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POSSIBILITY-BASED MULTIDISCIPLINARY DESIGN OPTIMIZATION IN THE FRAMEWORK OF SEQUENTIAL OPTIMIZATION AND RELIABILITY ASSESSMENT

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Abstract. *Reliability Based Multidisciplinary Design Optimization (RBMDO) has received increasing attention to reach high reliability and safety in complex and coupled systems. In early design stage of such systems, however, there are insufficient data to precisely construct the probability distributions required by the RBMDO and consequently RBMDO can not be carried out effectively. To deal with this case, the present work proposes Possibility Based Multidisciplinary Design Optimization (PBMDO) and a method of PBMDO within the framework of the Sequential Optimization and Reliability Assessment (PBMDO-SORA). The proposed method enables designers to solve MDO problems with insufficient information on the uncertainties associated with design inputs, and efficiently decreases the computational demand. The efficiency of the proposed method is illustrated with a mathematical example and an engineering design.*
Keywords: Sequential optimization and reliability assessment, Multidisciplinary design optimization, Possibility based multidisciplinary design optimization

1. **Introduction.** In the last two decades, the consideration of uncertainty has been a focus of engineering design for complex and coupled systems. Reliability Based Multidisciplinary Design Optimization (RBMDO) has received increasing attention because of requirements for high reliability and safety in complex and coupled systems [1-8]. In [8], a Sequential Optimization and Reliability Assessment (SORA) method for RBMDO was proposed. SORA is based on the idea of decoupling reliability analysis from design optimization [9]. By using the MPP obtained from the previous cycle, the constraint in deterministic optimization is modified to make sure the MPP of current cycle fall into the feasible region. After solving the deterministic optimization, a new design point is obtained and followed by reliability assessment to check up the feasibility of each probability constraint at the new design point. Generally, the whole process of solution will converge in a few cycles.

However, in the early design stage of complex and coupled systems, due to time, environment and human, etc, there are insufficient data to precisely construct probability distributions which are basics in RBMDO. When distributions of variables are constructed using the limited available data, the Reliability based Design Optimization (RBDO) may lead to an unsafe design [10]. Results will be even worse for design with multiple disciplines. Possibility based Design Optimization (PBDO) is powerful to deal with problems in which there are insufficient data about uncertainty [10-18]. Possibility approach can deal with uncertainty with insufficient data by defining a fuzzy variable corresponding to the limited available data [12,16,17]. If there are insufficient data to precisely construct probability distributions, it is recommended to use Possibility Based Multidisciplinary Design Optimization (PBMDO) instead of RBMDO. PBMDO under the framework of the SORA (PBMDO-SORA) is proposed in this paper. This method can efficiently solve MDO problems in which uncertainty in design inputs is described with insufficient data, and also can obviously decrease computational demand.

This paper is organized as follows. In Section 2, PBDO and Multidisciplinary Design Optimization (MDO) are briefly reviewed. In Section 3, the formulation of PBMDO discussed in this paper is provided. In Section 4, the proposed method of PBMDO-SORA is discussed in detail, including the strategy, procedure and formulation. In Section 5, a mathematical example and an engineering design are utilized to illustrate the efficiency of the proposed method, followed by the conclusions given in Section 6.

2. A Briefly Review of PBDO and MDO.

2.1. PBDO.

To deal with the case of insufficient information about uncertainty in single-discipline design, PBDO has recently been introduced in optimization design. PBDO method has two obvious computational advantages than RBDO [10-15]: first, when there are only a few data which can not be used to precisely construct probability distributions for design variables, the fuzzy design variables can be more easily defined than stochastic variables; second, fuzzy operations are simpler than the case of probability, especially when there are a number of design variables.

2.1.1. *Formulation of PBDO.* The mathematical formulation of PBDO is

$$\begin{aligned} &\min_{\mathbf{d}',\mathbf{X}'^N} f\left(\mathbf{d}',\mathbf{X}'^N,\mathbf{P}'^N\right) \\ &s.t.\quad \Pi\left(G_i(\mathbf{d}',\mathbf{X}',\mathbf{P}')>0\right)\le\alpha_t,\quad i=1,2,\cdots,n \\ &\qquad \mathbf{d}'^L\le\mathbf{d}'\le\mathbf{d}'^U,\quad \mathbf{X}'^{N,L}\le\mathbf{X}'^N\le\mathbf{X}'^{N,U} \end{aligned} \tag{1}$$

where $\mathbf{d}'$ is a vector of deterministic design variables, $\mathbf{X}'$ denotes a vector of fuzzy design variables. $\mathbf{P}'$ refers to a set of fuzzy parameters. $\mathbf{X}'^N$ are maximal grade points of fuzzy design variables. The maximal grade point is defined as: $\mathbf{X}'^N=\{x'|\max\{\Pi_{X'}(x')\}\}$. $f(\cdot)$ represents the objective function. $\Pi(G(\cdot)>0)$ is the possibility of failure with the failure mode defined as $G(\mathbf{d}',\mathbf{X}',\mathbf{P}')>0$, and $G(\cdot)$ is performance function. α_t is the target possibility of failure. n is the total number of constraints. "L" and "U" are lower and upper bounds, respectively [11].

