fitting the discrete cumulative probabilities of failure using least squares regression. A continuous time-varying failure rate is then derived from the estimated Weibull distribution.

Finally, the discretized time-varying failure rate for repaired samples over the planning horizon $[0, T]$, is determined from type II failure rates $\lambda^r(T_{n-1}, T_n)$, $n = 1, 2, \ldots, N$. The latter are calculated in Eq. (22), using the established copula function from step 1 of stage 2, $\Pr\{E_n^{\text{int}}\}$ from stage 1, and the failure event probability $\Pr\{E_{n-1}\}$ from Eq. (20).

4 A Vehicle Vibration Example

The vehicle vibration example of Fig. 3 is used to illustrate the proposed method. The vehicle travels over a stochastic terrain with a speed of 20 mile per hour (mph). There are two random variables representing the random parameters of the system and a random process $M(t)$ representing the road excitation. The two random variables are the damping coefficient b_s and the stiffness k_s. Both are normally distributed as $b_s \sim N(7000,\ 1400^2)\text{N/m/s}$ and $k_s \sim N(40,000,\ 4000^2)$ N/m, respectively. The sprung mass m_s, unsprung mass m_u, tire stiffness k_t, and tire damping b_t are deterministic parameters with $m_s = 100\,\text{kg}$, $m_u = 100\,\text{kg}$, $k_t = 40 \times 10^4\,\text{N/m}$, and $b_t = 4 \times 10^3\,\text{N/m/s}$.

We use the state-space approach to determine the vertical acceleration response $S(\mathbf{d}, \mathbf{X}, t) = \ddot{x}_s(t)$ of the sprung mass in g's.

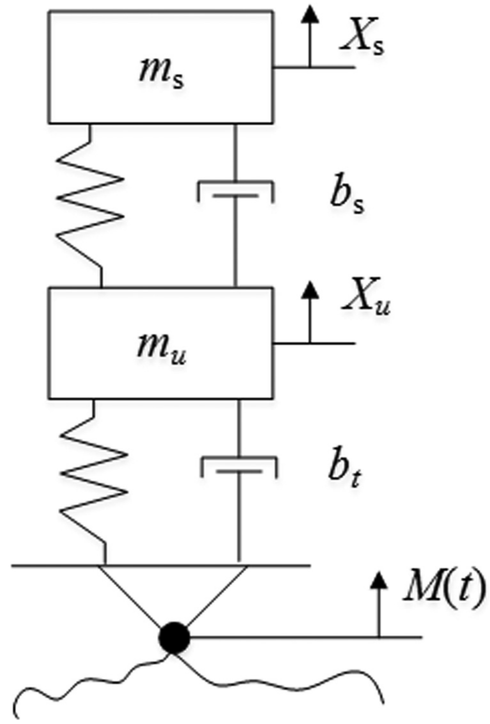

Fig. 3 Vehicle vibration model

Failure occurs if the magnitude of the vertical acceleration exceeds a threshold, $|S(\mathbf{d}, \mathbf{X}, t)| \ge S_t$ where $S_t = 3.5$.

A third-order autoregressive time-series model AR(3) represents the stochastic road elevation process $M(t)$ as

$$\begin{aligned} m(t_i) &= 1.2456m(t_{i-1}) - 0.2976m(t_{i-2}) - 0.1954m(t_{i-3}) \\ &\quad + \varepsilon_i(0, 0.5132^2) \end{aligned}$$

where $\varepsilon_i(0, 0.5132^2)$ is a Gaussian white noise with a standard deviation of 0.5132. The coefficients 1.2456, −0.2976, and −0.1954 are the three estimated feedback parameters.

The equations of motion

$$m_u\ddot{x}_u + (b_t + b_s)\dot{x}_u - b_s\dot{x}_s + (k_t + k_s)x_u - k_s x_s = k_t m + b_t\dot{m}$$

and

$$m_s\ddot{x}_s + b_s(\dot{x}_s - \dot{x}_u) + k_s(x_s - x_u) = 0$$

are transformed to a state-space form and the Runge–Kutta method is used to integrate them in time. The time step is $\Delta t = 0.01$ s. The same time step is used for the discretization of the stochastic road elevation.

The planning horizon is $T = 300s$ resulting in $T/\Delta t = 300/0.01 = 30,000$ random variables. The time interval is $\Delta T = T_n - T_{n-1} = 2s$ resulting in $(\Delta T/\Delta t) = (2/0.01) = 200$ random variables per time interval. We use 1000 trajectories to estimate the probabilities of failure for each time interval $[T_{n-1}, T_n]$ $(n = 1, \ldots, 150; T_0 = 0)$ using SPA.

