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Non-probabilistic reliability sensitivity analysis of the model of structural systems with interval variables whose state of dependence is determined by constraints

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Abstract

Non-probabilistic reliability sensitivity analysis for structural systems plays an important role in determining key design variables that affect structural reliability strongly. Traditional non-probabilistic model assumes that all interval variables are mutually independent. However, this assumption may not be true in practical engineering. In this article, the dependency of interval variables is introduced into the non-probabilistic model by using both inequality and equality constraints. The non-probabilistic index model and optimization method for structural systems with interval variables, whose state of dependence is determined by constraints, are proposed on the basis of the existing non-probabilistic index theory. The linear optimization model is alternative when nonlinear optimization model cannot find any solution. Non-probabilistic reliability sensitivity analysis model and optimization method for structural systems, with the interval variables whose state of dependence is determined by constraints, are established based upon the finite difference theory. The proposed method is demonstrated via several examples.

Keywords

Reliability, reliability sensitivity analysis, non-probabilistic model, interval variables, dependency

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Introduction

Reliability sensitivity analysis has been widely used in both reliability analysis and reliability-based design with the aim of identifying how sensitive the reliability of systems/components with respect to the characteristics of uncertain variables is. It plays an important role in determining the key design variables that affect structural reliability strongly.[1–3] Most existing reliability sensitivity analysis methods are based on the probability and statistics theories. That is, every statistical parameter involved is perfectly determined and all the parameters associated with probabilities and distributions are precisely known. The methodologies for calculating reliability sensitivity based on probability theory have been well established.[3,4] However, owing to impact from various uncertainties in practical engineering, especially in the early design stage of products, lack of data or imprecise information are inevitable,[5,6] especially for aviation and aerospace systems with very limited samples that prohibit to conduct expensive experiments.[7] Therefore, traditional probability-based reliability sensitivity methods are difficult to handle these situations. It is necessary to explore new reliability sensitivity analysis theories and methods.

In order to calculate the reliability sensitivity, the reliability analysis models for structural systems should be established first. In the 1990s, Elishakoff and Ben-Haim suggested using convex models to represent uncertainty.[8,9] Based upon the convex model theory, non-probabilistic reliability principle was first introduced by

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Ben-Haim, which is different from the traditional probability-based reliability theory.[9] Recently, based on interval arithmetic, a novel non-probabilistic reliability model for structural systems was presented by Guo et al.[10] In their studies, the non-probabilistic reliability index for structural systems was defined,[11] and furthermore they proposed some algorithms to calculate the non-probabilistic reliability index.[12] However, in the methods proposed by Guo et al.,[10–12] all interval variables were assumed to be mutually independent. Although some non-probabilistic reliability models have many advantages and have been successfully applied in engineering practices, an obvious limitation of aforementioned methods is that all interval variables in structural systems are assumed to be mutually independent, which is not always true in practical engineering. Nowadays, there are some research efforts on sensitivity analysis under non-probabilistic uncertainty,[13–17] and sensitivity analysis with dependent variables.[18–20] In this work, a novel non-probabilistic reliability index model and reliability sensitivity method for structural systems, with interval variables whose state of dependence is determined by constraints, is proposed.

The rest of this article is organized as follows. 'Interval arithmetic and its dependent models' provides a brief introduction of interval arithmetic and its dependent models. 'Non-probabilistic reliability index model for structural systems with interval variables whose state of dependence is determined by constraints' proposes a non-probabilistic reliability index model for structural systems with interval variables whose state of dependence is determined by constraints. The details of the proposed non-probabilistic reliability sensitivity method are elaborated in 'Non-probabilistic reliability sensitivity analysis for structural systems with interval variables whose state of dependence is determined by constraints'. Two numerical examples are presented in 'Illustrative examples and discussions' to demonstrate the proposed method. 'Conclusions' presents brief discussions and conclusions.