2.1.2. *Performance measure approach (PMA) in PBDO.* The formulation of PBDO with PMA [12,13] is

$$\begin{aligned} &\min_{\mathbf{d}',\mathbf{X}'^N} f\left(\mathbf{d}',\mathbf{X}'^N,\mathbf{P}'^N\right) \\ &s.t.\quad G_{\Pi_i}(\mathbf{d}',\mathbf{X}',\mathbf{P}')\le 0,\quad i=1,2,\cdots,n \\ &\qquad \mathbf{d}'^L\le\mathbf{d}'\le\mathbf{d}'^U,\quad \mathbf{X}'^{N,L}\le\mathbf{X}'^N\le\mathbf{X}'^{N,U} \end{aligned} \tag{2}$$

where $G_{\Pi_i}(\mathbf{d}',\mathbf{X}',\mathbf{P}')$ denotes the value of the ith performance function at its Most Possible Point (MPP, from now on, MPP means Most Possible Point). The constraint $G_{\Pi_i}(\mathbf{d}',\mathbf{X}',\mathbf{P}')\le 0$ is equivalent to $\Pi(G_i(\mathbf{d}',\mathbf{X}',\mathbf{P}')>0)\le\alpha_t$ [12]. The explanations of the symbols in this formulation are same as those in Section 2.1.1.

The fuzzy variables and parameters are assumed non-interactive. The membership function of each fuzzy variable and parameter satisfies three properties: unity; strong convexity; boundedness (detail definitions can be found in [12]). Evaluation of the possibility constraints needs a fuzzy analysis using PMA [12,13], which is formulated as

$$\begin{aligned} &\max_{\mathbf{U}} \; G(\mathbf{U}) \\ &s.t. \quad \|\mathbf{U}\|_\infty \le 1-\alpha_t \end{aligned} \tag{3}$$

where $\mathbf{U}$ is the vector of standard fuzzy variables and parameters transformed from those fuzzy ones in X space. A standard fuzzy variable has the isosceles triangular membership function as follows:

$$\Pi_U(u) = \begin{cases} u+1 & -1 \le u \le 0 \\ 1-u & 0 \le u \le 1 \end{cases} = 1-|u|, \quad |u| \le 1 \tag{4}$$

The transformation is

$$U = \begin{cases} \Pi_X(x)-1 & x \le X^N \\ 1-\Pi_X(x) & x > X^N \end{cases} \tag{5}$$

where $\Pi_X(\cdot)$ is the membership function of fuzzy variable X. It should be noted that in this paper all fuzzy variables and parameters are assumed to be non-interactive.

2.2. **MDO.** The mathematical formulation of MDO is

$$\begin{aligned} &\min f(\mathbf{X}'', \mathbf{Y}'') \\ &s.t. \quad g^{(i)}(\mathbf{X}''_s, \mathbf{X}''_i, \mathbf{Y}''_{\bullet i}) \ge 0 \\ &\qquad\; h^{(i)}(\mathbf{X}''_s, \mathbf{X}''_i, \mathbf{Y}''_{\bullet i}) = 0 \\ &\qquad\; \mathbf{Y}''_{i\bullet} = (y''_{ij}, j \ne i) = \mathbf{Y}''_{i\bullet}(\mathbf{X}''_s, \mathbf{X}''_i, \mathbf{Y}''_{\bullet i}) \\ &\qquad\; i = 1, 2, \cdots, nd \end{aligned} \tag{6}$$

where $\mathbf{X}'' = (\mathbf{X}''_s, \mathbf{X}''_1, \mathbf{X}''_2, \cdots, \mathbf{X}''_{nd})$ is a vector of design variables. $\mathbf{X}''_s$ is a vector of sharing variables, $\mathbf{X}''_i$ are local input variables to discipline i, and nd is the total number of disciplines. $\mathbf{Y}'' = (\mathbf{Y}''_{i\bullet},\ i = 1 \sim nd)$ represents a vector of linking variables. $\mathbf{Y}''_{i\bullet} = (y''_{ij};\ j \ne i,\ j = 1 \sim nd)$ is a vector of output linking variables obtained from the ith discipline, and y''_{ij} is the output of discipline i and input of discipline j. $\mathbf{Y}''_{\bullet i}$ is the input to the ith discipline. $f(\cdot)$ is the objective function. $g^{(i)}(\cdot)$ and $h^{(i)}(\cdot)$ are inequality and equality constraint functions in discipline i.

In RBMDO, uncertainties with variables are characterized with probability density functions. While in PBMDO, because of insufficient data, all the uncertainties with variables are characterized with membership functions based on possibility theory.