Figure 4 shows the cumulative probability of failure by integrating the probabilities of failure over each time interval without/with considering the correlation among them based on Monte Carlo simulation (MCS) with 1,000,000 samples. We observe that the cumulative probability of failure from MCS without considering correlation is greater than that with correlation. The reason is that an item could fail several times during the planning horizon but the population of items is fixed if the correlation is not considered, leading to an increase of probability of failure. The

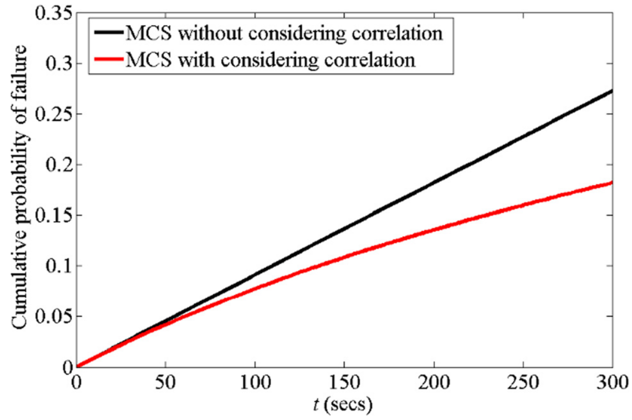

Fig. 4 Estimated cumulative probability of failure from MCS with/without considering correlation

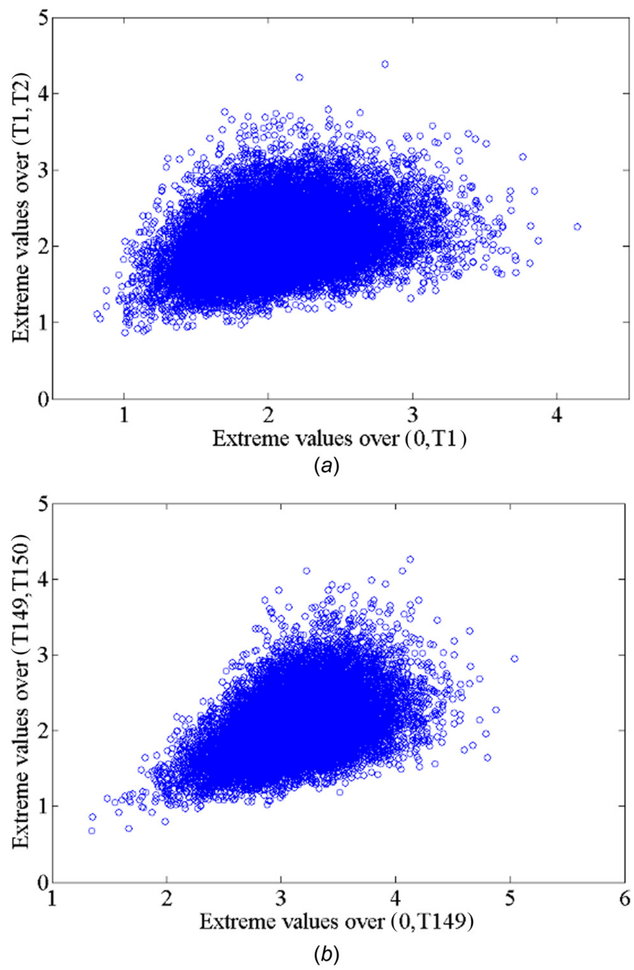

Fig. 5 Scatter diagram for first ($r_{\Sigma 1}$ and r_2) (*a*) and last ($r_{\Sigma 149}$ and r_{150}) (*b*) data pairs

difference with and without correlation varies depending on the correlation. Therefore, the correlation can be used as an index to illustrate failure dependence for dynamic systems as time progresses.

Because there are 150 time intervals, $(T/\Delta T) - 1 = (300/2) - 1 = 149$ copula functions must be estimated to represent the time-dependent correlation. A data pair is defined by $r_{\Sigma(n-1)}$ and r_n. Figure 5 shows the scatter diagram for data pairs $r_{\Sigma 1}$ and r_2 (first pair), and $r_{\Sigma 149}$ and r_{150} (last pair), respectively. To select the proper copula among Clayton, Gumbel, Frank, Gaussian, Ali-Mikhail-Haq (AMH), Farlie–Gumbel– Morgenstern (FGM), Arch 12, and Arch 14, a Bayesian approach is used based on 1000 simulated trajectories. We find that the Clayton copula is suitable for this example. Figure 6 shows the estimated relative weights of the Bayesian approach for data pairs $r_{\Sigma 1}$ and r_2, and $r_{\Sigma 149}$ and r_{150} verifying the selection of Clayton copula.