Interval arithmetic and its dependent models

A closed bounded interval $[Y^L, Y^U] = (Y^L \leqslant Y \leqslant Y^U, Y \in \Re)$ is called an interval number, where $\Re$ denotes a real number. Y is called an interval variable, whose lower and upper bounds are Y^L and Y^U, respectively. The midpoint $\bar{Y}$ and radius Y^r, can be calculated as

$$\bar{Y} = \frac{Y^L + Y^U}{2}, Y^r = \frac{Y^U - Y^L}{2} \tag{1}$$

According to equation (1), interval variable Y can be written in the following standardized form

$$Y = \bar{Y} + Y^r\delta \tag{2}$$

where $\delta \in [-1, 1]$ is called the standardized unit interval variable. Then the midpoints and radius for interval vector $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ can be expressed as

$$\bar{\mathbf{Y}} = (\bar{Y}_1, \bar{Y}_2, \cdots, \bar{Y}_n), \mathbf{Y}^r = \left(Y_1^r, Y_2^r, \cdots, Y_n^r\right) \tag{3}$$

For interval variables Y_1 and Y_2, the four basic algebraic operations are provided[21,22]

$$Y_1 + Y_2 = [\underline{Y}_1 + \underline{Y}_2, \bar{Y}_1 + \bar{Y}_2] \tag{4}$$

$$Y_1 - Y_2 = [\underline{Y}_1 - \bar{Y}_2, \bar{Y}_1 - \underline{Y}_2] \tag{5}$$

$$\begin{aligned} Y_1 \bullet Y_2 = [&\min(\underline{Y}_1\underline{Y}_2, \underline{Y}_1\bar{Y}_2, \bar{Y}_1\underline{Y}_2, \bar{Y}_1\bar{Y}_2), \\ &\max(\underline{Y}_1\underline{Y}_2, \underline{Y}_1\bar{Y}_2, \bar{Y}_1\underline{Y}_2, \bar{Y}_1\bar{Y}_2)] \end{aligned} \tag{6}$$

$$Y_1/Y_2 = [(\underline{Y}_1, \bar{Y}_1) \bullet (1/\bar{Y}_2, 1/\underline{Y}_2), (0 \notin Y_2)] \tag{7}$$

Since the dependency of interval variables is extremely complicated, it cannot be modeled by correlation coefficients directly. In engineering practices, the dependency of random variables can be modeled via Pearson correlation coefficients when interval variables are usually assumed to be mutually independent. In this article, we assume that the dependency of interval variables can be determined by using both inequality and equality constraints. For example, the length, width, height, and density of a cantilever are all interval variables, which are represented by Y_1, Y_2, Y_3, and Y_4, respectively. The cross-section of the beam is a constant s. Furthermore, in order to control the whole weight of the cantilever, its weight should be less than a given constant w. Therefore, the dependency of the interval variables can be determined by the following equality and inequality constraints

$$h(Y_1, Y_2) = Y_1Y_2 - s = 0 \tag{8}$$

$$g(Y_1, Y_2, Y_3, Y_4) = Y_1Y_2Y_3Y_4 - w \leqslant 0 \tag{9}$$

When performing non-probabilistic reliability analysis and reliability sensitivity analysis, we should consider the dependency of the interval variables Y_1, Y_2, Y_3, and Y_4, because they satisfy the constraints obviously. In this article, we assume that the state of dependence of interval variables can be determined by equality and inequality constraints

$$h(\mathbf{Y}) = 0 \tag{10}$$

$$g(\mathbf{Y}) \leqslant 0, m(\mathbf{Y}) \geqslant 0 \tag{11}$$

where $h(\mathbf{Y})$, $g(\mathbf{Y})$, and $m(\mathbf{Y})$ are functions with interval variables; $h(\mathbf{Y}) = 0$, $g(\mathbf{Y}) \leqslant 0$, and $m(\mathbf{Y}) \geqslant 0$ are equality and inequality constraints for interval variables, respectively.

Since the inequality $m(\mathbf{Y}) \geqslant 0$ can be transformed into $-m(\mathbf{Y}) \leqslant 0$ easily, we only consider $h(\mathbf{Y}) = 0$ and $g(\mathbf{Y}) \leqslant 0$ in this article.