3. **PBMDO.** The mathematical formulation of PBMDO discussed in this paper is

$$\begin{aligned} &\min_{DV} f(\mathbf{d}_s, \mathbf{d}, \mathbf{X}_s^N, \mathbf{X}^N, \mathbf{P}^N, \mathbf{Y}^N) \\ &s.t. \quad \Pi(G^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s, \mathbf{X}_i, \mathbf{P}_i, \mathbf{Y}_{\bullet i}) > 0) \le \alpha_t \\ &\qquad\; g^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s^N, \mathbf{X}_i^N, \mathbf{P}_i^N, \mathbf{Y}_{\bullet i}^N) \le 0 \\ &\qquad\; i = 1, 2, \cdots, nd \\ &\qquad\; \mathbf{d}_s^L \le \mathbf{d}_s \le \mathbf{d}_s^U,\ \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \\ &\qquad\; \mathbf{X}_s^{N,L} \le \mathbf{X}_s^N \le \mathbf{X}_s^{N,U},\ \mathbf{X}^{N,L} \le \mathbf{X}^N \le \mathbf{X}^{N,U} \\ &DV = \{\mathbf{d}_s, \mathbf{d}, \mathbf{X}_s^N, \mathbf{X}^N\} \end{aligned} \tag{7}$$

where

$$\mathbf{Y} = (y_{ij}; i, j = 1 \sim nd, j \neq i)$$
$$y_{ij} = y_{ij}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s, \mathbf{X}_i, \mathbf{P}_i, \mathbf{Y}_{\bullet i})$$

where DV denotes design variables. $\mathbf{d}_s$ denote sharing deterministic variables which have no uncertainty; $\mathbf{d} = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{nd}\}$, and $\mathbf{d}_i$ are deterministic variables in discipline i. nd is total number of disciplines. $\mathbf{X}_s$ are sharing fuzzy variables with the maximal grade points as $\mathbf{X}_s^N$; $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{nd}\}$ is a vector composed of local fuzzy variables of each discipline, and $\mathbf{X}^N$ is its maximal grade point. $\mathbf{P} = \{\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{nd}\}$ is a vector composed of fuzzy parameters of each discipline, and $\mathbf{P}^N$ is its maximal grade point. $\mathbf{Y}$ denotes a vector of linking variables, y_{ij} is output of discipline i and input to discipline j. $\Pi(G^{(i)} > 0) \leq \alpha_t$ is possibility constraint in discipline i with the failure mode defined as $G^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s, \mathbf{X}_i, \mathbf{P}_i, \mathbf{Y}_{\bullet i}) > 0$, and $G^{(i)}(\cdot)$ is performance function. $g^{(i)} \leq 0$ is common constraint in discipline i.

Directly solving the PBMDO problem, there are three nested loops: minimizing the objective function in the outer loop; performing possibility analysis to check feasibilities of possibility constraints in the middle loop; performing multidisciplinary analysis to obtain consistency among multiple disciplines. To efficiently solve PBMDO problems, PBMDO within the framework of SORA (PBMDO-SORA) is proposed in the next section.

4. PBMDO-SORA.

In this section, PBMDO-SORA is discussed in detail including strategy, procedure and formulations.

4.1. Strategy of PBMDO-SORA.

For efficiently solving PBMDO problems, two technologies are adopted.

(1) PMA. PMA is more efficient than evaluating the actual reliability in RBDO [8,9], and PMA is also found efficient for PBDO [10,12-15]. Hence it is adopted in PBMDO.

(2) SORA. In this paper, with the idea of SORA, PBMDO is solved by series of cycles of deterministic MDO solution and possibility analysis. In each cycle, possibility analysis follows the deterministic MDO.

4.2. Procedure of PBMDO-SORA.

The PBMDO-SORA contains four basic steps:

Step 1: Set the initial values for design variables as $\mathbf{d}_s^{(0)}$, $\mathbf{d}^{(0)}$, $\mathbf{X}_s^{N,(0)}$, $\mathbf{X}^{N,(0)}$; $k = 1$.

Step 2: Solve the deterministic MDO. The results are optimal value of each deterministic variable $\mathbf{d}_s^{(k)}$, $\mathbf{d}^{(k)}$ and the maximal grade point of each fuzzy variable $\mathbf{X}_s^{N,(k)}$, $\mathbf{X}^{N,(k)}$ in the kth cycle. In the first cycle, the value of MPP in each deterministic constraint relevant to the possibility constraint is set to be equal to $\mathbf{X}_s^{N,(0)}$, $\mathbf{X}^{N,(0)}$, $\mathbf{P}^N$. From the second cycle, constraints are modified with the MPPs of previous cycle when possibility constraints are not all satisfied and the value of objective function is not stable.

Step 3: Perform possibility analysis. First of all, each fuzzy variable and parameter should be transformed into standard fuzzy ones using Equation (5) with its maximal grade point. The results of possibility analysis are MPP and value of performance function at the MPP of each possibility constraint. In the formulation of possibility analysis, consistencies among disciplines are treated as extra constraints similar as in [8].

