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Bayesian framework for probabilistic low cycle fatigue life prediction and uncertainty modeling of aircraft turbine disk alloys

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ABSTRACT

Probabilistic life prediction of aircraft turbine disks requires the modeling of multiple complex random phenomena. Through combining test data with technological knowledge available from theoretical analyses and/or previous experimental data, the Bayesian approach gives a more complete estimate and provides a formal updating approach that leads to better results, save time and cost. The present paper aims to develop a Bayesian framework for probabilistic low cycle fatigue (LCF) life prediction and quantify the uncertainty of material properties, total inputs and model uncertainty resulting from choices of different deterministic models in a LCF regime. Further, based on experimental data of turbine disk material (Ni-base superalloy GH4133) tested at various temperatures, the capabilities of the proposed Bayesian framework were verified using four fatigue models (the viscosity-based model, generalized damage parameter, Smith–Watson–Topper (SWT) and plastic strain energy density (PSED)). By updating the input parameters with new data, this Bayesian framework provides more valuable performance information and uncertainty bounds. The results showed that the predicted distributions of fatigue life agree well with the experimental data. Further it was shown that the viscosity-based model and the SWT model yield more satisfactory probabilistic life prediction results for GH4133 under different temperatures than the generalized damage parameter and PSED ones based on the same available knowledge.

1. Introduction

As a critical flight safety component of aircraft engines, the turbine disk is subjected to high temperature, corrosive and oxidative conditions, and its failure often leads to catastrophic results. Low cycle fatigue (LCF) at high temperature is a key failure mode of these turbine disks. Along with the requirement for high thrust-weight ratio and reliability of aircraft engines, the designed stress level of turbine disks has been greatly increased. The need to reduce their aging maintenance cost and downtime drives the increasing attention to the probabilistic life prediction methods. All these factors generate new challenges to accurately predict the LCF life of turbine disks, thus, a general probabilistic LCF life prediction framework is worthwhile to establish.

The fatigue life shows a random behavior, and is affected by uncertainties regarding the following items: material properties, model errors, parameter estimates, load variation and structural component properties in engineering [1]. Therefore, the uncertainties due to these sources should be addressed directly for life prediction. Compared with deterministic analyses, probabilistic methods model the load variation and input parameters as distributions and predict distributions of performance. By quantifying the corresponding probability distributions, the uncertainty is propagated through the probabilistic-based model to predict the probability distribution of fatigue life. In principle, the case model parameters formalized, as a prior credibility using Bayes' theorem based on the available knowledge will make a more accurate fatigue life prediction. Probabilistic methods have recently been widely used to account for the uncertainty in the fatigue life prediction of structures or materials, including fatigue crack propagation [2–4], simulation of stress–strain level of turbine disk using finite element analysis [5], stress-life (S-N) fatigue data analysis [6] and structural reliability modeling using Bayesian updating [7], probabilistic fatigue life prediction using DARWIN [8–10] and AFGROW software [11] and accounting for model uncertainty [12] and considering microstructure [13] and considering damaging and strengthening of low amplitude loads [14]. However, few attempts have been made in the past to consider the uncertainty of total inputs and model uncertainty in a LCF regime. As engineering structural systems become more complex, the dependence of structural analysis on physics-based model

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Nomenclature

b_p	mean, error of model to the real value
b_t	mean, error of experiment to the real value
C	material constants representing the material energy absorption capacity
$\mathbf{D}$	vector of data
E	Young's modulus
F_p	multiplicative error of model to the real value
F_t	multiplicative error of experiment to the real value
F_{pt}	multiplicative error of experiment to the model prediction
$L(.)$	likelihood function
$LN(.)$	lognormal distribution function
N_f	number of cycles to failure
$\tilde{N}_f$	mean prediction life
N_{fp}	model prediction
N_{ft}	experimental result
N_{real}	real fatigue life
n'	cyclic strain hardening exponent
s	model parameter, natural logarithm standard deviation of life cycles
s_p	standard deviation, error of model to the real value
s_t	standard deviation, error of experiment to the real value
R_ε	strain ratio
α	material constant representing the fatigue exponent
ϕ	material constant
$\Delta\varepsilon_t$	total strain range
$\Delta\varepsilon_p$	plastic strain range
$\Delta\varepsilon_{in}$	inelastic strain range
$\dot{\varepsilon}$	strain rate
υ_d	dynamic viscosity
σ_{max}	maximum stress
σ_{min}	minimum stress
σ_m	mean stress
$\Delta\sigma$	stress range
$\boldsymbol{\xi}$	vector of parameters
$\pi(\boldsymbol{\xi}\|\mathbf{D})$	posterior joint distribution of parameters
$\pi_0(\boldsymbol{\xi})$	prior joint distribution of parameters

increases. According to the situation that more diverse models are being used to analyze an engineering system, model uncertainty, which is the uncertainty involved in selecting the best model from a set of possibilities, is unavoidably accompanied by the creation of different life prediction models for the same system. In particular, the uncertainty of the error in model prediction as well as model uncertainty should be incorporated into a response prediction [15].

A clear understanding of LCF behavior at high temperature is very important for the design, selection, and safety assessment of turbine disks. LCF at high temperature is an interactive mechanism arising from various processes such as time-independent plastic strain, time-dependent creep, dynamic strain aging, and oxidation. Over the past several decades, the issue of predicting LCF life of high temperature components has been an area of interest. To improve the accuracy of LCF life prediction, researchers have presented several models [16–21]. Due to the complex damage mechanisms involved, a unified model that can provide accurate LCF life predictions does not exist [22–23].