Non-probabilistic reliability index model for structural systems with interval variables whose state of dependence is determined by constraints

Traditional non-probabilistic reliability index model

Let performance function of a system be denoted by $f(\mathbf{Y})$, $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ is a vector consisting of all mutually independent interval variables. Function $f(\mathbf{Y}) = 0$ is called the limit-state function. If $f(\mathbf{Y}) > 0$, the system is safe. If $f(\mathbf{Y}) < 0$, the system fails. When $f(\mathbf{Y})$ is a continuous function $M = f(\mathbf{Y})$ is an interval variable, and its midpoint and radius are denoted by $\bar{M}$ and M^r, respectively. The non-probabilistic index η is defined as[10]

$$\eta = \frac{\bar{M}}{M^r} \tag{12}$$

From equation (12), if $\eta > 1$, $\forall Y_i \in [Y_i^L, Y_i^U](i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) > 0$ and the system is safe. If $\eta < -1$, $\forall Y_i \in [Y_i^L, Y_i^U](i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) < 0$ and the system falls in the failure domain. If $-1 \leqslant \eta \leqslant 1$, $\forall Y_i \in [Y_i^L, Y_i^U](i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) < 0$ or $f(\mathbf{Y}) > 0$, and then the system is in the uncertain domain. Generally, a larger value of η indicates the system is more reliable. From equation (2), we can transform the vector $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ into the standardized unit interval space by using $\mathbf{Y} = \bar{\mathbf{Y}} + \mathbf{Y}^r\boldsymbol{\delta}$. Guo et al.[11] extended the definition of the non-probabilistic index η, and the extended η measured by $\|\bullet\|_\infty$ is the shortest distance from coordinate origin to the limit-state function in the standardized unit interval space

$$\eta = \min(\|\boldsymbol{\delta}\|_\infty) \tag{13}$$

which should satisfy the constraint in

$$M = f(\mathbf{Y}) = F(\boldsymbol{\delta}) \tag{14}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_n)$ is a standardized unit interval vector that comes from an interval vector $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$.

In order to calculate non-probabilistic index η, we can expand the function $f(\mathbf{Y})$ with linear Taylor expansion at the midpoints $\bar{\mathbf{Y}} = (\bar{Y}_1, \bar{Y}_2, \cdots, \bar{Y}_n)$. The expansion process can be expressed as

$$f(\mathbf{Y}) \approx f(\bar{\mathbf{Y}}) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial Y_i}\right|_{\bar{\mathbf{Y}}} (Y_i - \bar{Y}) = f(\bar{\mathbf{Y}}) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial Y_i}\right|_{\bar{\mathbf{Y}}} Y_i^r \delta_i \tag{15}$$

From equations (12) and (15), the non-probabilistic index η can be computed by

$$\eta \approx \frac{f(\bar{\mathbf{Y}})}{\sum_{i=1}^{n} \left.\frac{\partial f}{\partial Y_i}\right|_{\bar{\mathbf{Y}}} Y_i^r} \tag{16}$$

Generally, when the performance function is a highly nonlinear function, error owing to approximation by using equation (16) is very large. There are three methods that can be used to calculate non-probabilistic index η, including definitional method, transformational method, and optimization method.[12] It has been proved that the optimization method is more robust and efficient among them. $M = f(\mathbf{Y})$ is an interval variable with the lower and upper bounds M^L and M^U, respectively. From equations (1), (12) and the optimization method,[12] we have

$$\eta = \frac{(M^U + M^L)}{(M^U - M^L)} \tag{17}$$

where

$$M^L/M^U = \min/\max f(\mathbf{Y}) \quad \begin{cases} s.t. \\ \mathbf{Y}^L \leqslant \mathbf{Y} \leqslant \mathbf{Y}^U \end{cases} \tag{18}$$

In the standardized unit interval space, equation (18) can be rewritten by

$$M^L/M^U = \min/\max F(\boldsymbol{\delta}) \quad \begin{cases} s.t. \\ -\mathbf{1} \leqslant \boldsymbol{\delta} \leqslant \mathbf{1} \end{cases} \tag{19}$$

Non-probabilistic reliability index model under dependent interval variables

In the traditional non-probabilistic model, all interval variables are assumed to be mutually independent. In this subsection, we will propose a novel non-probabilistic reliability index model for structural systems with interval variables whose dependence is determined by constraints.