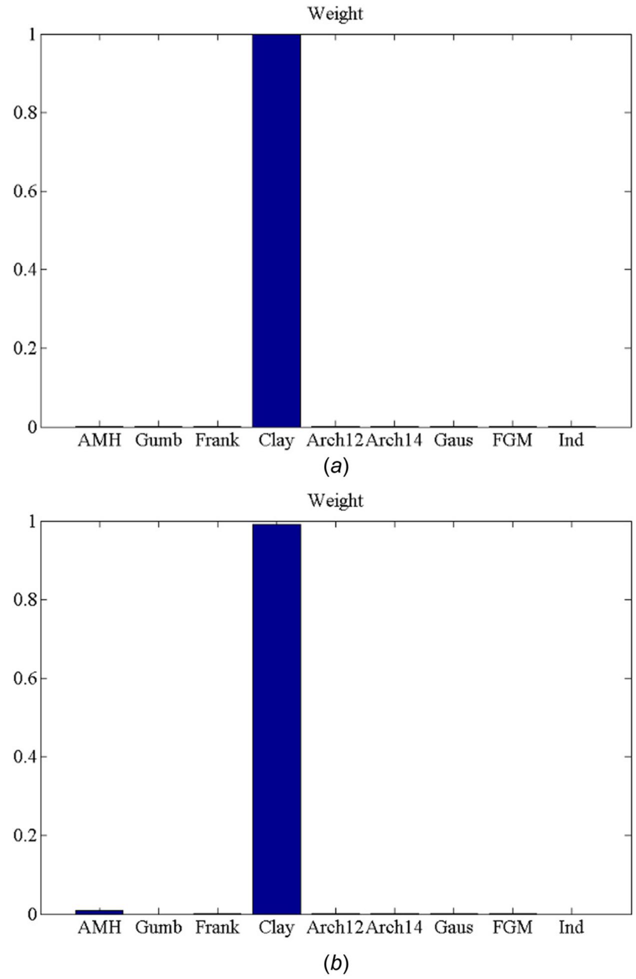

Fig. 6 Copula weights results for first (r_1 and r_2) (*a*) and last ($r_{\Sigma 149}$ and $r_{\Sigma 150}$) (*b*) data pairs

After selecting the Clayton copula, we use the same 1000 trajectories to build the time-dependent copula functions among different data pairs and to subsequently estimate the time-varying failure rate. Figure 7 shows the Clayton copula parameter for all data pairs showing an increasing trend as time progresses which indicates that the extreme values over the time interval $[T_{n-1}, T_n]$ are not only correlated with those over the previous time interval $[T_{n-2}, T_{n-1}]$ but also with those over the time intervals $[0, T_1], [T_1, T_2], \ldots, [T_{n-3}, T_{n-2}]$. This is in contrast to an intuitive expectation that the extreme values over the time interval $[T_{n-1}, T_n]$ are correlated only with those over the previous time interval $[T_{n-2}, T_{n-1}]$.

Figure 8 shows the cumulative probability of failure from Eq. (20) using the probabilities of failure $\Pr(E_n^{\text{int}})$ and the time-dependent Clayton copula function. Four different runs are performed to illustrate its variability. The figure also shows the average cumulative probability of failure of the four runs. The estimated cumulative probability of failure from MCS with 1,000,000 trajectories is used as a reference for accuracy