Combining probabilistic methods with different fatigue models, it is possible to predict LCF life and to evaluate different possible sources of uncertainty for turbine disks. Various LCF life prediction models have been proposed for assessing the life of structures or materials, include viscosity-based model [19], SWT (Smith–Watson–Topper) [24], plastic strain energy density (PSED) [25], generalized damage parameter (GDP) [26] and thermodynamic entropy (TE) [27]. To account for the scarcity and scatter of material properties and fatigue test results, the uncertainty of model structure itself and its predictions must be characterized. Probabilistic methods such as Bayesian approach are used to quantitatively account for uncertainty in fatigue predictions without relying upon overly conservative safety factors. The combined effects of these uncertainties lead to a significant scatter in the actual fatigue life of mechanical components. Thus, this paper proposes a probabilistic LCF life prediction framework for turbine disk alloys using Bayes' theorem, by considering all available data that contribute to uncertainties associated with those predictions.

The paper is organized as follows: in Section 2, the authors' previously developed viscosity-based model [19] capabilities are addressed more extensively to account for the effects of time-dependent damage mechanisms on the LCF life. Then, a Bayesian framework is presented by incorporating the uncertainty of material properties, total inputs and model uncertainty into probabilistic LCF life prediction. In Section 3, based on the concept of white-box approach used for assessing fire simulation code uncertainty in [28], the uncertainty in probabilistic LCF life prediction is modeled using Bayesian inference. In Section 4, this probabilistic life prediction framework was verified using four different models with experimental data of GH4133 under different temperatures. Finally, conclusions are presented in Section 5.

2. Bayesian framework for probabilistic LCF life prediction

2.1. A viscosity-based model for LCF life prediction

Though several strain energy based methods for predicting LCF life at high temperature have been developed [29–31], the authors' previous work [19,32–33] has clearly showed that it is possible: (1) to correlate the fatigue-creep damage and the life with the viscosity-based parameter E_p; (2) to reflect the effects of time-dependent damage mechanisms on the LCF life; (3) to identify the main influential factor of LCF life, the maximum stress and stress range at minimum stress $\sigma_{max} \leq 0$ and mean stress at minimum stress $\sigma_{min} > 0$. In this section, further development and modifications to these issues are in progress which will make the prediction of LCF life via Bayes' theorem with high accuracy, simplification and wide application scope.

In this paper, a trapezoid load diagram, as shown in Fig. 1, was used to analyze the conditions of most alloys under high temperature and cyclic loading. In Fig. 1, T_{du},T_{dl},T' and T'' represent the tensile hold time, compressive hold time, tension-going time and compression-going time respectively in one loading cycle when

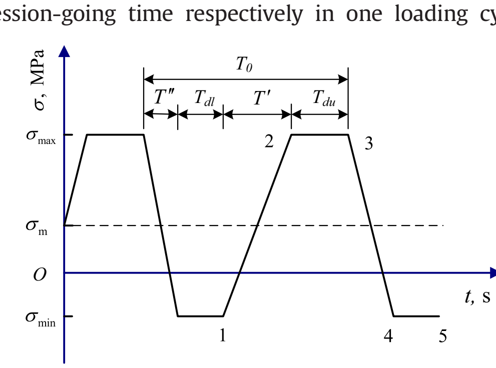

Fig. 1. Stress-time with trapezoidal loading waveform.

$\sigma_{max}>0$ and $\sigma_{min}<0$. T_{dl} is the tensile hold-time when $\sigma_{min}>0$. T_0 and T are the total time period and the period time not including the hold time, respectively, where $T=T'+T''$.

According to the assumption made in [34], the effect of compressive hold-time on the LCF life can be ignored usually, as creep damage of most materials is sensitive to the tensile hold-time instead of compressive hold-time [22]. Similar with the energy criterion proposed in [35], the viscosity-based parameter E_p accumulated per cycle under fatigue-creep interaction can be described by the stress area under loading waveforms, and above the zero-stress line. The viscosity-based parameter E_p per cycle can be calculated by

$$E_p=T_{du}\sigma_{max}+(T_{dl}+T)\sigma_{min}H(\sigma_{min})+\frac{T}{2}f(\sigma_{max},\sigma_{min}) \tag{1}$$

and the stress conversion function $f(\sigma_{max},\sigma_{min})$

$$f(\sigma_{max},\sigma_{min})=\begin{cases}\Delta\sigma, & \sigma_{min}>0\\ \dfrac{\sigma_{max}^2}{\Delta\sigma}, & \sigma_{min}\le 0\end{cases} \tag{2}$$

where $H(\sigma_{min})$ is the unit step function of σ_{min} and defined as

$$H(\sigma_{min})=\begin{cases}1, & \sigma_{min}\ge 0\\ 0, & \sigma_{min}<0\end{cases} \tag{3}$$

Based on ductility exhaustion theory, Goswami [36–37] developed a ductility model based on the assumption that deformation under LCF can be represented in terms of a viscous behavior. The dynamic viscosity should account for the strain range effects and can be presented based on the fundamental viscosity concept. The dynamic viscosity υ_d is defined as [37–38]

$$\upsilon_d = \Delta\sigma\left(\Delta\varepsilon_t/\dot{\varepsilon}\right) \tag{4}$$

According to the physical significance of the parameter E_p in Eq. (1) and the dynamic viscosity υ_d in Eq. (4), note that the latter is included in the former, and the essential difference between them is that the former includes the tensile elastic energy input which causes no damage compared with the latter. In order to estimate the real ductility exhausted during the fatigue process, similar with the total strain energy density method [39], an improved viscosity-based parameter E_r is presented by using the parameter E_p and the tensile elastic energy input ΔW_{FL} which causes no damage,

$$E_r=E_p-\Delta W_{FL}T_0 \tag{5}$$

where the tensile elastic energy input which causes no damage [39], ΔW_{FL}, was determined by

$$\Delta W_{FL}=\sigma_{\lim}^2/2E \tag{6}$$

where $\sigma_{\lim}$ is the fatigue limit of material.

