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Fatigue Reliability Analysis of Turbine Disk Alloy Using Saddlepoint Approximation

Abstract: In this paper, a new fatigue reliability analysis method based on saddlepoint approximation (SPA) was proposed for calculating the probability of failure of turbine disk alloy in a low cycle fatigue (LCF) regime. Firstly, two LCF life prediction models based on total strain energy density and Support Vector Regression (SVR) metamodel are presented for turbine disk alloy GH4133 under different loading conditions at 250 °C. Compared with the SWT model, modified Walker model and Response Surface (RS) model, the predicted lives by the proposed models are within a factor of ±2 and a factor of ±1.1 respectively. Secondly, based on the fatigue design criteria, the probabilities of failure are calculated using SPA for the explicit and implicit performance functions using two proposed LCF models and viscosity-based model. These three models have provided the reliability design rules for GH4133. Finally, the failure probabilities curves between SPA and the designed fatigue lives are achieved. The reliability analysis results were found to be in good agreement with the calculated results of test data. These results show that SPA is very apt for the fatigue reliability analysis of turbine disk under different loading conditions using only a small number of samples without any distribution assumptions for random variables. Moreover, it can be used to estimate the system's probability of failure with a large number of random variables or high nonlinearity of performance functions. The effectiveness and accuracy of the combination of the fatigue models and SPA for fatigue reliability analysis are verified using three examples.

Keywords: LCF, SPA, life prediction, total strain energy density, fatigue reliability analysis

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1 Introduction

Fatigue failure is one of the main failure modes for the aero engine components. As the main durability and fracture critical components of an aero engine, turbine disk is subjected to great centrifugal forces and thermal stresses as well as vibration stresses interaction during operations [1]. For the turbine disk, low cycle fatigue (LCF), creep and their interaction are the most principal reasons that induce the life consumption of turbine disk, which is one of the most serious and costly issues, and their service lives will directly influence the life and reliability of an aero engine [2]. Fatigue reliability of high-temperature components of turbine engines is directly affected by various uncertainties, such as variations in loading, material properties, physical dimensions and environments. According to uncertainty analysis in design process, reliability issues on LCF and low cycle fatigue-creep (LCF-C) life prediction can be addressed early to prevent the occurrence of failure events that may lead to a catastrophic consequence, which plays an important role in the material selection and design of these high temperature components.

LCF life prediction has become a considerable research subject for aero engines. Early efforts have been focused on the strain-based and/or energy-based approaches for fatigue life prediction [3–9]. Through considering the effects of mean stresses on fatigue life, a LCF life prediction model based on the total strain energy density is established for turbine disk alloy GH4133 under 250 °C. Compared with SWT model and modified Walker model, lives predicted by the total strain energy density model are in good agreement with those tested and within a factor of ±2. Until now, there does not exist a robust method that can be applied for a wide class of materials and loading conditions [10], nor there exists a universal model of fatigue damage and quite a few 'general' models are suitable for actual application [1]. Besides, many 'advanced' models are so complicated that are not conventional to apply in engineering. Thus, when subjected to LCF and creep interaction, by considering the effects of loading waveform, ratchet effect and mean stress on fatigue life, Zhu et al. [11] proposed a viscosity-based LCF-C life prediction model for turbine disk alloy. Compared with other life prediction model, it has higher accuracy and robustness.

A significant number of uncertainties can be found in engineering practice such as epistemic and aleatory uncertainties. Although surrogate model or metamodel technology has been extensively applied for the field of engineering optimization, it is rare that the surrogate model research applicable for uncertainty problem. Compared with other surrogate, the precision and robustness in metamodel based on support vector regression (SVR) is better than other traditional models [12–13]. Even though SVR method has been applied to a certain degree in function approximation and surrogate model [14–19], its application can be seen a little to uncertainty surrogate model and still imperfection, especially in the surrogate model research for uncertainty existed in fatigue reliability analysis of engineering components. In view of uncertainties existed in the reliability assessment of turbine disk and the relationship between uncertainties and the properties of components is complex or unknown, and it is often difficult to obtain their explicit function. In this paper, through finite element method (FEM) analysis in [20] and Manson-Coffin equation, LCF life prediction model of metamodel based on SVR and response surface (RS) model are respectively built.

Many methods have been put forward for failure probability calculation and uncertainty analysis in engineering, such as the First-Order Reliability Method (FORM), the Second-Order Reliability Method (SORM), Monte Carlo Simulation (MCS) and saddlepoint approximation (SPA). Compared with FORM and SORM, SPA requires neither solving the derivative of the performance function nor the most probable point (MPP) search, neither does it normal space to non-normal space transformation. It produces much more precise probability estimation, especially in high required reliability system [21]. Available research indicates that SPA is apt for large-scale complex components and system as well as high nonlinearity performance function [22–24]. Though SPA has been more studied in the field of statistics, it is employed in reliability domain still imperfect. Since it is difficult to obtain large amounts of data and to define the probability distribution of random variables is often difficult. Compared with MCS, the advantages of SPA is that it can not only be employed to known distribution of random variables, but also not need suppose distribution, only a small quantity of data required, then reliability analysis can be performed. Since complex component such as turbine disk expensively manufacture and costly tests, LCF test for turbine disk is characterized as a significant small sample. Although test data for turbine disk is small sample, high level of reliability and confidence degree is required. In this paper, SPA is utilized to assess the reliability and calculate probabilities of failure using two proposed models and viscosity-based model.

The effectiveness and feasibility of SPA for fatigue reliability are illustrated with three examples. In examples 1 and 2, SPA is used for calculating failure probabilities of two explicit performance functions which are derived from LCF life prediction model based on total strain energy density and a viscosity-based LCF life prediction model and with higher nonlinearity. In example 3, LCF implicit models for strain ranges are developed using metamodel based on SVR, then, predicted lives are derived. Finally, SPA is used to calculate probabilities of failure through constructing two implicit performance functions.

2 LCF life prediction models

Fatigue cracks of turbine disk often occurred mainly located at the groove, serration root or blade root of integral bladed disk, pin holes, bolt holes and center holes. Stress concentration and big plastic deformation as well as LCF and creep of shorter life can be easily introduced around these areas. Hence, LCF life prediction model with high precision and excellent reliability estimation approach are significant for aero engines' practical application. During the design process, a physics-based fatigue life prediction model and failure assessment methodology that is easy to use would enable the designers to evaluate the fatigue behavior of their preliminary designs and modifications before arriving at a final design, which would greatly improve design precision and save potential design cost.