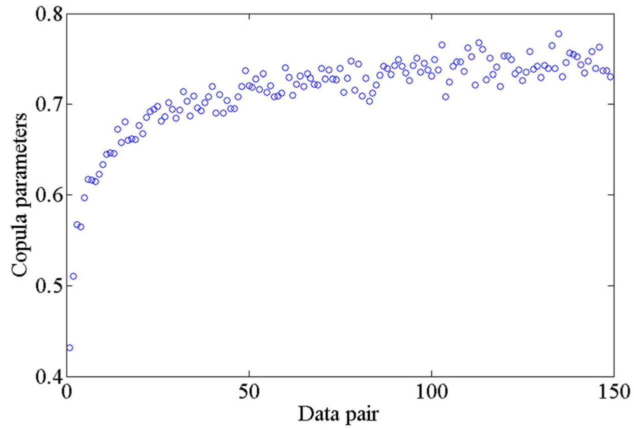

Fig. 7 Parameters of time-dependent Clayton copula functions

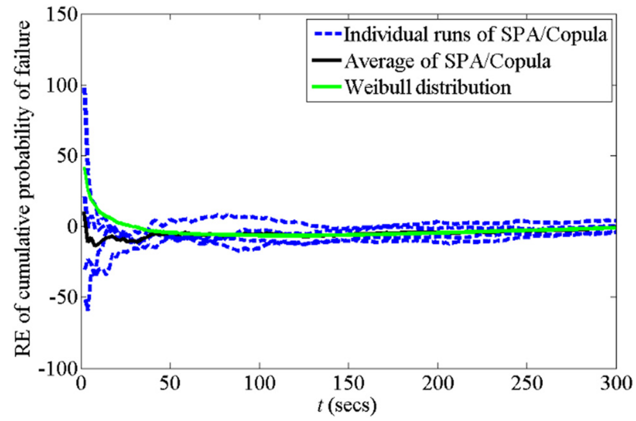

Fig. 10 RE of cumulative probability of failure. RE means relative error.

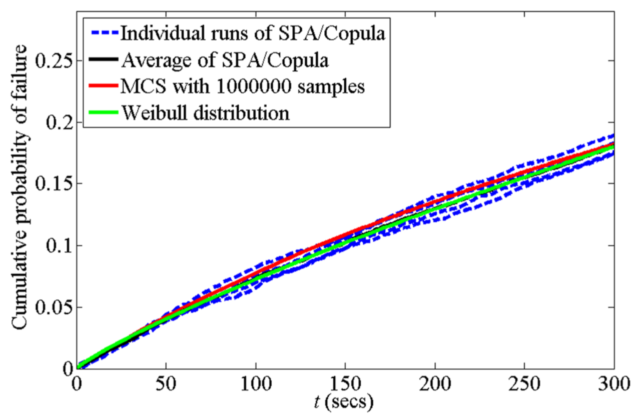

Fig. 8 Estimated cumulative probability of failure

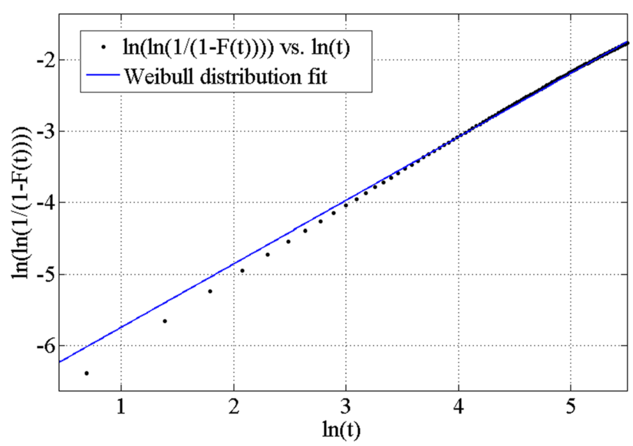

Fig. 9 Kolmogorov–Smirnov goodness of fit

comparison. To obtain a smooth estimation of the failure rate, we use a Weibull distribution to fit the cumulative probabilities of failure calculated from Eq. (20). We observe that the cumulative probability of failure from Weibull distribution and the average cumulative probability of failure from SPA are very close to the MCS estimate.

To test the suitability of the Weibull distribution, Fig. 9 shows the Kolmogorov–Smirnov goodness of fit. The Weibull CDF is transformed to a linear model and the $\ln[1/(1-F(t))]$ is plotted against $\ln t$ where $F(t)$ is the cumulative probability of failure using the conditional probabilities and the time-dependent copula function. We see that the Weibull distribution is a good fit. The goodness of fit indices sum of squares due to error, R-square, adjusted R-square, and root mean squared error are equal to 0.01009, 0.9999, 0.9999, and 0.008229, respectively, indicating that the Weibull distribution is suitable for the lifetime of the dynamic system. The two Weibull parameters are $\alpha = 0.8828$ and $\beta = 775.9$.