Substituting Eq. (1) and Eq. (6) into Eq. (5) results in the following equation:

$$E_r=\begin{cases}T_{du}\sigma_{max}+(T_{dl}+T)\sigma_{min}+\dfrac{T}{2}\Delta\sigma-\dfrac{\sigma_{\lim}^2}{2E}T_0, & \sigma_{min}>0\\ T_{du}\sigma_{max}+\dfrac{T}{2}\dfrac{\sigma_{max}^2}{\Delta\sigma}-\dfrac{\sigma_{\lim}^2}{2E}T_0, & \sigma_{min}\le 0\end{cases} \tag{7}$$

A power law relationship exists between the improved viscosity-based damage function, $\Delta W_r=\Delta\varepsilon_{in}(E_r)^{\phi}$, and the number of cycles to failure,

$$\Delta\varepsilon_{in}(E_r)^{\phi}N_f^{\alpha}=C \tag{8}$$

where the inelastic strain range $\Delta\varepsilon_{in}$ can be replaced by the plastic strain range $\Delta\varepsilon_p$ under pure fatigue loading.

Based on Eq. (8), it should be noted that the improved viscosity-based model (VBM) enable to describe the damaging process during the LCF as a dependence on loading parameters. This equation describes the average behavior, and the life in different tests varies around this average life. In order to describe the variation, a probabilistic LCF life prediction framework will be developed in Section 2.2.

2.2. Probabilistic LCF life prediction framework using Bayes' theorem

In engineering, the test or service data of some equipment are hard to get or may even be inaccessible, such as aircraft engines. In this study, the material properties were modeled as distributions, model parameters and perturbed inputs for the probabilistic methods were incorporated into the physical or mechanical model. As the Bayesian approach can potentially give more accurate estimates by combining test data with technological knowledge available from theoretical studies and/or previous experimental data, this section will focus on the physical and statistical model updating using Bayes' theorem.

The Bayesian inference is a technique used to update a given state of knowledge, and expresses a decrease in uncertainty gained by an increase in knowledge. In the Bayesian analysis, the estimation of a vector of parameters ξ is updated from its prior probability distribution function (PDF) using the observed data, $\mathbf{D}$. The Bayesian inference on $\mathbf{D}$ is obtained as

$$\pi(\xi|\mathbf{D})=\frac{\pi_0(\xi)L\left(\mathbf{D}|\xi\right)}{\int_{\xi}\pi_0(\xi)L\left(\mathbf{D}|\xi\right)d\xi} \tag{9}$$

where $\pi_0(\xi)$ is the prior distribution of parameters ξ, $\xi=\{C,\alpha,\phi,s\}$ is the vector of the model parameters C,α,ϕ and s in Eq. (8). $L\left(\mathbf{D}|\xi\right)$ is the likelihood function of the observed data $\mathbf{D}$. $\pi(\xi|\mathbf{D})$ is the posterior joint distribution of ξ.

By combining with a subjective prior distribution, the data are represented in the form of a lognormal likelihood function. Through replacing the log-mean of the lognormal PDF with log life derived from Eq. (8), the lognormal likelihood function used in the model parameter uncertainty steps is shown as follows:

$$L\left(\mathbf{D}|\{C,\alpha,\phi,s\}\right)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}sN_f}\exp\left(-\frac{1}{2}\left[\frac{\ln\ (N_f)-\frac{1}{\alpha}\ln(C)+\frac{1}{\alpha}\ln(\Delta\varepsilon_{in})+\frac{\phi}{\alpha}\ln\ (E_p-\Delta W_{FL}T_0)}{s}\right]^2\right) \tag{10}$$

where s is a model parameter equal to the natural logarithm standard deviation of life cycles.

The intercept parameters of the improved viscosity-based model (i.e. C, α, ϕ and s) automatically take into account any possible non-zero mean for error. Combining this likelihood with the PDFs developed to represent the prior state of knowledge leads to an estimation of the posterior by Eq. (9). Based on the current state of knowledge, the prior of the model parameters can be defined as either informative or non-informative. Informative Bayesian inference assumes that a subject is able to express one's personal knowledge about an unknown quantity ξ in a quantitative manner. The prior distribution $\pi_0(\xi)$ reflects the best available knowledge of the distribution of parameters ξ, which should contain the following information:

(1) physics, engineering, and related/additional information;
(2) mathematical or physical models;

(3) expert's judgments and
(4) corporate memory, historical data.

If there is a lack of information on the parameters, then the prior is defined as a uniform distribution using a rough idea of the range of the parameter values. Different kinds of prior functions can arise depending on the degree of initial personal knowledge about model parameters.

In order to make a fatigue life prediction under a given stress loading, the mean prediction life $\tilde{N}_f$ by the improved VBM can be estimated as

$$\tilde{N}_f = \int_{C,\alpha,\phi,s} \pi(\{C,\alpha,\phi,s\}|\mathbf{D}) \left(C^{\frac{1}{\alpha}} (\Delta\varepsilon_{in})^{-\frac{1}{\alpha}} (E_p - \Delta W_{FL} T_0)^{-\frac{\phi}{\alpha}} \right) dC\, d\alpha\, d\phi\, ds \tag{11}$$