Assume that the state of dependence of interval variables can be determined by inequality constraints $g_k(\mathbf{Y}_p) \leqslant 0$ and equality constraints $h_l(\mathbf{Y}_q) = 0$, $(k, l \geqslant 0)$.p and q denote the numbers of interval variables in inequality and equality constraints, $\mathbf{Y}_p = (Y_1, Y_2, \cdots, Y_p; p \leqslant n)$, $\mathbf{Y}_q = (Y_1, Y_2, \cdots, Y_p; q \leqslant n)$. From equation (1) and the definition of the non-probabilistic index in equation (12), the non-probabilistic index $\tilde{\eta}$ for structural systems with interval variables whose dependence is determined by constraints can be expressed as

$$\tilde{\eta} = \frac{\bar{\tilde{M}}}{\tilde{M}^r} = \frac{\left(\tilde{M}^U + \tilde{M}^L\right)}{\left(\tilde{M}^U - \tilde{M}^L\right)} \tag{20}$$

where $\bar{\tilde{M}}$,$\tilde{M}^r$,$\tilde{M}^L$ and $\tilde{M}^U$ denote the midpoint, radius, lower bound, and upper bound for the performance function $M = f(\mathbf{Y})$, respectively. From equation (20), if $\tilde{\eta} > 1$, $\forall Y_i \in [Y_i^L, Y_i^U] (i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) > 0$ and the system is safe. If $\tilde{\eta} < -1$, $\forall Y_i \in [Y_i^L, Y_i^U] (i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) < 0$ and the system falls in the failure domain. If $-1 \leqslant \tilde{\eta} \leqslant 1$, $\forall Y_i \in [Y_i^L, Y_i^U] (i = 1, 2, \cdots, n)$, we have $f(\mathbf{Y}) < 0$ or $f(\mathbf{Y}) > 0$, therefore, the system is in the uncertain domain.

From equation (20), we know that the main tasks for calculating non-probabilistic index $\tilde{\eta}$ are to determine the values of $\tilde{M}^L$ and $\tilde{M}^U$, which can be given by

$$\tilde{M}^L \Big/ \tilde{M}^U = \min/\max f(\mathbf{Y}) \qquad (21)$$
$$\begin{cases} s.t. \\ g_k(\mathbf{Y}_p) \leqslant 0 (k = 1, 2, \cdots, n_p) \\ h_l(\mathbf{Y}_q) = 0 (l = 1, 2, \cdots, n_q) \\ \mathbf{Y}^L \leqslant \mathbf{Y} \leqslant \mathbf{Y}^U \end{cases}$$

In the standardized unit interval space, equation (21) can be rewritten as

$$\tilde{M}^L \Big/ \tilde{M}^U = \min/\max F(\boldsymbol{\delta}) \qquad (22)$$
$$\begin{cases} s.t. \\ G_k(\boldsymbol{\delta}_p) \leqslant 0 (k = 1, 2, \cdots, n_p) \\ H_l(\boldsymbol{\delta}_q) = 0 (l = 1, 2, \cdots, n_q) \\ \mathbf{-1} \leqslant \boldsymbol{\delta} \leqslant \mathbf{1} \end{cases}$$

where $G_k(\boldsymbol{\delta}_p) = g_k(\mathbf{Y}_p)$, and $H_l(\boldsymbol{\delta}_q) = h_l(\mathbf{Y}_q)$.

There are many nonlinear optimization algorithms that can be used to solve both equations (21) and (22), such as sequential quadratic programming (SQP).[23] The optimization model in equation (21) can be expressed as

$$L(\mathbf{Y}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = f(\mathbf{Y}) - \boldsymbol{\lambda}^T g_k(\mathbf{Y}_p) - \boldsymbol{\gamma}^T h_l(\mathbf{Y}_q) \qquad (23)$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ are Lagrange multipliers. For more details about the SQP, please see Nocedal and Wright.[23]