Table 1 RE of cumulative probability of failure

	RE of cumulative probability of failure (%)					
Time (s)	Run 1	Run 2	Run 3	Run 4	Average	Weibull
2	100.1	49.76	22.97	32.23	10.26	41.61
4	39.85	59.38	0.93	23.36	10.95	25.02
6	18.84	32.55	7.95	26.57	8.08	18.42
8	15.45	33.60	2.43	35.62	11.05	14.56
10	1.52	17.51	2.79	30.67	10.36	10.51
12	5.70	15.28	0.43	29.73	9.93	8.92
14	5.19	4.35	0.72	34.19	8.16	7.08
16	1.45	0.11	0.47	29.21	7.76	6.25
…	…	…	…	…	…	…
288	5.14	3.75	0.01	4.18	1.42	1.46
290	4.95	3.62	0.13	4.48	1.49	1.37
292	4.83	4.13	0.44	4.32	1.15	1.31
294	4.72	4.12	0.30	3.96	1.06	1.22
296	4.45	3.98	0.46	3.54	0.89	1.15
298	4.03	4.05	0.33	3.30	0.74	1.08
300	3.91	4.11	0.31	3.20	0.67	1.03

Figure 10 and Table 1 show the relative error (RE) of the cumulative probability of failure from SPA/Copula and Weibull distribution methods with respect to MCS. The relative error decreases, reaching an asymptotic value of approximately 1%.

Figure 11 shows the corresponding mean failure rate for each time interval $[T_{n-1}, T_n]$ from four individual runs using Eqs. (17)–(21). It also shows the average curve of the four individual runs. The solid curve shows the smooth time-varying failure rate from the estimated CDF of Weibull distribution. The estimated mean time-varying failure rate from MCS with 1,000,000 trajectories is used as a reference for accuracy comparison. We observe that the estimated time-varying failure rate oscillates around the reference provided by MCS with 1,000,000 trajectories.

Figure 12 and Table 2 show the relative error (RE) of type I failure rate using the SPA/Copula and Weibull distribution methods with respect to MCS. The relative error from Weibull distribution is less than approximately 10%, which is deemed acceptable.

Figure 13 shows the estimated type II failure rate over each time interval $[T_{n-1}, T_n]$ from four individual runs using Eq. (22). It also provides the average of the four individual runs. Type II

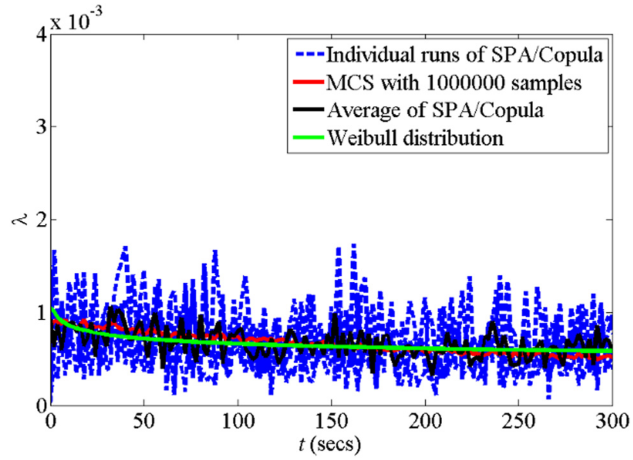

Fig. 11 Estimated type I failure rate $\lambda(t)$

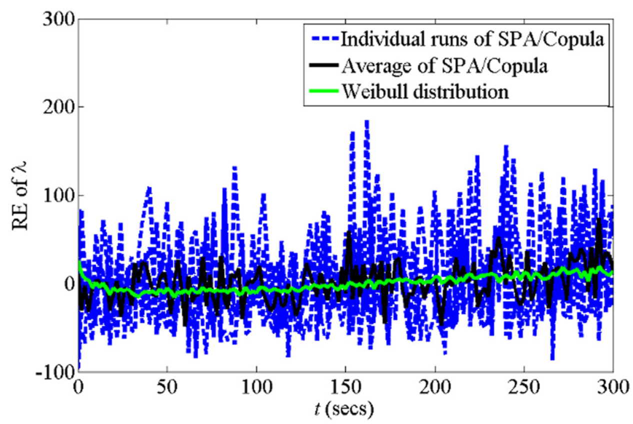

Fig. 12 RE of type I failure rate $\lambda(t)$. RE means relative error.