In practice, the greater computational burden is usually associated to Bayesian methods when compared to the classical methods for these equations. Therefore Markov Chain Monte Carlo (MCMC) simulations are used for most Bayesian analyses. The MCMC method provides an alternative in which samples can be generated from the posterior directly, and we obtain sample estimates of the quantities of interest, thereby performing the integration implicitly. It generates a sample set $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \cdots, \xi_m\}$, $\xi_i = \{C_i, \alpha_i, \phi_i, s_i\}$, $i = 1, 2, 3, \cdots, m$, which represents the posterior density of parameters C, α, ϕ and s. The PDF of LCF life can be readily predicted based on a given $\boldsymbol{\xi}$. For this study, a Bayesian updating procedure has been constructed to estimate the parameters C, α, ϕ and s in Eq. (10). Similarly, the developed probabilistic life prediction framework can be expanded to other fatigue life prediction models by updating the Eqs. (10) and (11). Given a general formulation of a fatigue criterion, which is expressed mathematically as a relation between the number of cycles to failure N_f of the structure and some function of its material properties, structure size, loading waveform, and damage driving parameters (e.g. stress, strain or force),

$$N_f = k\left(\Phi(P_{(\cdot)}, \cdots, \sigma, \varepsilon)\right)^q \tag{12}$$

where $P_{(.)}$ denotes the parameters related to the structure, such as material properties, structure size, loading waveform; k and q are material dependent constants. According to Eq. (11), the mean prediction life $\tilde{N}_f$ based on a given fatigue criterion can be obtained as

$$\tilde{N}_f = \int_{\xi,s} \pi(\{\xi, s\}|\mathbf{D}) \left(k\left(\Phi(P_{(\cdot)}, \cdots, \sigma, \varepsilon)\right)^q \right) d\xi\, ds \tag{13}$$

where the lognormal likelihood function to be used with Eq. (13) is shown as follows:

$$L(\mathbf{D}|\{\xi, s\}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} s N_f} \exp\left(-\frac{1}{2} \left[\frac{\ln(N_f) - \ln\left(k\left(\Phi(P_{(\cdot)}, \cdots, \sigma, \varepsilon)\right)^q\right)}{s} \right]^2 \right) \tag{14}$$

In this analysis, characterizing the posterior distribution through sampling simulation methods using the MATLAB platform to run the necessary MCMC simulation performs the Bayesian inference. The posterior distribution contains an updated statement of the uncertainty in $\boldsymbol{\xi}$ in light of subsequently acquired data modeled as a probability distribution.

3. Uncertainty modeling using the white-box approach

In order to understand physical behaviors and predict the response of a physical system, developing a life prediction model is the process of idealizing the complicated load conditions into a relatively simple form through making some assumptions. Uncertainties in the values used as input for a life prediction model are propagated in this model to find the effects on the output uncertainty. In order to obtain the best possible overall estimation of uncertainty, the uncertainty of inputs must be considered and estimated. In this study, input uncertainties are developed using the available information reported from experiments or other sources of information. For cases in which uncertainty is not reported, expert judgment given prior experience with similar experiments and test equipment can be used to develop a PDF for the inputs. Bayesian inference is used a second time to characterize the total uncertainties associated with inputs, model and model parameters, which represents the continuing research of the model uncertainty analysis in [28]. The comparison of the model predictions with experimental results is also considered in this approach.

In the black-box approach, both the model prediction and experimental result are considered to be independent representations of the physical reality of interest being predicted [28,40]. In a LCF regime, since the model prediction, experiment result, and physical reality of interest have the same sign (all positive), the ratio of real fatigue life and model prediction or experimental results is simply proved to be a random variable with lognormal distribution, and will be used to represent the likelihood of data in the form of multiplicative errors as shown in Eqs. (15) and (16)

$$\frac{N_{real,i}}{N_{ft,i}} = F_{t,i} \quad F_t \sim LN(b_t, s_t) \tag{15}$$

and

$$\frac{N_{real,i}}{N_{fp,i}} = F_{p,i} \quad F_p \sim LN(b_p, s_p) \tag{16}$$

where $F_{t,i}$ and $F_{p,i}$ are the experimental error and the model prediction error respectively.

The relationship between the experimental and model uncertainty is

$$\frac{N_{ft,i}}{N_{fp,i}} = \frac{F_{p,i}}{F_{t,i}} = F_{pt,i} \tag{17}$$

Assuming independency of F_p, F_t leads to

$$F_{pt} \sim LN\left(b_p - b_t, \sqrt{s_p^2 + s_t^2}\right) \tag{18}$$

Using the observed number of cycles to failure $\mathbf{N_{ft}} = \{N_{ft,1}, N_{ft,2}, \cdots, N_{ft,n}\}$ and model predictions $\mathbf{N_{fp}} = \{N_{fp,1}, N_{fp,2}, \cdots, N_{fp,n}\}$, the likelihood function used for the prior $\pi_0(b_p, s_p)$ is

$$L(N_{ft,i}, N_{fp,i}, b_t, s_t | b_p, s_p) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\left(\frac{N_{ft,i}}{N_{fp,i}}\right)\sqrt{s_p^2 + s_t^2}} \exp\left(-\frac{1}{2} \frac{\left[\ln\left(\frac{N_{ft,i}}{N_{fp,i}}\right) - (b_p - b_t)\right]^2}{s_p^2 + s_t^2} \right) \tag{19}$$

Thus, the resulting posterior joint distribution for the black-box approach using Eq. (9) is

$$\pi(b_p, s_p | N_{ft,i}, N_{fp,i}, b_t, s_t) = \frac{\pi_0(b_p, s_p) L(N_{ft,i}, N_{fp,i}, b_t, s_t | b_p, s_p)}{\int_{s_p}\int_{b_p} \pi_0(b_p, s_p) L(N_{ft,i}, N_{fp,i}, b_t, s_t | b_p, s_p)\, db_p\, ds_p} \tag{20}$$

where $\pi(b_p, s_p | N_{ft,i}, N_{fp,i}, b_t, s_t)$ is the posterior joint distribution of parameters.

In the current uncertainty analysis, $F_{pt,i}$ is a PDF resulting from the combination of multiple model predictions paired with a single experimental result. The resulting posterior of this analysis is much more complex than that shown in Eq. (20). In order to account for this new distribution, F_{pt} will be multiplied by the PDF of model predictions and integrated over each distribution resulting from independent cases as follows:

$$F_{pt} \sim \int_{b_p, s_p} LN\left(b_p - b_t, \sqrt{s_p^2 + s_t^2}\right) g(b_p, s_p) db_p \, ds_p \quad (21)$$

where $g(b_p, s_p)$ is the joint PDF of parameters b_p and s_p.