2.1 LCF life prediction model based on total strain energy density

Assessment precision of fatigue life mainly depends on the reasonable selection of damage parameter. During cyclic plasticity a certain amount of energy is stored in the material as hysteresis energy and failure will occur when the cumulative energy is equal to the energy required for fracture [25]. Cyclic plastic strain energy is a significant parameter to characterize the fatigue damage. Fatigue failure in metals whether high cycle fatigue (HCF) or LCF are both attributed to the increasing accumulation of cyclic plastic strain energy. Once cumulative cyclic strain energy approach to a critical value, crack nucleates, initiate to propagate till total fracture failure [26]. Fatigue fracture mechanism shows that fatigue fracture of material is mainly caused by interaction of cyclic stresses, tensile stresses and plastic deformations. Most of fatigue life prediction models are mainly linked fatigue lives with stresses, strains and strain energy density. In addition, damage models based on energy criterion are mainly involved with cyclic plastic strain energy and total strain energy. On account of elastic strain energy associated with the tensile stress and the impacts of mean stresses on fatigue lives of turbine disk, cyclic plastic strain energy does not involve this kind of influence. Hence, in order to account for the effects of mean stress on fatigue life, total strain energy density should include cyclic plastic strain energy and tensile elastic strain energy. In general, total strain energy theory has a clear physical meaning and high precision and can account for the nature of fatigue damage. Lots of research have been done for cyclic strain energy theory, Ellyin [27–28] presented a damage parameter using total strain energy density, and it can be modeled using $\Delta W_t = \Delta W_p + \Delta W_e^+$. But the effects of mean stresses and tensile stresses on the fatigue lives of material can not be taken into account within this model. Kujawski [29] proposed a life formula, but the physical meaning of involving mean stress function is not clear in the formula, and β only can be obtained by experimental formula approximation. Chen [30] deduced a modified energy model of LCF life prediction model by entropy conservation and energy conservation law. To account for the effects of mean stresses on fatigue life of high pressure tube steel, Koh [31] proposed a fatigue life prediction model based on cyclic total strain energy density concept. In view of the shortcomings of these models when applied to turbine disk, and the action crack nucleation and propagation stress amplitude and maximum stress build the foundation of *S-N* curve, accounting for the effect of stress amplitude [32] and mean stress on the life of turbine disk. In this paper, a LCF life prediction model is established by using the physics based damage parameter of total strain energy density. On account of mean stress effects on life evaluation, mean stress should be reckoned in tensile elastic strain energy for total strain energy calculation. For a Masing type material, the cyclic plastic strain energy density is

$$\Delta W_p = 4\frac{1-n'}{1+n'}K'\left(\frac{\Delta\varepsilon_p}{2}\right)^{1+n'} \tag{1}$$

Tensile elastic strain energy density is given by

$$\Delta W_e^+ = \begin{cases} \dfrac{(\sigma_a+\sigma_m)^2}{2E}, & for\ \sigma_{min}<0 \\ \dfrac{2\sigma_a^2}{E}, & for\ \sigma_{min}\ge 0 \end{cases} \tag{2}$$

thus, the total strain energy density based life prediction model can be achieved by

$$\left(\Delta W_p + \Delta W_e^+\right) N_f^{\alpha} = C \tag{3}$$

Rearranging the various items in Eq. (3), predicted life N_{fp} by energy-based LCF prediction model can be calculated as

$$N_{fp} = \begin{cases} \left(\dfrac{C}{\left[4\dfrac{1-n'}{1+n'}K'\left(\dfrac{\Delta\varepsilon_p}{2}\right)^{1+n'} + \dfrac{(\sigma_a+\sigma_m)^2}{2E}\right]}\right)^{1/\alpha}, & for\ \sigma_{min}<0 \\ \left(\dfrac{C}{\left[4\dfrac{1-n'}{1+n'}K'\left(\dfrac{\Delta\varepsilon_p}{2}\right)^{1+n'} + \dfrac{2\sigma_a^2}{E} + \sigma_{min}\Delta\varepsilon_t\right]}\right)^{1/\alpha}, & for\ \sigma_{min}\ge 0 \end{cases} \tag{4}$$

To validate the feasibility and accuracy of the proposed energy-based model, this LCF life prediction model is verified using the test data in [1]. The temperature of center holes and pin holes of turbine disk is about 250 °C during operation and fatigue critical locations are around the intersecting locations of radial pin holes and center holes. During operation, the deformation of turbine disk is mainly under elastic deformation, but in some cases, stresses concentration sites around pin holes have introduced plastic deformation, which often leads to crack initiation and propagation. According to stress-strain relationship, it's worth noting that the fatigue strength and life of turbine disk relates to the maximum stress and strain in strain concentration site. Thus, the turbine disk's life depends on the life of pin holes. The LCF test data under constant amplitude axial loading of turbine disk alloy GH4133 were supplemented in [1] at 250 °C under $R=-1$ and $R=0.44$, respectively. Loading frequency is $f=0.25$–0.5 Hz and loading waveform is triangular wave.

Material constants K' and n' of GH4133 in Eq. (1) are 2211.0227 *MPa* and 0.1235119 respectively. For $\sigma_{min}<0$ and

$\sigma_m = 0$, by fitting the test data, the life prediction model for GH4133 can be expressed as

$$\left[6899.6\left(\frac{\Delta\varepsilon_p}{2}\right)^{1.1235119} + \frac{\sigma_a{}^2}{2E}\right] \cdot N_f{}^{0.673671} = 1114.749 \quad (5)$$

For GH4133 under $R = 0.44$, for $\sigma_{\min} < 0$, by fitting the test data, the life prediction model can be expressed as

$$\left[5804.4\left(\frac{\Delta\varepsilon_p}{2}\right)^{1.0971544} + \frac{\sigma^2{}_{max}}{2E}\right]^{0.570502} = 373.4627 \quad (6)$$

Correlations between the total strain energy density and the life of turbine disk at 250 °C under different loading conditions are observed as seen in Figs. 1–2. From Fig. 1, note that the predicted results are in good agreement with the test ones when $R = -1$. A fitting curve of total strain energy density versus fatigue life is plotted in Fig. 2 when $R = 0.44$, a noticeable deviation from fitting curve can be observed for the predicted results. In general, due to the mean stress relaxation, the imposed mean strain has no effect on the fatigue life at constant strain amplitude. Moreover, a stable non-zero tensile mean stress often reduces the fatigue life but a compression one increases it due to mean strain action [33]. Therefore, the effects of mean stress have a significant impact on the fatigue life of turbine disk.