Step 4: Check convergence. If constraints are all satisfied and the value of objective function is stable ($G^{(i)} \leq 0$, $i = 1 \sim nd$; $|f(k) - f(k-1)| \leq \varepsilon$, where ε is an arbitrary small positive value), stop the process of solution; otherwise, set $k = k+1$ and go to Step 2 with the MPPs obtained in Step 3.

If possibility constraint $\Pi(G^{(i)} > 0) \leq \alpha_t$ is not satisfied in Cycle $k-1$ which means that the value of performance function at its MPP satisfies $G^{(i)} > 0$, then the MPP $\left(\mathbf{X}_s^{*,(i),(k-1)}, \mathbf{X}^{*,(i),(k-1)}, \mathbf{P}^{*,(i),(k-1)}\right)$ obtained from possibility analysis in Cycle $k-1$ will be used to construct deterministic constraint in deterministic MDO for Cycle k. To make sure the feasibility of the possibility constraint, its MPP should fall into the deterministic feasible region. Let $\mathbf{S}$ be the shift vector. The shift is based on the idea of SORA in [9] as:

$$\begin{aligned} &\mathbf{S}_s^{(i)} = \mathbf{X}_s^{N,(k-1)} - \mathbf{X}_s^{*,(i),(k-1)} \\ &\mathbf{S}_j^{(i)} = \mathbf{X}_j^{N,(k-1)} - \mathbf{X}_j^{*,(i),(k-1)} \\ &i, j = 1 \sim nd \end{aligned} \tag{8}$$

where $\mathbf{S}_s^{(i)}$, $\mathbf{S}_j^{(i)}$ are shift vectors of $\mathbf{X}_s$, $\mathbf{X}_j$ in discipline i, respectively. $\mathbf{X}_s^{*,(i),(k-1)}$, $\mathbf{X}_j^{*,(i),(k-1)}$ are MPPs of $\mathbf{X}_s$, $\mathbf{X}_j$ in discipline i of the $(k-1)$th cycle, respectively.

The values of parameters of MPP directly substitute $\mathbf{P}$ in function $G^{(i)}$. The deterministic constraint in the kth deterministic MDO is as:

$$G^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s^N - \mathbf{S}_s^{(i)}, \mathbf{X}_i^N - \mathbf{S}_i^{(i)}, \mathbf{P}_i^{*,(i),(k-1)}, \mathbf{Y}_{\bullet i}^{*,(i)}) \le 0 \tag{9}$$

4.3. **Formulations in PBMDO-SORA.** The mathematical formulations of deterministic MDO and possibility analysis mentioned in the previous section will be provided and discussed in the following subsections.

4.3.1. *Deterministic MDO of the kth cycle.* The deterministic MDO of the kth cycle is

$$\begin{aligned} &\min_{DV} f(\mathbf{d}_s, \mathbf{d}, \mathbf{X}_s^N, \mathbf{X}^N, \mathbf{P}^N, \mathbf{Y}^N) \\ &s.t. \quad G^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s^N - \mathbf{S}_s^{(i)}, \mathbf{X}_i^N - \mathbf{S}_i^{(i)}, \mathbf{P}_i^{*,(i),(k-1)}, \mathbf{Y}_{\bullet i}^{*,(i)}) \le 0 \\ &\qquad g^{(i)}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s^N, \mathbf{X}_i^N, \mathbf{P}_i^N, \mathbf{Y}_{\bullet i}^N) \le 0 \\ &\qquad i = 1 \sim nd \\ &\qquad y_{ij}^N = y_{ij}(\mathbf{d}_s, \mathbf{d}_i, \mathbf{X}_s^N, \mathbf{X}_i^N, \mathbf{P}_i^N, \mathbf{Y}_{\bullet i}^N) \quad i, j = 1 \sim nd; \quad j \ne i \\ &\qquad y_{jm}^{*,(i)} = y_{jm}(\mathbf{d}_s, \mathbf{d}_j, \mathbf{X}_s^N - \mathbf{S}_s^{(i)}, \mathbf{X}_j^N - \mathbf{S}_j^{(i)}, \mathbf{P}_j^{*,(i),(k-1)}, \mathbf{Y}_{\bullet j}^{*,(i)}) \\ &\qquad i, j, m = 1 \sim nd; \quad m \ne j \\ &\qquad \mathbf{d}_s^L \le \mathbf{d}_s \le \mathbf{d}_s^U, \ \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \\ &\qquad \mathbf{X}_s^{N,L} \le \mathbf{X}_s^N \le \mathbf{X}_s^{N,U}, \ \mathbf{X}^{N,L} \le \mathbf{X}^N \le \mathbf{X}^{N,U} \\ &DV = \{\mathbf{d}_s, \mathbf{d}, \mathbf{X}_s^N, \mathbf{X}^N, \mathbf{Y}^N, \mathbf{Y}^*\} \end{aligned} \tag{10}$$