An important step in developing a meaningful probabilistic model is the accurate inference of the joint distribution of model parameters. The research on unpaired data has been made in previous uncertainty analysis work. In order to quantify the uncertainty surrounding the unknown of interest based on expert opinions and evaluate the impact of the number of experts on the accuracy of aggregated estimate, Shirazi [41] proposes a posterior for dealing with different expert judgments. In his research, multiple estimations by experts are compared to a single "true value". Similarly, by considering the multiple estimations made by experts as the multiple model predictions and the "true value" as the same as the experimental result, Shirazi's posterior can be extended in the current research.

To evaluate multiple model predictions for one true value, the distribution of error can be marginalized in terms of parameters $\boldsymbol{\xi}$, which by itself is a variable symbolized by a variability distribution of $f(\boldsymbol{\xi})$. This hyper distribution can be characterized by hyper-parameters, $\boldsymbol{\omega}$, leads to a distribution of error $f(\boldsymbol{\xi}|\boldsymbol{\omega})$.

Under the assumption of independence among those model predictions, we can get

$$L(N_{ft,i}, N_{fp,ik}, b_t, s_t|\boldsymbol{\omega}) = \prod_{i=1}^{N} \left(\prod_{k=1}^{M_i} \int_{b_p, s_p} L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p) f(b_p, s_p|\boldsymbol{\omega}) db_p \, ds_p \right) \quad (22)$$

For each test $i = (1, 2, \cdots, N)$, the model prediction $k = (1, 2, \cdots, M_i)$ of $N_{real,i}$ is $N_{fp,ik}$. Then the hyper-parameters are $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_m)$. As represented in Table 1, the model prediction error term has two dimensions of (i, k) to cover all i tests.

Estimating the hyper-parameters $\boldsymbol{\omega}$ using likelihood function $L(N_{ft,i}, N_{fp,ik}, b_t, s_t|\boldsymbol{\omega})$ and data in Table 1

$$\pi(\boldsymbol{\omega}|N_{ft,i}, N_{fp,ik}, b_t, s_t) = \frac{\prod_{i=1}^{N} \left(\prod_{k=1}^{M_i} \int_{b_p, s_p} L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p) f(b_p, s_p|\boldsymbol{\omega}) db_p \, ds_p \right) \pi_0(\boldsymbol{\omega})}{\int_{\boldsymbol{\omega}} \prod_{i=1}^{N} \left(\prod_{k=1}^{M_i} \int_{b_p, s_p} L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p) f(b_p, s_p|\boldsymbol{\omega}) db_p \, ds_p \right) \pi_0(\boldsymbol{\omega}) \, d\boldsymbol{\omega}} \quad (23)$$

Moreover, the desired posterior distribution of error given the evidence becomes the expected distribution, which is estimated by eliminating the aleatory uncertainty over $\boldsymbol{\omega}$, the resulting posterior specific to this analysis becomes

$$\begin{aligned} f(b_p, s_p|N_{ft,i}, N_{fp,ik}, b_t, s_t) &= \int_{\boldsymbol{\omega}} f(b_p, s_p|\boldsymbol{\omega}) \pi(\boldsymbol{\omega}|N_{ft,i}, N_{fp,ik}, b_t, s_t) d\boldsymbol{\omega} \\ &= \int_{\boldsymbol{\omega}} f(b_p, s_p|\boldsymbol{\omega}) \frac{\prod_{i=1}^{N} \left(\prod_{k=1}^{M_i} \int_{b_p, s_p} L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p) f(b_p, s_p|\boldsymbol{\omega}) db_p \, ds_p \right) \pi_0(\boldsymbol{\omega})}{\int_{\boldsymbol{\omega}} \prod_{i=1}^{N} \left(\prod_{k=1}^{M_i} \int_{b_p, s_p} L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p) f(b_p, s_p|\boldsymbol{\omega}) db_p \, ds_p \right) \pi_0(\boldsymbol{\omega}) \, d\boldsymbol{\omega}} d\boldsymbol{\omega} \end{aligned} \quad (24)$$

where N experiments will be updated with the M_i model predictions of the ith experiment. The likelihood $L(N_{ft,i}, N_{fp,ik}, b_t, s_t|b_p, s_p)$ to be used with Eq. (24) is shown in Eq. (19).

By given an error as defined in Eq. (15), the experimental results (true values) are uncertain. Computing the posterior predictive distribution of fatigue life using Eq. (24), the combined effects of those uncertainties associated with inputs, models, model parameters and model outputs are considered for LCF life prediction. As represented in Fig. 2 and Eq. (24), the posterior distribution of error F_p can be obtained through multi-source uncertain information fusion. Moreover, when new information reported from experiments or other sources are available, the model prediction error can be updated using Eq. (24). The mean or median of the posterior is compared with the real value in order to determine if and how much the formulated likelihood function has been able to reduce the error of model prediction. In this section, an approach to evaluate uncertainties during the life predictions using Bayesian inference is developed, as depicted in Fig. 2. This methodology will be verified by the LCF life data of GH4133 in Section 4.

4. Probabilistic LCF life predictions and output updating

To verify the feasibility and prediction capability of the probabilistic LCF life prediction framework, the proposed methodology using different LCF life prediction models was applied to experimental results of turbine disk material GH4133 [42,43]. The heat treatment conditions of this alloy are austenitization (8 h at 1080 °C, air-cooled) and tempering (16 h at 750 °C, air-cooled). LCF data were obtained from Beijing Institute of Aeronautical Materials, China. Details of mechanical properties of the materials, test conditions, and strain-life data are reported in [42,43].