To verify the prediction ability of the proposed model, the predicted lives by the proposed model are compared

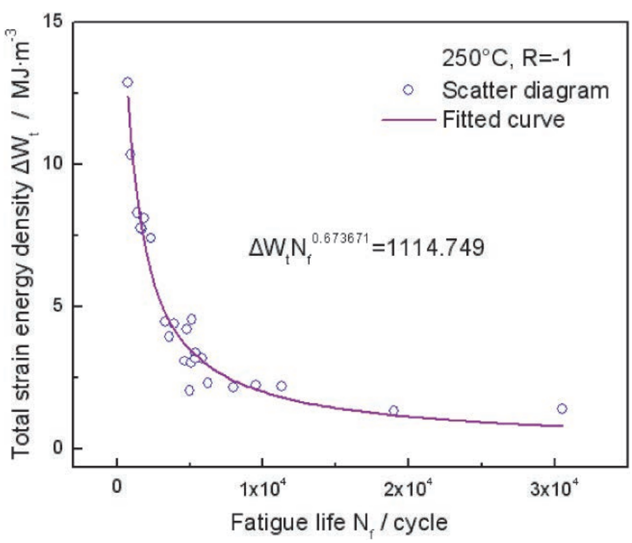

Fig. 1: The fitting curve of total strain energy density versus lives tested for GH4133 under symmetric loading

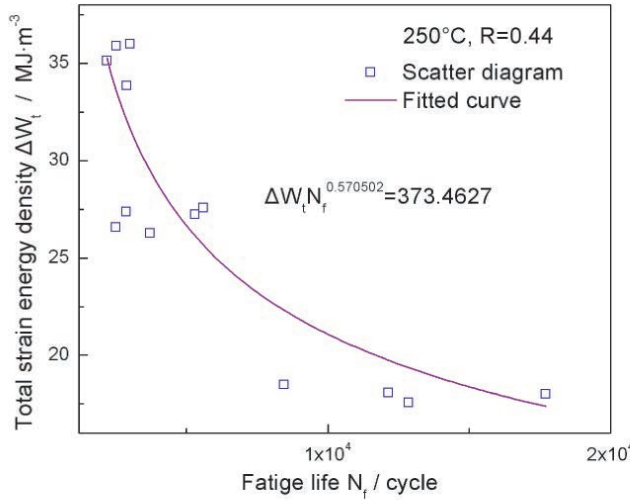

Fig. 2: The fitting curve of total strain energy density versus lives tested for GH4133 under non-symmetrical loading

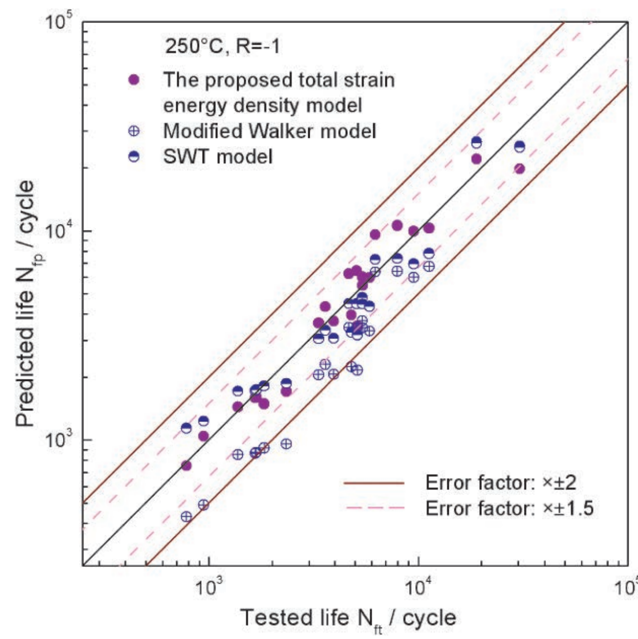

Fig. 3: Comparison between lives predicted by the proposed model, Modified Walker model, SWT model and those tested for GH4133 at 250 °C under symmetric loading

with the predicted lives by SWT model as well as modified Walker model under different strain ratios. N_{fp} and N_{ft} of these three models are compared as shown in Fig. 3 and Fig. 4.

When $R = -1$, from Fig. 3, results show that all the predicted cyclic lives by the proposed model and SWT model fall into a scatter band of ±2 and nearly within a factor of ±1.5, while about 20 out of 23, 9 out of 23 cyclic lives predicted by the modified Walker model are within a factor of

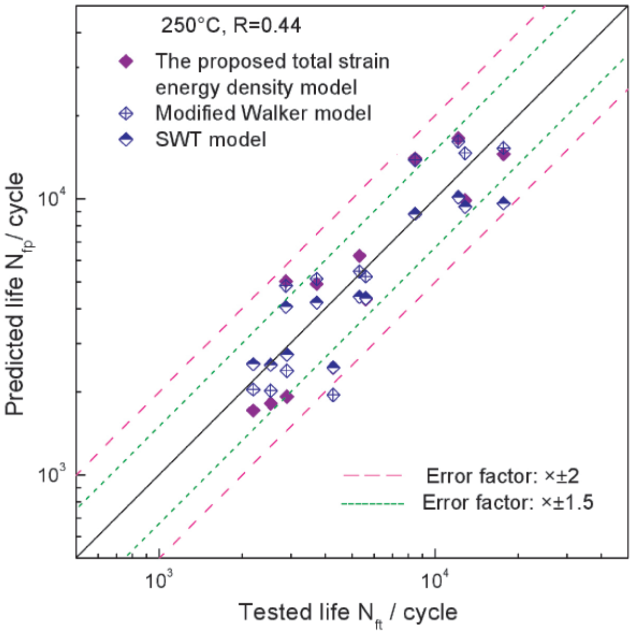

Fig. 4: Comparison between lives predicted by the proposed model, Modified Walker model, SWT model and those tested for GH4133 at 250 °C under non-symmetrical loading

±2 and a scatter band of ±1.5, respectively. In Fig. 4, it should be noted that the predicted lives by the proposed model and SWT model are all within a factor of ±2, and about 11 out of 13 cyclic lives are predicted within a scatter band of ±1.5 to the test ones, but about 12 out of 13 predicted lives form modified Walker model are within a factor of ±2 and about 11 out of 13 predicted lives are within a scatter band of ±1.5. Comparing the scatter band and standard deviation of these methods, results indicated that the proposed model has a better predictability than others. The correlation of predicted life and experimental results is satisfactory under different loading conditions.