where $\mathbf{Y}^N = \left\{y_{ij}^N;\ i, j = 1 \sim nd,\ j \ne i\right\}$ are maximal grade points of $\mathbf{Y}$. $\mathbf{Y}^*$ denotes a vector of linking variables at MPPs, corresponding to possibility constraint. $\mathbf{Y}_{\bullet i}^{*,(i)}$ corresponds to the possibility constraint in discipline i. $G^{(i)} \le 0$ is the deterministic constraint corresponding to the possibility constraint in discipline i. $\mathbf{S}_s^{(i)}$, $\mathbf{S}_i^{(i)}$, $\mathbf{S}_j^{(i)}$ are shift vector of $\mathbf{X}_s$, $\mathbf{X}_i$, $\mathbf{X}_j$ in discipline i, respectively. $\mathbf{P}_i^{*,(i),(k-1)}$ are MPP of $\mathbf{P}_i$ in discipline i in the $(k-1)$th cycle.

The equality constraints for achieving consistency among disciplines are modified with $\mathbf{X}_s^{*,(i),(k-1)}$, $\mathbf{X}_j^{*,(i),(k-1)}$, $\mathbf{P}_j^{*,(i),(k-1)}$ as $y_{jm}^{*,(i)} = y_{jm}\left(\mathbf{d}_s, \mathbf{d}_j, \mathbf{X}_s^N - \mathbf{S}_s^{(i)}, \mathbf{X}_j^N - \mathbf{S}_j^{(i)}, \mathbf{P}_j^{*,(i),(k-1)}, \mathbf{Y}_{\bullet j}^{*,(i)}\right)$ $i, j, m = 1 \sim nd;\ m \ne j$. $\mathbf{S}_s^{(i)}$, $\mathbf{S}_j^{(i)}$ are shift vectors obtained from Equation (8) relevant to $\mathbf{X}_s$, $\mathbf{X}_j$ in discipline i. $\mathbf{X}_j^{*,(i),(k-1)}$, $\mathbf{P}_j^{*,(i),(k-1)}$ are MPPs obtained in the $(k-1)$th cycle corresponding to $\mathbf{X}_j$, $\mathbf{P}_j$ in discipline i.

4.3.2. *Possibility analysis of the kth cycle.* The possibility analysis of the kth cycle using PMA is

$$\begin{aligned}
&\max_{DV} G^{(i)}(\mathbf{d}_s^k, \mathbf{d}_i^k, \mathbf{U}_s^{(i),k}, \mathbf{U}_i^{(i),k}, \mathbf{U}_{\mathbf{P}i}^{(i),k}, \mathbf{Y}_{\bullet i}^{(i)}) \\
&s.t. \quad ||(\mathbf{U}_s^{(i),k}, \mathbf{U}^{(i),k}, \mathbf{U}_{\mathbf{P}}^{(i),k})||_\infty \le 1 - \alpha_t \\
&\qquad\; y_{jm}^{(i)} = y_{jm}(\mathbf{d}_s^k, \mathbf{d}_j^k, \mathbf{U}_s^{(i),k}, \mathbf{U}_j^{(i),k}, \mathbf{U}_{\mathbf{P}j}^{(i),k}, \mathbf{Y}_{\bullet j}^{(i)}) \\
&\qquad\; j, m = 1 \sim nd; \quad m \ne j \\
&DV = \{\mathbf{U}_s^{(i),k}, \mathbf{U}^{(i),k}, \mathbf{U}_{\mathbf{P}}^{(i),k}, \mathbf{Y}^{(i)}\} \\
&\qquad\; i = 1 \sim nd
\end{aligned} \tag{11}$$

where $\mathbf{d}_s^k$, $\mathbf{d}_i^k$, $\mathbf{d}_j^k$ are optimal values of deterministic variables obtained after solving deterministic MDO. The superscript k indicates variables in the kth cycle, and "(i)" denotes variables and parameters corresponding to the possibility constraint in discipline i. $\mathbf{U}_s^{(i),k}$, $\mathbf{U}_i^{(i),k}$, $\mathbf{U}_{\mathbf{P}i}^{(i),k}$, $\mathbf{U}_j^{(i),k}$, $\mathbf{U}_{\mathbf{P}j}^{(i),k}$ are respectively transformed from $\mathbf{X}_s$, $\mathbf{X}_i$, $\mathbf{P}_i$, $\mathbf{X}_j$, $\mathbf{P}_j$. $\mathbf{Y}^{(i)}$ is linking variables at MPP, corresponding to possibility constraint in discipline i. $\mathbf{Y}_{\bullet i}^{(i)}$, a vector of linking variables, is input to discipline i. $\mathbf{U}^{(i),k} = \left\{\mathbf{U}_1^{(i),k}, \mathbf{U}_2^{(i),k}, \cdots, \mathbf{U}_{nd}^{(i),k}\right\}$, $\mathbf{U}_{\mathbf{P}}^{(i),k} = \left\{\mathbf{U}_{\mathbf{P}_1}^{(i),k}, \mathbf{U}_{\mathbf{P}_2}^{(i),k}, \cdots, \mathbf{U}_{\mathbf{P}_{nd}}^{(i),k}\right\}$.