In some certain circumstances, if equation (21) is difficult to be solved, we can transform it into a linear optimization model. $g_k(\mathbf{Y}_p)$ and $h_l(\mathbf{Y}_q)$ can be expanded by linear Taylor expansion at the midpoints $\bar{\mathbf{Y}}_p$ and $\bar{\mathbf{Y}}_q$, and then we have

$$g_k(\mathbf{Y}_p) \approx g_k(\bar{\mathbf{Y}}_p) + \sum_{i=1}^{p} \frac{\partial g_k}{\partial Y_i}\bigg|_{\bar{\mathbf{Y}}_p} (Y_i - \bar{Y}_i) = g_k(\bar{\mathbf{Y}}_p) + \sum_{i=1}^{p} \frac{\partial g_k}{\partial Y_i}\bigg|_{\bar{\mathbf{Y}}_p} {Y_i}^r \delta_i \qquad (24)$$

$$h_l(\mathbf{Y}_q) \approx h_l(\bar{\mathbf{Y}}_q) + \sum_{j=1}^{q} \frac{\partial h_l}{\partial Y_j}\Big|_{\bar{\mathbf{Y}}_q} (Y_j - \bar{Y}_j) = h_l(\bar{\mathbf{Y}}_q) + \sum_{j=1}^{q} \frac{\partial h_l}{\partial Y_j}\bigg|_{\bar{\mathbf{Y}}_q} {Y_j}^r \delta_j \qquad (25)$$

From equations (15), (24), and (25), the nonlinear optimization model in equation (21) can be approximated by using a linear optimization model, which is given by

$$\tilde{M}^L \Big/ \tilde{M}^U \approx \min/\max \left[f(\bar{\mathbf{Y}}) + \sum_{i=1}^{n} \frac{\partial f}{\partial Y_i}\bigg|_{\bar{\mathbf{Y}}} (Y_i - \bar{Y}_i) \right]$$
$$\begin{cases} s.t. \\ g_k(\bar{\mathbf{Y}}_p) + \sum_{i=1}^{p} \frac{\partial g_k}{\partial Y_i}\Big|_{\bar{\mathbf{Y}}_p} (Y_i - \bar{Y}_i) \leqslant 0 (k = 1, 2, \cdots, n_p) \\ h_l(\bar{\mathbf{Y}}_q) + \sum_{j=1}^{q} \frac{\partial h_l}{\partial Y_j}\Big|_{\bar{\mathbf{Y}}_q} (Y_j - \bar{Y}_j) = 0 (l = 1, 2, \cdots, n_q) \\ \mathbf{Y}^L \leqslant \mathbf{Y} \leqslant \mathbf{Y}^U \end{cases} \qquad (26)$$

Non-probabilistic reliability sensitivity analysis for structural systems with interval variables whose state of dependence is determined by constraints

Reliability sensitivity analysis is to study how sensitive the reliability is with respect to the characteristics of uncertain variables. Non-probabilistic reliability sensitivity is defined as the rate of variation in non-probabilistic index η owing to the variation of midpoint and radius of interval variables,[7] which can be expressed as

$$\left(\frac{\partial \eta}{\partial \bar{Y}}, \frac{\partial \eta}{\partial Y^r} \right) \qquad (27)$$

From equation (27), non-probabilistic reliability sensitivity for structural systems with interval variables whose state of dependence is determined by constraints can be defined as

$$\left(\frac{\partial \tilde{\eta}}{\partial \bar{Y}}, \frac{\partial \tilde{\eta}}{\partial Y^r} \right) \qquad (28)$$

Since $\tilde{\eta}$ cannot be expressed via an explicit function, equation (28) has no analytical solutions. In this situation, the finite difference method is considered because it can handle this problem easily.[1,4] In engineering practices, equation (28) can be calculated by using the finite difference method

$$\frac{\partial \tilde{\eta}}{\partial \bar{Y}_i} \approx \frac{\tilde{\eta}(\bar{Y}_i + \Delta \bar{Y}_i) - \tilde{\eta}(\bar{Y}_i)}{\Delta \bar{Y}_i} \qquad (29)$$

$$\frac{\partial \tilde{\eta}}{\partial Y_i^r} \approx \frac{\tilde{\eta}(Y_i^r + \Delta Y_i^r) - \tilde{\eta}(Y_i^r)}{\Delta Y_i^r} \qquad (30)$$

where $\Delta \bar{Y}_i$ and ΔY_i^r are the small step size of $\bar{Y}_i$ and Y_i^r, respectively. From equation (20), we have