The tests were performed under axial total strain control with a triangular fully reversed waveform, using an axial extensometer placed on the specimen. Numerous tests were carried out with various conditions: mechanical strain range of 0.5–1.4% for isothermal LCF at temperature 400 °C and 500 °C under strain ratio $R_\varepsilon = -1$ respectively.

In the probabilistic analyses, the prior distributions of material properties and input variables were determined from the test conditions [43], theoretical and experimental data analysis in [44,45], as shown in Tables 2 and 3.

In order to obtain the estimated parameters for the VBM model, the natural log of both sides of Eq. (8) were taken to transform it into the general linear regression model

$$\ln (N_f) = C_1 + A \ln (\Delta\varepsilon_{in}) + B \ln (E_r) \quad (25)$$

Table 1
Model prediction errors for tests.

Tests $(i=1,2,\cdots,N)$	Model prediction $(k=1,2,\cdots,M_i)$	Real value $(i=1,2,\cdots,N)$	Model prediction error $\left(F_{p,ik}=\frac{N_{real,i}}{N_{fp,ik}}\right)$
1	$[N_{fp,11},N_{fp,12},\cdots,N_{fp,1M_1}]$	$N_{real,1}$	$[F_{p,11},F_{p,12},\cdots,F_{p,1M_1}]$
2	$[N_{fp,21},N_{fp,22},\cdots,N_{fp,2M_2}]$	$N_{real,2}$	$[F_{p,21},F_{p,22},\cdots,F_{p,2M_2}]$
3	$[N_{fp,31},N_{fp,32},\cdots,N_{fp,3M_3}]$	$N_{real,3}$	$[F_{p,31},F_{p,32},\cdots,F_{p,3M_3}]$
⋮	⋮	⋮	⋮
N	$[N_{fp,N1},N_{fp,N2},\cdots,N_{fp,NM_N}]$	$N_{real,N}$	$[F_{p,N1},F_{p,N2},\cdots,F_{p,NM_N}]$

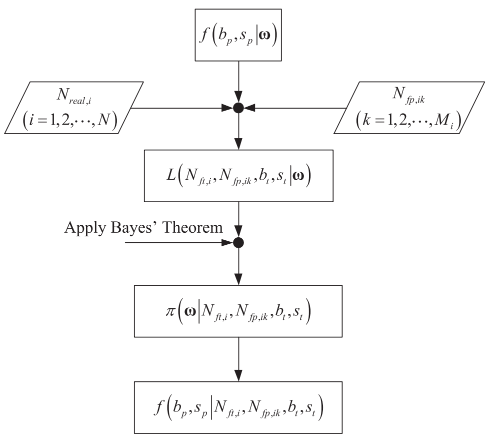

Fig. 2. Uncertainty modeling using Bayesian inference.

Table 2
Input uncertainties.

Input variables	Uncertainty (%)
$\Delta\varepsilon_p$	± 0.5
σ_{max}	± 1
T_0	± 1

Table 3
Random variables for material constants of GH4133.

Random variables	Distribution	Mean value (MPa)	Standard deviation
Young's modulus E	Normal	1.992×10^5	7.0×10^3
Stress endurance limit $\sigma_{\rm lim}$	Normal	4.207×10^2	17.33
Cyclic strain hardening exponent n'	Normal	0.1005	0.006093

Table 4
Summary statistics of model input parameters for the VBM.

Parameter	Mean	Standard Deviation	2.50%	Median	97.5%
A	−2.6714	0.051454	−2.9477	−2.8469	−2.736
B	−2.6472	0.048906	−2.7941	−2.6982	−2.6024
C_1	84.7816	0.047166	84.4699	84.5624	84.6548

where

$$C_1=\frac{1}{\alpha}\ln(C),\quad A=-\frac{1}{\alpha}\quad\text{and}\quad B=-\frac{\phi}{\alpha}$$

Based on the experimental results of GH4133, the marginal posterior distributions of model parameters (A,B and C_1) can be obtained using the prior likelihood in Eq. (10) and life model in

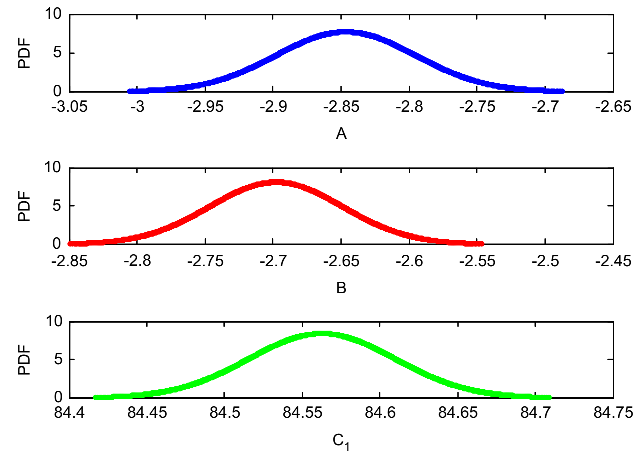

Fig. 3. Marginal distributions of model parameters (A, B and C_1) using MCMC simulation.

Table 5
White-box summary statistics using the VBM for experimental results.

Parameter	Mean	Standard Deviation	2.50%	Median	97.5%
b_p	0.054371	0.0011742	0.051888	0.054189	0.05649
s_p	0.097277	0.0069799	0.084257	0.097937	0.11162
F_p	1.0016	0.039565	0.92801	1.0054	1.0828

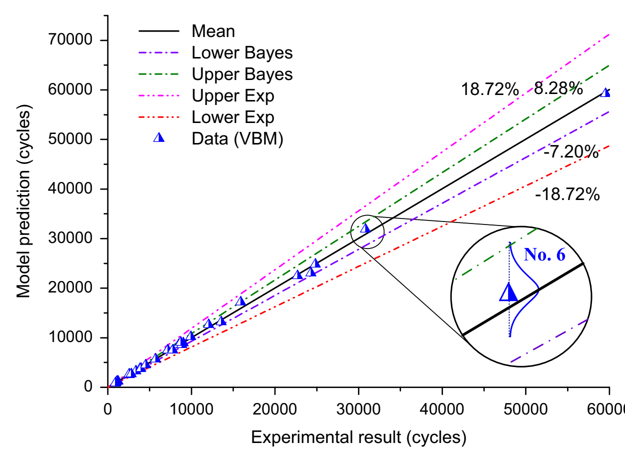

Fig. 4. Probabilistic LCF life prediction using the VBM.