2.2 LCF-C life prediction using viscosity-based model

Fatigue-creep interaction is one of the main reasons that lead to turbine disk failure [10]. Due to the LCF-C failure mechanism for turbine disk, its failure relates not only with plastic strain range, but also with material properties and loading factors, moreover with environmental factors. Through considering the effects of loading waveform, mean stress and ratcheting behavior on LCF-C life, a viscosity-based model for LCF-C life prediction under high temperature is developed [11].

$$N_f = k(E_p - T_0 \cdot \Delta W_{FL})^p (\Delta\varepsilon_{in}\sigma_{max})^q \tag{7}$$

where E_p is a viscosity-based parameter, and it is used to describe fatigue-creep damage. $E_p = T_{du}\sigma_{max} + (T_{dl}+T)\sigma_{min}H(\sigma_{min}) + \frac{T}{2}f(\sigma_{max},\sigma_{min})$, in one loading cycle when $\sigma_{max} > 0$ and $\sigma_{min} < 0$, T_{dl}, T_{du}, T' and T'' represent respectively the tensile hold time, compressive hold time, tension-going time and compression-going time. When $\sigma_{min} > 0$, T_{du} is also the tensile hold time. T_0 and T are the total time periods, where $T = T' + T''$ is the period time not containing the holding load time. $f(\sigma_{max},\sigma_{min})$ is stress conversion function, $H(\sigma_{min})$ is the unit step function of σ_{min}. The stress conversion function can is defined as

$$f(\sigma_{max},\sigma_{min}) = \begin{cases} \Delta\sigma, & for\ \sigma_{min} > 0 \\ \dfrac{\sigma^2{}_{max}}{\Delta\sigma}, & for\ \sigma_{min} \le 0 \end{cases} \tag{8}$$

ΔW_{FL} is strain energy density at the fatigue limit of the material which causes no damage, where σ_{lim} is the fatigue limit of material, E is Yong's modulus, $\Delta\varepsilon_{in}$ is the inelastic strain range.

To validate the feasibility and efficiency from the viscosity-based model of LCF-C life prediction for high temperature components, predicted lives for LCF-C are evaluated using test data from [1] with different strain ratios and temperature (400 °C and 500 °C) for GH4133. Using Eq. (7), a general model for LCF-C life prediction is derived as

$$N_f = 4.99462\times10^{31}(E_p - 1.7138\cdot\Delta W_{FL})^{-3.0382} \cdot(\Delta\varepsilon_{in}\sigma_{max})^{-0.370252} \tag{9}$$

where $\Delta W_{FL} = 0.444248\ \mathrm{MJ\cdot m^{-3}}$, $\sigma_{lim} = 420.7$ MPa, $E = 1.992\times10^5$ MPa. Owing to loading waveform being triangular wave, $T_{dl} = T_{du} = 0$, where $\Delta\varepsilon_{in}$ is inelastic strain range and $\Delta\varepsilon_c$ is creep strain range, E_p is simplified as

$$E_p = \begin{cases} T\sigma_{min} + \dfrac{T}{2}\Delta\sigma, & for\ \sigma_{min} > 0 \\ \dfrac{T}{2}\cdot\dfrac{\sigma^2{}_{max}}{\Delta\sigma}, & for\ \sigma_{min} \le 0 \end{cases} \tag{10}$$

Substituting E_p and ΔW_{FL} into Eq. (9), the expression set of N_f is

$$N_f = \begin{cases} 4.99462\times10^{31}(T\sigma_{min} + \frac{T}{2}\Delta\sigma - 7.613526*10^5)^{-3.0382}(\Delta\varepsilon_{in}\sigma_{max})^{-0.370252}, \\ \quad for\ \sigma_{min} > 0 \\ 4.99462\times10^{31}(\frac{T}{2}\cdot\frac{\sigma^2{}_{max}}{\Delta\sigma} - 7.613526*10^5)^{-3.0382}(\Delta\varepsilon_{in}\sigma_{max})^{-0.370252}, \\ \quad for\ \sigma_{min} \le 0 \end{cases} \tag{11}$$

When the temperature is 400 °C and 500 °C, T = 1.7138, $R = -1$, using Eq. (11) and LCF-C life can be obtained by

$$N_f = 4.99462 \times 10^{31} (0.42845 \cdot \sigma_{\max} - 7.613526 * 10^5)^{-3.0382} \cdot (\Delta\varepsilon_p \sigma_{\max})^{-0.370252} \tag{12}$$

The results show that the viscosity-based model has an excellent prediction effect compared with Goswami's ductility model and SWT model [11].

2.3 LCF life prediction using metamodel based on SVR

For structure reliability analysis using regression metamodel based on uncertainty, explicit performance function based on input random variables and response random variables is often difficult to be obtained such as turbine disk under thermal-mechanical fatigue loading. Thus, the performance function is often established as implicit function but not explicit function, this brings great difficulty in directly integral to calculate the failure probability as a result of structure failure region is often complex integral region. Therefore, lots of approximate calculation methods have been put forward for failure probability, mainly including the approximate probability method, Monte Carlo method and its variance reduction technique [34], stochastic finite element analysis method [35] and surrogate model [36], in which, metamodel method includes the RS method, Kriging method, Radial Basis Function (RBF) method, Artificial Neural Network (ANN) method and Support Vector Machine (SVM) method. At present, in view of the serious defect can be found in conventional surrogate models, the main cause is that one of the common theory foundation in traditional surrogate model is conventional statistics for studying approximation theory that sample numbers trend to infinite, but in fact, input random variables' samples are limited. Metamodel based on SVR is adopted by Clarke et al. [12] to classical engineering instance and compared with RS, RBF, regression for multi-variables and Kriging model. A regression modeling technique with excellent development prospects, SVR metamodel has several advantages compared with the other metamodels as follows: Firstly, it is established based on VC dimension theory of statistics and Structural Risk Minimization (SRM) principle, its algorithm theory is very rigorous, and it adopts convex quadratic programming optimization theory to seek for the global optimal solution of limited sample information; Secondly, for nonlinear problems, Kernel function technology is used in SVR algorithm for mapping nonlinear problem of low dimensional space into high dimensional Hilbert space, therefore a nonlinear problem is transformed into a linear problem. Furthermore, SVR model has excellent generalization ability due to the less number of training samples. On account of its high precision and it can address efficiently the inaccuracy problem of conventional surrogate model and computational complexity problem of complex components [13].

New implicit models of LCF strain ranges for turbine disk are developed based on SVR metamodel. According to the generalized stress-strength interference theory, random factors that influence turbine disk life have two aspects: one is due to the random variation in rotary speed, geometrical size, material properties, which cause stress/strain scatter in critical site around pin holes for turbine disk, the other is due to microstructure factors inside the material such as inhomogeneity of grain size and inhomogeneity of defect distribution, which influences micro crack incubation and cause the scatter of fatigue strength properties. Maximum rotary speed, material properties and interference fit tolerance of pins and holes as random input variables, central composite design is used based on the FEA in [20] to derive 15 finite element calculation point in the critical locations around pin holes, $\omega_{\max}$ represents the maximum rotary speed (rad/s), E is Young's modulus (MPa), T is virtue temperature for simulating interference fit tolerance of pins and holes (°C). On account of the difference in unit for three input random variables in the critical locations aside pin holes, which result in the large difference of orders of magnitude. Thus, after various input random variables for $\omega_{\max}$, E and T are normalized, Two LCF strain ranges implicit models are established based on SVR metamodel, in which, Gauss RBF $K(x, x') = \exp(-\frac{\|x - x'\|^2}{2\sigma^2})$ is adopted in this model. Finally, Strain ranges models are used to predict strain ranges by substituting the simulation test data into models. Predicted LCF strain ranges can be derived by restoring predicted normalized strain ranges data.