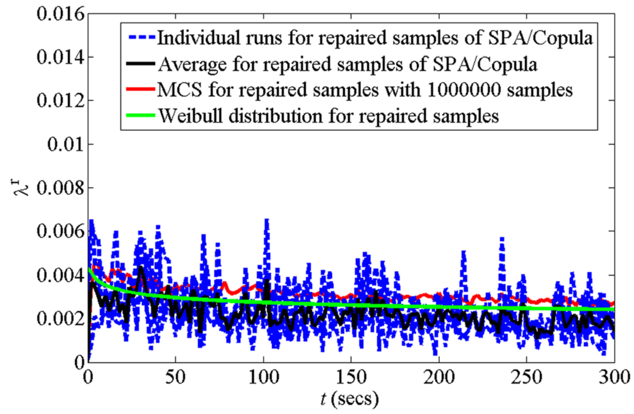

Fig. 13 Estimated type II failure rate $\lambda^r(t)$

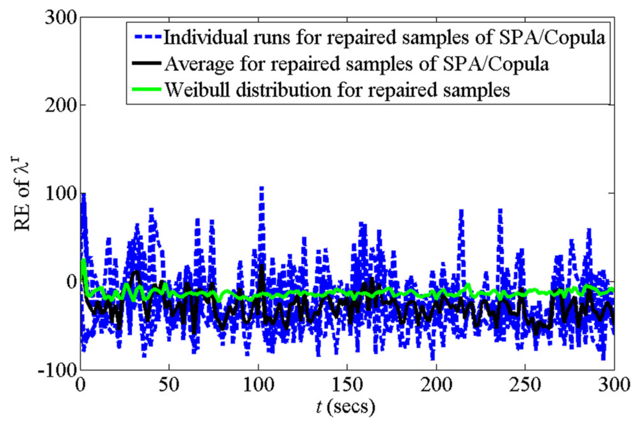

Fig. 14 RE of type II failure rate $\lambda^r(t)$

Table 2 RE of type I failure rate $\lambda(t)$

	RE of the type I failure rate $\lambda(t)$ (%)					
Time (s)	Run 1	Run 2	Run 3	Run 4	Average	Weibull
2	100.1	49.76	22.97	32.23	10.26	25.16
4	15.45	68.26	22.89	15.23	30.46	10.52
6	22.37	19.95	25.40	32.94	2.49	5.82
8	5.35	36.89	33.75	62.97	32.06	3.21
10	51.74	43.73	4.19	11.95	6.04	4.75
12	27.63	3.72	11.92	25.02	2.70	0.80
14	2.10	61.78	7.74	61.43	2.55	3.95
16	51.33	33.59	9.46	7.84	4.84	2.28
…	…	…	…	…	…	…
288	43.15	35.92	15.89	139.1	40.56	13.89
290	31.44	21.06	7.40	67.61	16.16	2.79
292	20.41	118.0	126.0	29.87	43.57	16.20
294	17.18	3.420	30.33	72.82	15.77	11.93
296	50.68	26.22	34.44	82.17	35.27	14.18
298	82.44	19.31	26.94	45.39	30.05	11.32
300	20.77	18.89	3.64	16.52	13.13	9.85

failure rate from MCS with 1,000,000 trajectories is used as a reference for accuracy comparison. The estimated time-varying failure rate oscillates around the MCS reference with 1,000,000 trajectories. Type II failure rate is similar with that of type I. The Weibull distribution is also used to fit the SPA/Copula average failure rate results.

Figure 14 shows the relative error of type II failure rate using SPA/Copula with MCS. The relative error of average and Weibull distribution is around 20%, which is greater than that of type I failure rate. The reason is that the denominator $\Pr\{E_{n-1}\}$ in Eq. (21) is much smaller than $\Pr\{\bar{E}_{n-1}\}$ and more samples are needed to improve the computational accuracy.

Comparison of type I and type II failure rates in Figs. 11 and 13 show that type II failure rate is larger. This means that the repaired samples have a greater risk of failure after a good-as-old maintenance is performed.

5 Summary and Conclusions

We propose a simulation-based method to estimate type I and type II failure rates of dynamic systems in this paper. The trajectories of the output random process are simulated at discrete times. The time of interest is partitioned into a series of time intervals in order to study the time-dependent failure correlation. The SPA is used to estimate accurately the small probabilities of failure over each time interval using a small number of simulated trajectories (1000 trajectories for probability of failure 0.0017). A time-dependent copula function to describe the failure correlation, is built with the same trajectories. Using a time-dependent copula function, we propose methods for two types of time-varying failure rate (type I and type II). Type I failure rate is based on the commonly used definition of failure rate, while type II is a new concept to describe the failure evolution of repaired samples considering a good-as-old virtual maintenance. A vehicle vibration example demonstrates that the proposed approach is computationally efficient and accurate for both type I and type II failure rate estimation of dynamic systems.

Future research will concentrate on improving the accuracy of type II failure rate and on using the concept of type II failure rate in maintenance scheduling.

Acknowledgment

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