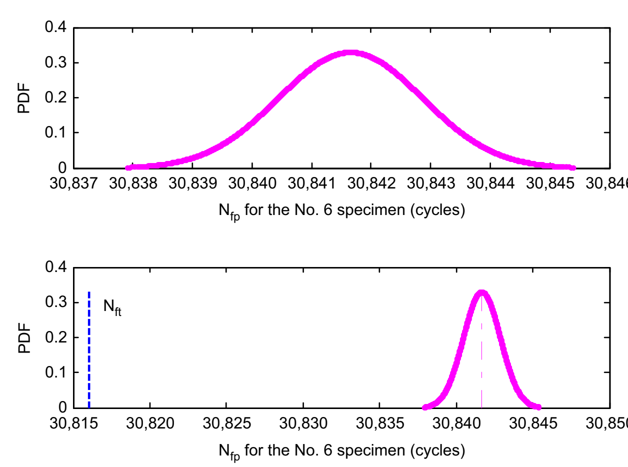

Fig. 5. Distribution of predicted life for the No. 6 specimen using the VBM.

Table 6
Black-box summary statistics using the VBM for experimental results.

Parameter	Mean	Standard deviation	2.50%	Median	97.5%
b_p	−0.01699	0.01819	−0.05251	−0.01699	0.0185
s_p	0.01663	0.01295	0.0006126	0.01376	0.04844
F_p	0.9836	0.02737	0.9305	0.9832	1.04

Table 7
White-box summary statistics using the GDP, SWT and PSED for experimental results.

Model	Parameter	Mean	Standard deviation	2.50%	Median	97.5%
GDP	b_p	0.064486	0.0067037	0.053373	0.066512	0.079651
	s_p	0.097364	0.0057106	0.087548	0.09874	0.10993
	F_p	0.99305	0.045837	0.89989	0.99117	1.08
SWT	b_p	0.07452	0.008781	0.059508	0.076718	0.093929
	s_p	0.086009	0.0057054	0.076582	0.087765	0.098947
	F_p	1.0023	0.040348	0.92629	1.0063	1.0843
PSED	b_p	0.070126	0.0039076	0.062386	0.070044	0.077703
	s_p	0.12958	0.0090468	0.11108	0.12881	0.14654
	F_p	0.98064	0.041357	0.89315	0.974	1.0552

Table 8
Black-box summary statistics using the GDP, SWT and PSED for experimental results.

Model	Parameter	Mean	Standard deviation	2.50%	Median	97.5%
GDP	b_p	−0.02576	0.0183	−0.06184	−0.02579	0.01021
	s_p	0.0173	0.01339	0.0006596	0.01431	0.04987
	F_p	0.9748	0.02769	0.9204	0.9745	1.031
SWT	b_p	−0.01629	0.01845	−0.05277	−0.01618	0.01975
	s_p	0.01684	0.013	0.0006458	0.0141	0.04869
	F_p	0.9842	0.02773	0.9295	0.984	1.04
PSED	b_p	−0.03802	0.01833	−0.07384	−0.03799	−0.00226
	s_p	0.01683	0.01303	6.15E−04	0.01401	0.04872
	F_p	0.963	0.02696	0.9098	0.9627	1.018

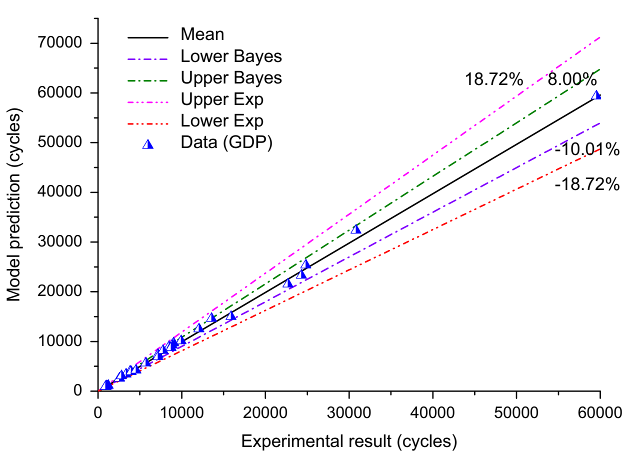

Fig. 6. Probabilistic LCF life prediction using the GDP model.

Eq. (25). The non-informative prior distributions for the model parameters are chosen to be uniform. The result of Bayesian analysis for the model parameters is listed in Table 4 and graphically in Fig. 3.

In order to output updating and verify the proposed probabilistic life prediction framework, the Bayesian inference is used to compare multiple model predictions with experimental results for the uncertainty modeling. The experimental uncertainty for the

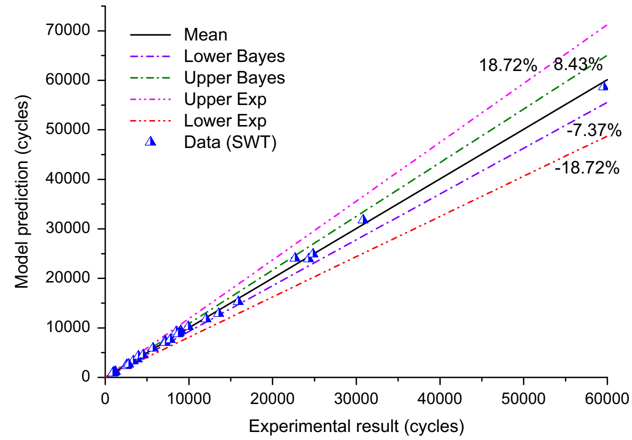

Fig. 7. Probabilistic LCF life prediction using the SWT model.