In order to compare the life prediction abilities of the metamodel based on SVR with RS model [37], it is indispensible to build RS models [38] of LCF strain ranges for turbine disk which are used to approximate the implicit performance function. Similarly, predicted normalized strain ranges data need to be restored into real predicted strain ranges which are substituted into Manson-Coffin equation to predict LCF lives. $\varepsilon - \mathrm{N}$ curve can be derived by fitting test data from [1] using Manson-Coffin equation for GH4133 under symmetric loading at 250 °C.

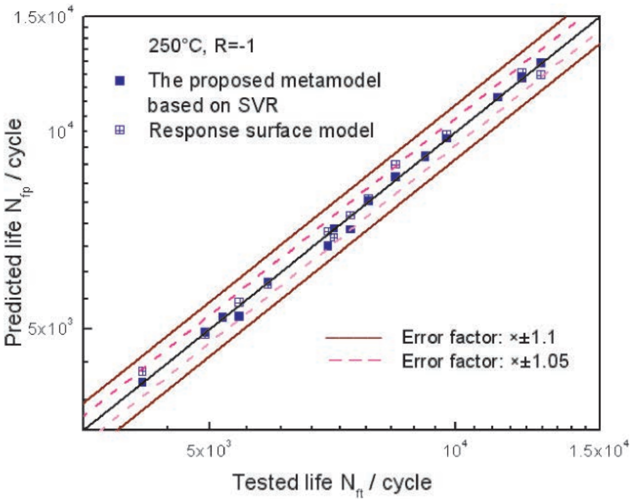

Fig. 5: Comparison between lives predicted by the proposed model, RS model and those tested for GH4133 at 250 °C under symmetric loading

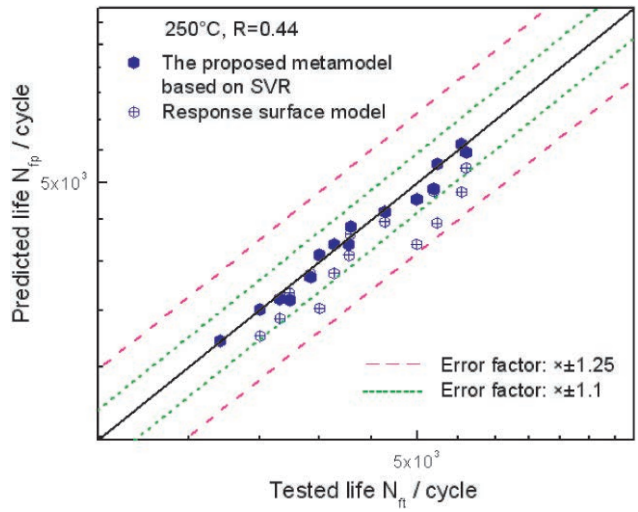

Fig. 6: Comparison between lives predicted by the proposed model, RS model and those tested for GH4133 at 250 °C under non-symmetrical loading

$$\frac{\Delta\varepsilon_t}{2} = \frac{\sigma_f'}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c = 0.02172(2N_f)^{-0.1748} + 4.75746(2N_f)^{-0.9921} \quad (13)$$

$\varepsilon - N$ curve can be obtained by Morrow's mean stress modified equation when $R_\varepsilon = 0.44$

$$\frac{\Delta\varepsilon_t}{2} = \frac{\sigma_f' - \sigma_m}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c = 0.02854176(2N_f)^{-0.2110} + 140.3921(2N_f)^{-1.4311} \quad (14)$$

The critical strain range can be obtained from the SVR and RS explicit model of ω_{max}, E and T. LCF life of turbine disk alloy can be predicted using Eqs. (13)–(14) based on the calculated strain range. Lives predicted by the metamodel based on SVR and RS model are compared with those tested lives, the correlations between the experimental and predicted fatigue lives under symmetric and non-symmetrical loading are given in Figs. 5–6, respectively. In Fig. 5, it can be found that the fatigue life correlation factor range is equal to ±1.1 and nearly within a scatter band of ±1.05 by LCF life prediction metamodel based SVR and RS model. Though predicted lives by RS model are in good agreement with those tested ones, it is a little conservative. In addition, it can be seen form Fig. 6 that lives predicted by RS model is fairly conservative. Though predicted lives are all within a factor of ±1.25, only about 11 out of 15 cyclic lives are predicted within a scatter band of ±1.1 to test ones when $R = 0.44$. However, predicted lives from the proposed metamodel are all within a factor of ±1.1. Therefore, a satisfactory correlation between the predicted and tested lives is observed as shown in Fig. 5 and Fig. 6 for metamodel based on SVR. In general, metamodel based on SVR can be used to predict LCF life and has higher accuracy.

3 Fatigue reliability analysis using saddlepoint approximation

SPA is originally proposed by Daniels [39] for approximating the distribution of random variables in 1954. It is an important and power algorithm for obtaining accurate probability density function (*pdf*) and cumulative density function (*cdf*). The applications of a range of distribution problem are discussed and propagated in [40–42]. In spite of the theory of SPA is quite complex, simple formulas have been developed for calculating *pdf* and *cdf*. Thus, it can be utilized fairly straightforward [43]. The *pdf* or *cdf* of structural response performance function is approximated by SPA, which use random variables to generate the cumulative generating function (*cgf*). As long as the *cgf* of basic random variables can be properly derived, the distribution form of basic variables is not restricted. Briefly speaking, its basic idea is to utilize the *cgf* property and inverse Fourier transform of random variable performance function to obtain the probability distribution of the performance function based on saddlepoint exponential power series expansion. SPA is suitable for the situations of the large number of random variables and high nonlinearity of the performance function as well as known or unknown distribution. It provides more computationally efficient solutions than the general MCS while maintaining high accuracy. Compared with FORM and SORM, the performance function of SPA can be replaced by the approximate range of first order Taylor series expansion and it is needless for SPA to search of MPP as well as a nonlinear transformation from non-normal space into normal space is also not required [43–48]. In this section, SPA will be simply introduced and the basic computation process will be given as follows.

3.1 Calculating the *cgf* of the performance function

In this section, the *cgf* of the performance function is calculated. The calculation process involves sampling for input random variables, evaluating the performance function and calculating cumulants to estimate the *cgf*. Suppose Y are response variables with the *pdf* $f_Y(y)$ and moment generating function (MGF) of Y is expressed as $\phi(t)=\int_{-\infty}^{\infty} e^{ty} f_Y(y)dy$, Moreover, the *cgf* of Y, i.e. K(t), is defined as $K(t)=\log[\phi(t)]$, where *log* is natural logarithm. When input n random variables samples, n output response variables y can be generated by the performance function, namely limited state function (LSF). Here, instituting input variables into the performance function leads to

$$y^i = g(X) = g(x_1{}^i, x_2{}^i, \ldots x_d{}^i)(i=1,2,\ldots n) \tag{15}$$

where $(x_1{}^i, x_2{}^i, \ldots x_d{}^i)$ is the ith sample of X, namely X^i.