It is implied that the performance function includes all design input because of the existences of linking variables. In Equation (11), the first constraint includes all fuzzy design inputs. The consistency among disciplines is treated as extra constraint.

The results of possibility analysis are MPP $\left(\mathbf{U}_s^{*,(i),(k)}, \mathbf{U}^{*,(i),(k)}, \mathbf{U}_{\mathbf{P}}^{*,(i),(k)}\right)$ and value of performance function at MPP $G^{(i)}$, $i = 1 \sim nd$. The MPP in the X-space $\left(\mathbf{X}_s^{*,(i),(k)}, \mathbf{X}^{*,(i),(k)}, \mathbf{P}^{*,(i),(k)}\right)$ can be obtained by the inverse Equation (5) based on $\mathbf{U}_s^{*,(i),(k)}$, $\mathbf{U}^{*,(i),(k)}$, $\mathbf{U}_{\mathbf{P}}^{*,(i),(k)}$.

5. **Numerical Examples.** In this section, a mathematical example and an engineering design are used to demonstrate the proposed method. The method proposed in this paper and the RBMDO-SORA are compared based on the obtained results. The efficiency of the proposed method is demonstrated through results.

5.1. **Mathematical example for PBMDO.** The mathematical example that is slightly modified from [8] is given as:

$$\begin{aligned}
&\min_{(d_s, d_1, d_2)} v\left(\mathbf{d}, \mathbf{x}^N\right) = \left(d_s + x_s^N\right)^2 + d_1^2 + d_2^2 \\
&s.t. \quad \Pi\{G_1(\mathbf{d}, \mathbf{x}) = x_1 - d_s - x_s - d_1 - d_2 > 0\} \le \alpha_t \\
&\qquad\; \Pi\{G_2(\mathbf{d}, \mathbf{x}) = d_s + x_s - 2d_1 + d_2 - x_2 > 0\} \le \alpha_t \\
&\qquad\; 0 \le d_s,\ d_1,\ d_2 \le 5
\end{aligned} \tag{12}$$

where the triangular memberships of x_s, x_1, x_2 are $(-0.9, 0, 0.9)$, $(3.5, 5, 6.5)$, $(0.7, 1, 1.3)$, respectively. It should be noted that in this problem x_s, x_1, x_2 are fuzzy parameters because their maximal grade points are fixed. The problem is decomposed into two subsystems as in [8] in Figure 1.

The results of RBMDO with $x_s \sim N(0, 0.3)$, $x_1 \sim N(5, 0.5)$, $x_2 \sim N(1, 0.1)$ solved by RBMDO-SORA and PBMDO are listed in the Table 1.

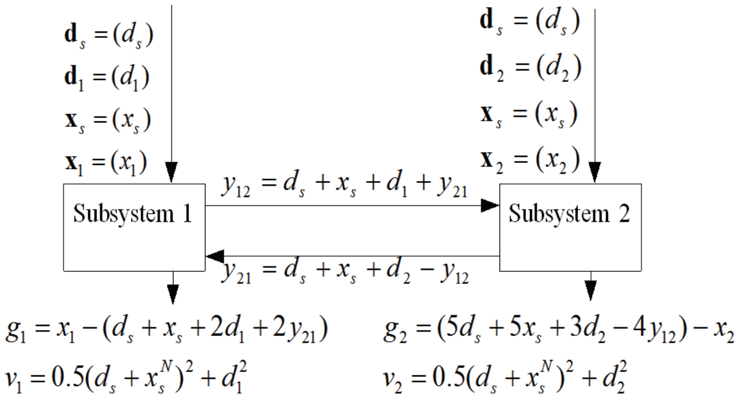

FIGURE 1. Mathematical problem

TABLE 1. Results of RBMDO-SORA and PBMDO-SORA

	Design Variables			Obj.			Number		
	d_s	d_1	d_2	v	G_1	G_2	n_1	n_2	k
RBMDO ($\beta = 3$)	2.2497	2.2498	2.2498	15.1843	0	–0.0513	451	635	
PBMDO ($\alpha_t = 0.0013$)	2.4326	2.5318	2.4326	18.2445	0	-1.7764×10^{-15}	354	354	3
($\alpha_t = 0.005$)	2.4303	2.5273	2.4303	18.2005	0	-1.7764×10^{-15}	354	354	3
($\alpha_t = 0.0001$)	2.4333	2.5332	2.4333	18.2588	0	-2.6645×10^{-15}	354	354	3

The start point is $(0, 0, 0)$. The second row shows results from [8]. n_1 is the number of discipline analysis for subsystem 1 and n_2 is for the other. When the target possibility of failure equals to the target probability of failure $\alpha_t = 0.0013 = 1 - \Phi(\beta) = 1 - \Phi(3)$, PBMDO provides a more conservative design. The reason is: since there is insufficient data to precisely construct probability distributions, membership functions are used based on possibility theory; this causes the difference between possibility analysis and probability analysis. In the possibility analysis, the constraint is a hypercube while in probability analysis is a hyper-sphere. The number of discipline analysis in PBMDO-SORA is in the same magnitude as that in RBMDO-SORA. From the values of objective function in the fifth column, the lower the target possibility of failure is, the more conservative the design will be. All possibility constraints are satisfied at each optimal design since values of performance functions at the MPPs are all not larger than zeros. PBMDO-SORA efficiently solves this problem with different target possibility of failure in three cycles.