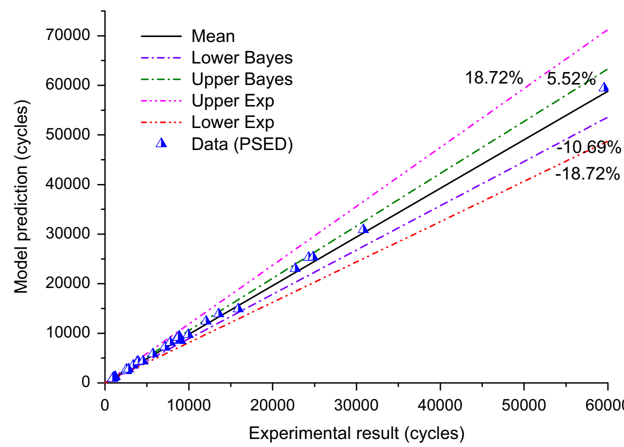

Fig. 8. Probabilistic LCF life prediction using the PSED model.

tests was determined to be approximately, 18.72% as given in [12,44]. Using the available LCF life data different from those used in model updating, the summary statistics for the marginal posterior PDFs of parameters (b_p, s_p) and the multiplicative error factor F_p for the viscosity-based model are shown in Table 5. According to the upper and lower bounds of F_p, the resulting estimated total uncertainty for model predictions using the VBM has an upper bound of +8.28% and lower bound of −7.20% as shown in Fig. 4.

With distributions over the inputs and model parameters developed, using the life prediction model with MCMC simulations results in a distribution of the predicted life. Using the No. 6 specimen in Fig. 4 as an example, Fig. 5 compares its tested life N_{ft} with the predicted life distribution N_{fp}. In order to compare the results using white-box approach, the model uncertainty were estimated by the black-box approach, and the summary statistics are shown in Table 6.

The capability of this new model was evaluated and compared with three other models, the GDP [26], SWT [24] and PSED [25] ones. Similarly, the summary statistics of the white-box results and black-box results using these three models are listed in Tables 7 and 8, respectively. And the probabilistic life prediction results are given in Figs. 6–8 respectively.

Figs. 4–8 show that all the predicted cyclic lives by these four models are in a factor of ±1.5 to the test ones. The value of F_p for the model predictions shows that to correct the model. The mean values of F_p for the GDP and PSED are less than 1 for the white-box approach, which suggests a bias in the model to over predict the LCF life. In the over prediction condition, the estimation of reality given the model prediction is expected to be lower. For the VBM and SWT model results listed in Table 7, however, the mean value of F_p is 1.0016 and 1.0023 respectively, which shows a bias to under predict the LCF life. Compared to the black-box results listed in Tables 6 and 8, using the distribution of white-box values results in larger uncertainty bounds for each model prediction. For the VBM, the uncertainty bound of F_p using white-box approach is $[+8.28\%, -7.20\%]$ and using black-box approach is $[+4\%, -6.95\%]$, respectively. This is expected, as the white-box approach accounts for the uncertainty of total inputs and model uncertainty, rather than the black-box method considers only the uncertainty of model parameters and model uncertainty.

Probabilistic life prediction using the viscosity-based model shows a good agreement with the experiment results by mean and bounds. The uncertainty bounds presented are those of the model estimation of reality. Using the white-box approach, the VBM method can predict the LCF life with tighter uncertainty bounds than the others, as $[+8.28\%, -7.20\%]$ for VBM, $[+8.43\%, -7.37\%]$ for SWT, $[+8\%, -10.01\%]$ for GDP and $[+5.52\%, -10.69\%]$ for PSED, which leads to better decision making and model selection based on the same available knowledge.

As aforementioned, one of the advantages of Bayesian inference is that the previous analysis can be updated with additional data. In this study, the proposed probabilistic LCF life prediction framework offers the capability to propagate the various uncertainties through a life prediction model to determine their combined effect on the distribution of fatigue life, which can be used to predict the LCF life for most metallic materials by quantifying the uncertainties associated with the total inputs and model uncertainty. Moreover, nested sampling using the MATLAB platform solved the complex Bayesian posterior calculations and the complete numerical solution of nonpaired data. Besides, the application of this probabilistic methodology to other cases such as random loading spectrum and updating with new data will be further evaluated.

5. Conclusions

In this paper, a probabilistic LCF life prediction framework using Bayesian inference is developed to systematically incorporate information from new data with the prior knowledge of the variability in the material properties, total inputs (model parameters and measured stress or strain etc.) and the model uncertainty resulting from choices of different deterministic models. To check the feasibility and validity of this methodology, the LCF test data of GH4133 under high temperature were compared with the predicted results by the viscosity-based model, GDP, SWT and PSED methods.

Through comparing the distribution of the multiplicative error F_p for each model, both the viscosity-based method and SWT model yield more satisfactory probabilistic life prediction results for GH4133 under different temperatures than the GDP and PSED ones. Moreover, the probabilistic life prediction using the viscosity-based method has a tighter uncertainty bounds than the others based on the same available knowledge. In the probabilistic LCF life prediction, the uncertainty bounds for the white-box analysis were wider than those for a black-box analysis. The larger bounds result from the recognition, quantification, and inclusion of inputs and parameter uncertainties associated with different deterministic models.

Through updating, the uncertainty in fatigue life can be reduced for individual components and the proposed framework provides more valuable information for assessing their updated remaining life distributions. In addition, it provides a theoretical basis for model selection based on the same available knowledge and output updating when new data are available. The proposed probabilistic framework appears to be an interesting alternative to the deterministic methods for LCF life prediction.

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