Furthermore, s_r can be calculated, where s_r is the first r-order power sum of the performance function sample value y, i.e. $s_r=\sum_{i=1}^{n}(y_i)^r$. High approximation accuracy can be obtained by choosing the first four moment of s_r, namely s_r ($r=1,2,3$ and 4). Once the sample set of y^i is generated, the cumulants of performance function can be calculated by

$$\begin{aligned}
k_1 &= \frac{s_1}{n}\\
k_2 &= \frac{ns_2 - s_1{}^2}{n(n-1)}\\
k_3 &= \frac{2s_1{}^3 - 3ns_1s_2 + n^2 s_3}{n(n-1)(n-2)}\\
k_4 &= \frac{-6s_1{}^4 + 12ns_1{}^2 s_2 - 3n(n-1)s_2{}^2 - 4n(n+1)s_1s_3 + n^2(n+1)s_4}{n(n-1)(n-2)(n-3)}
\end{aligned} \tag{16}$$

Therefore, the power of the *cgf* with random variable cumulants can be expanded as

$$K(t)=\sum_{i=1}^{r}\frac{k_r t_s{}^r}{r!}=k_1 t+\frac{k_2 t^2}{2!}+\ldots+\frac{k_r t^r}{r!}+\ldots. \tag{17}$$

3.2 Calculating the *pdf* and the *cdf* of performance function

This section attempts to solve saddlepoint and estimate the *pdf* and *cdf* as well as failure probability. Once the *cgf* of the performance function is obtained, it is easy to derive analytical solution of saddlepoint and derivative of the *cgf*. Seek the first-order partial derivative of $K(t)$, and saddlepoint t_s can be obtained by solving $K'(t)=y$ [21–22].

$$K_Y'(t)=k_1+\sum_{j=2}^{r}K_j\frac{t^{j-1}}{(j-1)!}=y \tag{18}$$

Then, the *pdf* of y is expressed as

$$f_Y(y)=\left[\frac{1}{2\pi K''(t_s)}\right]^{1/2} e^{\left[K(t_s)-t_s y\right]} \tag{19}$$

where $K''(\cdot)$ is the second-order partial derivative of the *cgf*, and *cdf* is given by

$$F_Y(y)=P\{Y\le y\}=\Phi(w)+\phi(w)\left(\frac{1}{w}-\frac{1}{v}\right)=\Phi\left[w+\frac{1}{w}\ln\left(\frac{w}{v}\right)\right] \tag{20}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the *cdf* and *pdf* of a standard normal distribution, respectively. In which, $w=\mathrm{sgn}(t_s)\{2[t_s y-K(t_s)]\}^{1/2}$, $v=t_s[K''(t_s)]^{1/2}$. When $y=0$, $-\left[w+\frac{1}{w}\ln\left(\frac{w}{v}\right)\right]$ in Eq. (20) can be referred as reliability index β in equivalent normal space.

Two problems may arise during numerical calculations, one is that Eq. (18) has r-1 roots and saddlepoint solution has quite a few real roots. The problem has been addressed by Wang [45], who proposed a simply improvement to Eq. (18) to ensure the approximation of $K_Y'(t)$ is monotonically increasing. The other problem is the singular value problem, when $w=\mathrm{sgn}(t_s)\{2[t_s y-K(t_s)]\}^{1/2}$ and $v=t_s[K''(t_s)]^{1/2}$ produce square root of negative value, which can be solved by reversing the symbol of the performance function [21].

4 Validations of the new method for fatigue reliability analysis

On account of many advantages of SPA, SPA is efficient for probability of failure calculation and its calculation results for failure probability are considered as exact results. Thus, using experimental data of turbine disk in [1], the accuracy of SPA for failure probability calculation need not be verified by MCS but only by test data. In this section, the proposed fatigue models are used to establish the performance functions and SPA is utilized to established performance function for reliability assessment. The performance function of each failure model must be properly defined for SPA calculation and evaluate the reliability of LCF life. In this paper, the performance function can be defined as $Y = g(X) = N_D - N_f$ by using the fatigue models in Section 2, and failure probability can be expressed as $P_f = P\{N_D - N_f \leq 0\}$. N_D is the designed fatigue life in the design procedure of structure, and N_f is the fatigue failure life calculated from the models such as the proposed energy based model in Eq. (5) and (6) and the viscosity-based model in Eq. (12) in Section 2 [49]. Failure probability is defined as the probability of N_D is less than N_f, namely the difference values of N_D and N_f is less than zero as failure criterion. If the designed life N_D is longer than the fatigue life N_f, turbine disk will not be damaged, i.e. it is reliable. In other words, N_f is not allowed to exceed N_D. Otherwise, failure will occur. SPA is used to calculate failure probabilities of LCF predicted lives of turbine disk from aforementioned implicit or explicit models, and the relationships between the designed fatigue lives and failure probabilities are shown quantificationally, which provides the reliability-based design rules quantificationally of the fatigue life for turbine disk alloy.

4.1 Fatigue reliability analysis using the energy-based life prediction model

To demonstrate the feasibility and accuracy of total strain energy density model using SPA, as samples of input random variables, relevant test data of turbine disk at 250 °C can be found [1] for the proposed model, SWT model and modified Walker model, and predicted lives by these models are used in SPA for fatigue reliability analysis. The performance function is determined by rewriting Eq. (5) in total strain energy density model under symmetric loading as

$$Y = g(X) = N_D - N_f = N_D - \sqrt[0.673671]{\frac{1114.749}{6899.6\left(\frac{\Delta\varepsilon_p}{2}\right)^{1.1235119} + \frac{\sigma_a{}^2}{2E}}} \tag{21}$$

The performance function is obtained by rearranging Eq. (6) under unsymmetrical loading

$$\begin{aligned} Y &= g(X) = N_D - N_f \\ &= N_D - \sqrt[0.570502]{\frac{373.4627}{5804.4\left(\frac{\Delta\varepsilon_p}{2}\right)^{1.0971544} + \frac{\sigma^2_{max}}{2E}}} \end{aligned} \tag{22}$$

From Eqs. (21)–(22), it is obviously that the nonlinearity of two performance functions for turbine disk is very high. As a result of life cycles under LCF is about between 1,000 and 100,000 cycles, the designed lives are divided into several phases in order to calculate failure probability of different designed lives.