5.2. **Design of a pressure vessel.** The example of pressure vessel design showed in Figure 2 is derived from [19]. The design variables are radius (R), length (L) and thickness (T). There are two parameters: internal pressure (P) and allowable tensile strength of the material (S_t). The objective is to maximize the internal volume while minimize the weight. In this paper, this problem is modified into an MDO problem.

The pressure vessel is designed by two design groups, and the coupled variables are thickness (T), length (L) and radius (R). The multidisciplinary systems and notations are given in Figure 3. T, R and L are all fuzzy variables, P and S_t are fuzzy parameters. Table 2 shows the membership functions of design variables and parameters.

Sharing design variables: $\mathbf{d}_s = \phi$. Sharing fuzzy variables: ϕ.
Subsystem 1: fuzzy variable is $\mathbf{X}_1 = \{T\}$; input linking variables are $\mathbf{Y}_{21} = \{y_{21,1}, y_{21,2}\} = \{R, L\}$; output linking variable is $\mathbf{Y}_{12} = \{y_{12}\} = \{T\}$; output is $Z_1 = \{v_1\}$, $v_1 =$

$\frac{4}{3}\pi\left(T^N + y_{21,1}^N\right)^3 + \pi\left(T^N + y_{21,1}^N\right)^2 y_{21,2}^N - \left[\frac{4}{3}\pi\left(y_{21,1}^N\right)^3 + \pi(y_{21,1}^N)^2 y_{21,2}^N\right]$. In this subsystem, the objective is to minimize the weight, which is equivalent to minimize the relevant volume. The possibility constraints in Subsystem 1 are those:

$$\Pi\{G_{11} = 5T - y_{21,1} > 0\} \le \alpha_t$$
$$\Pi\{G_{12} = T + y_{21,1} - 40 > 0\} \le \alpha_t$$

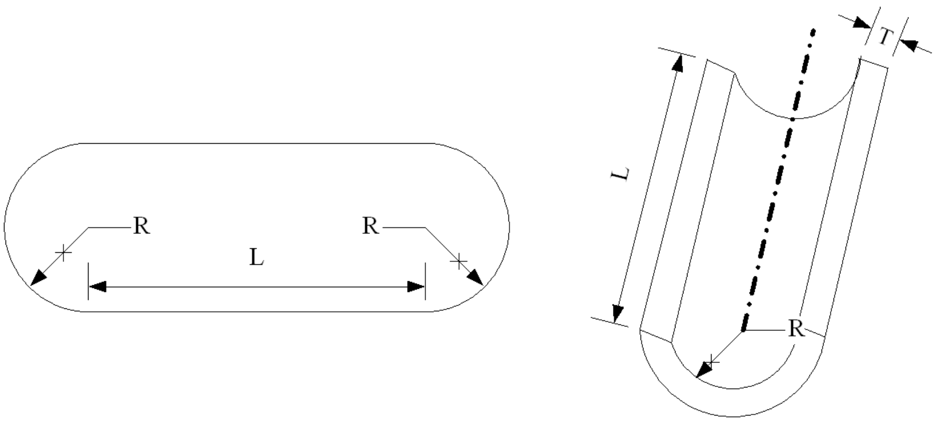

FIGURE 2. Pressure vessel

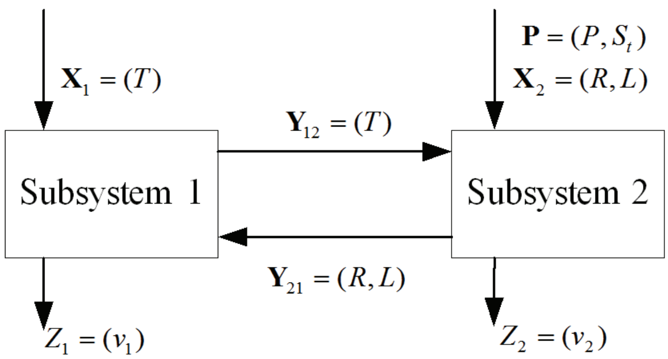

FIGURE 3. MDO problem of pressure vessel

TABLE 2. Membership functions of design variables and parameters

Variables or parameters	Maximal grade point	Deviation	Membership function	Low bound of maximal grade point	Up bound of maximal grade point
R		0.03	Triangular	0.1	36
T		0.03	Triangular	0.5	6.0
L		0.03	Triangular	0.1	140
P	3.89	1.167	Triangular		
St	40	12	Triangular		