$$\tilde{\eta}(\bar{Y}_i + \Delta \bar{Y}_i) = \frac{\left(\tilde{M}^L_{\bar{Y}_i + \Delta \bar{Y}_i} + \tilde{M}^U_{\bar{Y}_i + \Delta \bar{Y}_i} \right)}{\left(\tilde{M}^U_{\bar{Y}_i + \Delta \bar{Y}_i} - \tilde{M}^L_{\bar{Y}_i + \Delta \bar{Y}_i} \right)} \qquad (31)$$

$$\tilde{\eta}(Y_i^r + \Delta Y_i^r) = \frac{\left(\tilde{M}^L_{Y_i^r + \Delta Y_i^r} + \tilde{M}^U_{Y_i^r + \Delta Y_i^r} \right)}{\left(\tilde{M}^U_{Y_i^r + \Delta Y_{ii}^r} - \tilde{M}^L_{Y_i^r + \Delta Y_i^r} \right)} \qquad (32)$$

From equation (21), $\tilde{M}^L_{\bar{Y}_i+\Delta\bar{Y}_i}$ and $\tilde{M}^U_{\bar{Y}_i+\Delta\bar{Y}_i}$ can be given by

$$\left(\tilde{M}^L_{\bar{Y}_i+\Delta\bar{Y}_i}\Big/\tilde{M}^U_{\bar{Y}_i+\Delta\bar{Y}_i}\right)=\min/\max f(\mathbf{Y})$$
$$\begin{cases} s.t. \\ g_k(\mathbf{Y}_p)\leqslant 0\left(k=1,2,\cdots,n_p\right) \\ h_l(\mathbf{Y}_q)=0\left(l=1,2,\cdots,n_q\right) \\ Y_i^L+\Delta\bar{Y}_i\leqslant Y_i\leqslant Y_i^U+\Delta\bar{Y}_i \\ Y_j^L\leqslant Y_j\leqslant Y_j^U(j=1,2,\cdots,n,j\neq i)\end{cases} \tag{33}$$

$\tilde{M}^L_{Y_i^r+\Delta Y_i^r}$ and $\tilde{M}^U_{Y_i^r+\Delta Y_i^r}$ can be expressed as

$$\left(\tilde{M}^L_{Y_i^r+\Delta Y_i^r}\Big/\tilde{M}^U_{Y_i^r+\Delta Y_i^r}\right)=\min/\max f(\mathbf{Y})$$
$$\begin{cases} s.t. \\ g_k(\mathbf{Y}_p)\leqslant 0\left(k=1,2,\cdots,n_p\right) \\ h_l(\mathbf{Y}_q)=0\left(l=1,2,\cdots,n_q\right) \\ Y_i^L-\Delta Y_i^r\leqslant Y_i\leqslant Y_i^U+\Delta Y_i^r \\ Y_j^L\leqslant Y_j\leqslant Y_j^U(j=1,2,\cdots,n,j\neq i)\end{cases} \tag{34}$$

From the linear optimization model in equation (26), $\tilde{M}^L_{\bar{Y}_i+\Delta\bar{Y}_i}$ and $\tilde{M}^U_{\bar{Y}_i+\Delta\bar{Y}_i}$ can be given by

$$\tilde{M}^L_{\bar{Y}_i+\Delta\bar{Y}_i}\Big/\tilde{M}^U_{\bar{Y}_i+\Delta\bar{Y}_i}\approx\min/\max\left[f(\bar{\mathbf{Y}})+\sum_{i=1}^{n}\left.\frac{\partial f}{\partial Y_i}\right|_{\bar{\mathbf{Y}}}(Y_i-\bar{Y}_i)\right]$$
$$\begin{cases} s.t. \\ g_k(\bar{\mathbf{Y}}_p)+\sum_{i=1}^{p}\left.\frac{\partial g_k}{\partial Y_i}\right|_{\bar{\mathbf{Y}}_p+\Delta\bar{\mathbf{Y}}_i}(Y_i-\bar{Y}_i)\leqslant 0\left(k=1,2,\cdots,n_p\right) \\ h_l(\bar{\mathbf{Y}}_q)+\sum_{j=1}^{q}\left.\frac{\partial h_l}{\partial Y_j}\right|_{\bar{\mathbf{Y}}_q+\Delta\bar{\mathbf{Y}}_i}(Y_j-\bar{Y}_j)=0\left(l=1,2,\cdots,n_q\right) \\ Y_i^L+\Delta\bar{Y}_i\leqslant Y_i\leqslant Y_i^U+\Delta\bar{Y}_i \\ Y_{j_1}^L\leqslant Y_{j_1}\leqslant Y_{j_1}^U(j_1=1,2,\cdots,n,j_1\neq i)\end{cases} \tag{35}$$