To verify the feasibility and efficiency of the proposed model based on SPA, calculated failure probabilities from test lives are regarded as standard results. SPA is used for failure probabilities calculation of the performance functions in the energy-based life prediction model, SWT model and modified Walker model. It can be seen from Fig. 7 that calculated failure probabilities for aforementioned three models are all identical to the experimental results. From Fig. 8, compared with SWT model and

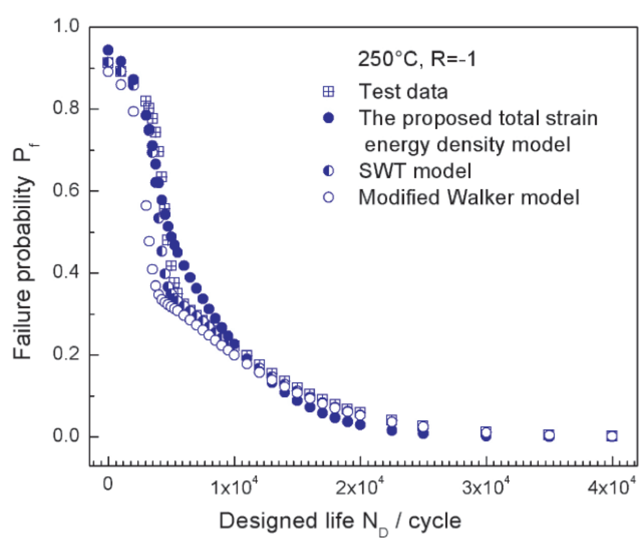

Fig. 7: The relation curve between failure probability and the designed life of turbine disk for test data, SWT model, the proposed model and Modified Walker model at 250 °C under symmetric loading

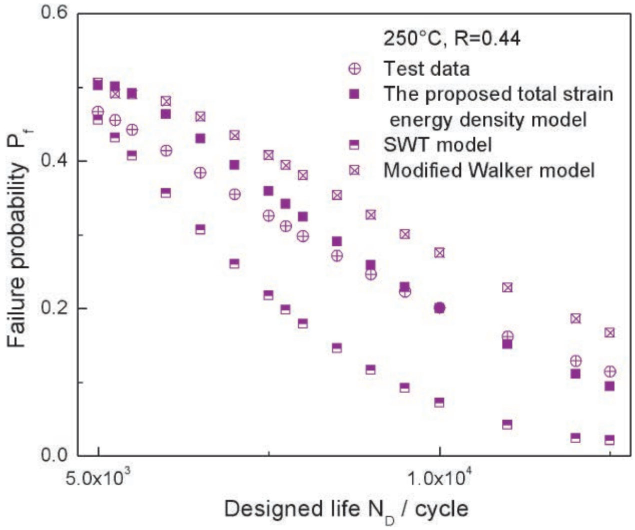

Fig. 8: The relation curve between failure probability and the designed life of turbine disk for test data, SWT model, the proposed model and Modified Walker model at 250 °C under non-symmetrical loading

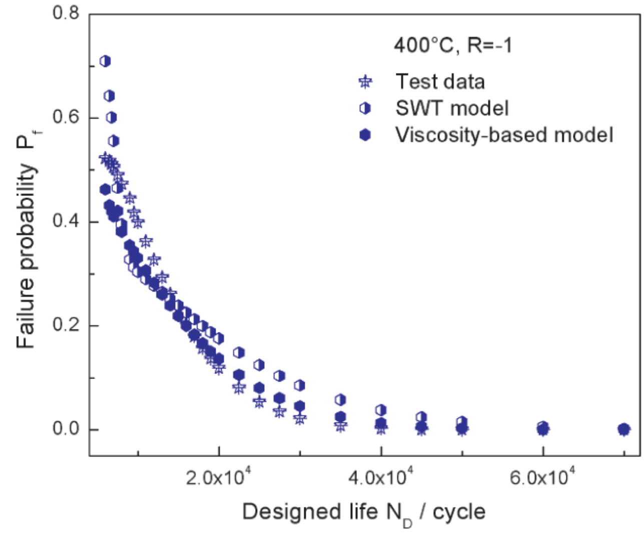

Fig. 9: The relation curve between failure probability and the designed life of turbine disk for test data, SWT model and proposed mode at 400 °C under symmetric loading

modified Walker model, it can be seen that failure probabilities from the proposed model are more approximate for GH4133 and obviously the scatter are smaller than SWT's and modified Walker models' scatters. Besides, it also can be seen from Fig. 7 and Fig. 8 that the scatters caused by mean stresses under non-symmetric loading are obviously greater than that under symmetric loading.

4.2 Fatigue reliability analysis using the viscosity-based life prediction model

The feasibility and efficiency of SPA for fatigue reliability analysis are verified using the viscosity-based model. As samples of input random variables, two sets of test data of turbine disk alloy at 400 °C and 500 °C are supplied [1] for the proposed model and SWT model under symmetric loading. Two performance functions can be obtained by Eq. (12) as

$$Y = N_D - 4.99462\times 10^{31}(0.42845\cdot\sigma_{max} - 7.613526*10^5)^{-3.0382} \cdot(\Delta\varepsilon_p\sigma_{max})^{-0.370252} \tag{23}$$

In addition, the designed lives are divided into quite a few phases to calculate probabilities of failure for different designed lives.

Failure probabilities of test lives are regarded as standard results in order to compare with failure probabilities for SWT model and viscosity-based model. Failure probabilities are calculated using SPA for the performance functions by SWT model and viscosity-based model. From the correlation curve between failure probabilities and the designed lives of turbine disk in Fig. 9 and Fig. 10, it can be seen that failure probabilities from the viscosity-based model are almost identical to the experimental results, and the analysis results also show that when the designed fatigue lives increase, the probabilities of failure that the designed fatigue lives are less than the predicted lives from the proposed models are continuously monotonically decreasing. At the same time, it also can be seen that

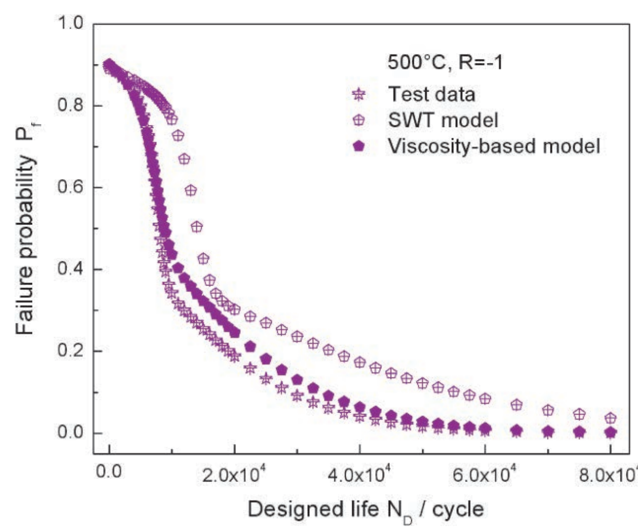

Fig. 10: The relation curve between failure probability and the designed life of turbine disk for test data, SWT model and proposed model at 500 °C under symmetric loading

the scatter of failure data at 500 °C are obviously larger than that at 400 °C, namely, the higher temperature are, the larger scatter of reliability assessment would be, this phenomenon is mainly caused by creep under high temperature.