Subsystem 2: fuzzy variables are $\mathbf{X}_2 = \{R, L\}$; input fuzzy parameters are $\mathbf{P} = \{P, S_t\}$; input linking variable is $\mathbf{Y}_{12} = \{y_{12}\} = \{T\}$; output linking variables are $\mathbf{Y}_{21} = \{y_{21,1}, y_{21,2}\} = \{R, L\}$; output is $Z_2 = \{v_2\}$, $v_2 = \frac{4}{3}\pi\left(R^N\right)^3 + \pi\left(R^N\right)^2 L^N$. In this subsystem, the

objective is to maximize internal volume. The possibility constraints in Subsystem 2 are those:

$$\Pi\left\{G_{21} = \frac{PR}{y_{12}} - S_t > 0\right\} \le \alpha_t$$
$$\Pi\{G_{22} = L + 2R + 2y_{12} - 150 > 0\} \le \alpha_t$$

The whole objective, v, is to minimize $v_1 - v_2$. To compare PBMDO with RBMDO, the results of PBMDO and RBMDO with $T \sim N(\mu_T, 0.01)$, $R \sim N(\mu_R, 0.01)$, $L \sim N(\mu_L, 0.01)$, $P \sim N(3.89, 0.389)$, $S_t \sim N(40, 4)$ are listed in Tables 3 and 4.

Table 3. Design of PBMDO and RBMDO

	T^N	R^N	L^N	v	v_1	v_2	n_1	n_2	k
PBMDO $\alpha_t = 0.0013$	6.0000	33.0538	71.7427	-2.0177×10^5	1.9575×10^5	3.9752×10^5	2484	2484	4
$\alpha_t = 0.005$	6.0000	33.1353	71.5802	-2.0310×10^5	1.9619×10^5	3.9929×10^5	2193	2193	3
$\alpha_t = 0.0001$	6.0000	33.0274	71.7953	-2.0134×10^5	1.9560×10^5	3.9694×10^5	2464	2464	4
	μ_T	μ_R	μ_L	v	v_1	v_2	n_1	n_2	k
RBMDO $\beta = 3$	5.2475	34.7100	69.9949	-2.6187×10^5	1.7822×10^5	4.4009×10^5	3428	3428	5

Table 4. Values of performance functions at relevant MPPs

	G_{11}	G_{12}	G_{21}	G_{22}
PBMDO $\alpha_t = 0.0013$	–2.8740	–0.8863	-3.5527×10^{-15}	0
$\alpha_t = 0.005$	–2.9562	–0.8050	-3.5527×10^{-15}	0
$\alpha_t = 0.0001$	–2.8474	–0.9126	0	0
	G_{11}	G_{12}	G_{21}	G_{22}
RBMDO $\beta = 3$	–8.3194	1.8666×10^{-11}	-2.7477×10^{-11}	8.6914×10^{-11}

Table 3 lists the design, corresponding objective function values and the number of disciplinary analysis of PBMDO at different possibility of failure and RBMDO. Same as the results in the mathematical problem, when the target possibility of failure equals to target probability of failure, PBMDO provides a more conservative design than that of RBMDO. But the number of discipline analysis in PBMDO-SORA is in the same magnitude as that in RBMDO-SORA. Meanwhile, the lower the target possibility of failure is, the more conservative the optimal design will be. The PBMDO-SORA efficiently solves this problem with different target possibility of failure in a few cycles.

Table 4 shows value of performance function of each possibility constraint at corresponding Most Possibility Point (MPP) and that of each probability constraint at the Most Probability Point (MPP). At each optimal design point, the values of performance function of each possibility constraint at its MPP are all not larger than zeros which indicates that all possibility constraints are satisfied. At the optimal design point of RBMDO, the values of G_{12} and G_{22} at their MPPs are almost equal to zeros which indicates that the corresponding probability constraints are active.

6. **Conclusions.** If sufficient data is not available to construct probability distributions, it is recommended to use PBMDO instead of RBMDO. A method, PBMDO-SORA, is proposed in this paper. This method can efficiently solve MDO problems in which uncertainty in design variables is described with insufficient data, and can also decrease the computational demand. In this method, the PBMDO problem is solved by sequential deterministic MDO and possibility analysis. The deterministic MDO is reconstructed with the MPP obtained in the previous cycle to improve the feasibilities of possibility constraints. After solving the deterministic MDO, possibility analysis is applied to analyze the feasibility of each possibility constraint at the new design point. Based on this, in each cycle the MDO solution and possibility analysis are sequential but not nested.

As illustrated in the demonstrative examples, PBMDO-SORA can efficiently solve PBMDO problems in a few cycles. When target possibility of failure is set to be same as probability of failure, PBMDO provides more conservative design. The reason is: since there is insufficient data to precisely construct probability distributions, membership functions are used based on possibility theory; this causes the difference between possibility analysis and probability analysis. In the possibility analysis, the constraint is a hypercube while in probability analysis is a hyper-sphere. For a lower target possibility of failure, the result will be more conservative. However, the number of discipline analysis in PBMDO-SORA is in the same magnitude as that in RBMDO-SORA.

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