4.3 Fatigue reliability analysis using the metamodel based on SVR

This section attempts to validate the accuracy and effectiveness of SPA applied to metamodel based on SVR for fatigue reliability analysis. Metamodel based on SVR is compared with RS model. Predicted life values are calculated from two LCF life prediction models, which aims to establish the performance function Y, and SPA is applied for failure probability analysis of fatigue life. Calculated failure probabilities from test data are regarded as standard values. It can be seen from Figs. 11 and 12, the calculated results of the probabilities of failure using the proposed models in this paper are almost identical to the results from the test data, which validate the rationality and feasibility that SPA is employed to the proposed models. Furthermore, the analysis results also show that with the designed fatigue lives decrease, the probabilities of failure that the designed fatigue lives are less than the predicted lives from the proposed models are continuously monotonically increasing. Besides, the scatter of data, which is caused by mean stress under non-symmetric loading, is obviously larger than that under symmetric loading.

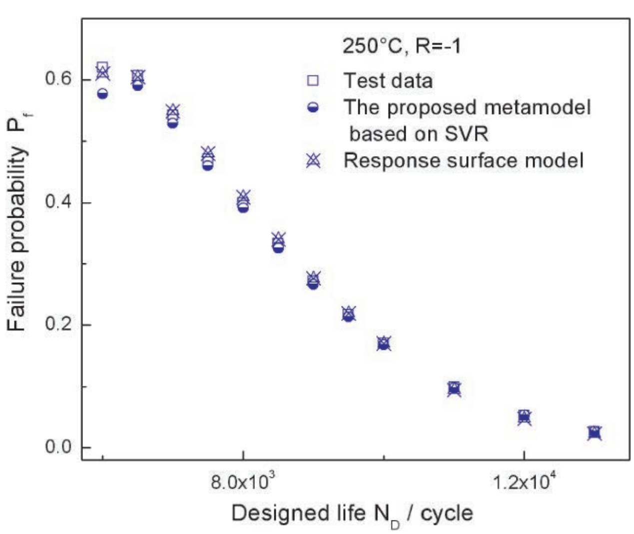

Fig. 11: The relation curve between failure probability and the designed life of turbine disk for SVR metamodel and RS model at 250 °C under symmetric loading

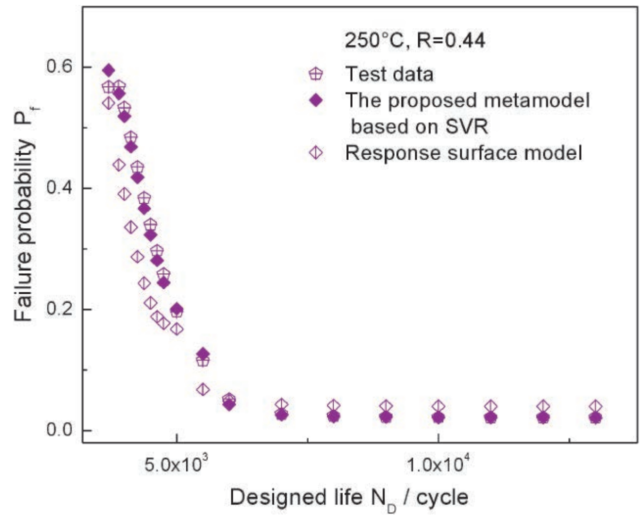

Fig. 12: The relation curve between failure probability and the designed life of turbine disk for SVR model and RS model at 250 °C under non-symmetrical loading

According to the applicable conditions of the proposed method, it is valid for most metallic materials under uniaxial loading. However, most engineering components are subjected to complex loading conditions at which stress-strain cycles fluctuate with time. Thus, the application of this method under multiaxial loading and different materials needs to be further evaluated.

5 Conclusions

The aim of current research was to present a new approach for fatigue reliability analysis of turbine disk alloy in a low cycle fatigue regime. In this paper, in order to construct the explicit and implicit performance functions and calculate the failure probabilities of turbine disk alloy, two LCF life prediction models based on total strain energy density and Support Vector Regression (SVR) metamodel are developed under different loading conditions, as well as a viscosity-based model is introduced under high temperatures. Through using the total strain energy density as the damage parameter, the effects of mean stresses are involved into the proposed total strain energy model under non-symmetrical loading and the predicted lives from the proposed models are all within a scatter band of ±2. Furthermore, the metamodel based on SVR has the higher accuracy of life prediction. The predicted lives are within a factor of ±1.1, and it is adaptable to the implicit performance function. Though the nonlinearity is high in three LCF life prediction models and with a small sample tests input random variables for turbine disk, SPA is used for analyzing failure probabilities by establishing the performance functions using three LCF life models and has achieved excellent results, which verifies the excellent effectiveness and accuracy of SPA and fatigue models. The combination of the proposed models and high precise uncertainty analysis algorithm provides powerful guarantee for high reliability analysis of mechanical components by characterizing the uncertainty of input random variables.

In the next step, it is necessary to seek more accurate LCF life prediction model under non-symmetrical loading to reduce the impact of model uncertainty on its reliability. In addition, by considering different stress and strain ratios, SPA used for reliability analysis of different turbine disk materials or a wider class of materials will be further evaluated. Finally, based on the fatigue failure model, through considering the probability distribution of test data and statistical distribution of input random variables on its reliability, it plays an important role for reducing failure probability to a lower level and ensuring its high reliability.

Nomenclature

A	coefficient of strain energy density damage equation
α	exponent of strain energy density damage equation
ΔW_e^+	elastic tensile strain energy density per cycle
ΔW_p	plastic strain energy density per cycle
ΔW_t	total strain energy density per cycle
$\Delta\varepsilon_{in}$	inelastic strain range
$\Delta\varepsilon_\mathrm{e}$	elastic strain range
$\Delta\varepsilon_\mathrm{p}$	plastic strain range
$\Delta\varepsilon_c$	creep strain range
$\Delta\varepsilon_\mathrm{t}$	total strain range
$\sigma_{\min}$	minimum stress
$\sigma_{\max}$	maximum stress
σ_m	mean stress
σ_a	stress amplitude
$\Delta\sigma$	stress range
E	Young's modulus
n'	cyclic strain hardening exponent
K'	cyclic strength coefficient
σ_f'	fatigue strength coefficient
ε_f'	fatigue ductility coefficient
b	fatigue strength exponent
c	fatigue ductility exponent
R	strain ratio ($\varepsilon_{\min} / \varepsilon_{\max}$)
N_f	cycles to failure
N_{ft}	tested life cycles
N_{fp}	predicted life cycles

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