where $\bar{\mathbf{Y}}_p=\left(\bar{Y}_1,\bar{Y}_2,\cdots,\bar{Y}_p\right)$, $\bar{\mathbf{Y}}_q=\left(\bar{Y}_1,\bar{Y}_2,\cdots,\bar{Y}_q\right)$.

$\tilde{M}^L_{Y_i^r+\Delta Y_i^r}$ and $\tilde{M}^U_{Y_i^r+\Delta Y_i^r}$ can be expressed as

$$\tilde{M}^L_{Y_i^r+\Delta Y_i^r}\Big/\tilde{M}^U_{Y_i^r+\Delta Y_i^r}\approx\min/\max\left[f(\bar{\mathbf{Y}})+\sum_{i=1}^{n}\left.\frac{\partial f}{\partial Y_i}\right|_{\bar{\mathbf{Y}}}(Y_i-\bar{Y}_i)\right]$$
$$\begin{cases} s.t. \\ g_k(\bar{\mathbf{Y}}_p)+\sum_{i=1}^{p}\left.\frac{\partial g_k}{\partial Y_i}\right|_{\bar{\mathbf{Y}}_p}(Y_i-\bar{Y}_i)\leqslant 0\left(k=1,2,\cdots,n_p\right) \\ h_l(\bar{\mathbf{Y}}_q)+\sum_{j=1}^{q}\left.\frac{\partial h_l}{\partial Y_j}\right|_{\bar{\mathbf{Y}}_q}(Y_j-\bar{Y}_j)=0\left(l=1,2,\cdots,n_q\right) \\ Y_i^L-\Delta Y_i^r\leqslant Y_i\leqslant Y_i^U+\Delta Y_i^r \\ Y_{j_1}^L\leqslant Y_{j_1}\leqslant Y_{j_1}^U(j_1=1,2,\cdots,n,j_1\neq i)\end{cases} \tag{36}$$

It should be noted that equations (35) and (36) are just a supplement for equations (33) and (34) under the situation that equations (33) and (34) do not have any solution.

Illustrative examples and discussions

In this section, two examples are provided to demonstrate the applications of the proposed method as well as its effectiveness. All parameters are treated as interval variables in the examples. The traditional non-probabilistic methods where interval variables are assumed to be mutually independent are also presented to make a comparison.

Example 1: a cantilever

Consider a cantilever, shown in Figure 1.[10] Two loads p_1 and p_2 are applied to the cantilever, and the distances between the loads and the fixed end are b_1 and b_2, respectively. The performance function is given by

$$M=m_{cr}-p_1b_1-p_2b_2$$

where m_{cr} is the critical limit bending moment. The sum of two loads should not be more than 7.5 kN, that is, the dependency of the interval variables p_1 and p_2 is determined by inequality constraint $g(p_1,p_2)=p_1+p_2-7.5\leqslant 0$. The details of interval variables are given in Table 1. Both non-probabilistic reliability index and non-probability reliability sensitivity for a structural system, with interval variables whose state of dependence is determined by constraint, are given in Tables 2 and 3.

(If interval variables are mutually independent, the non-probabilistic reliability index can be calculated by the method in Guo et al.,[12] and the non-probabilistic reliability sensitivity can be calculated by the method in Li et al.[7])

From Tables 2 and 3, we know that the results calculated without considering dependency are quite different from that with considering dependency. For example, the midpoint sensitivity of interval variable p_1 is -0.2714 without considering dependency, while it is 0.1044 with considering dependency. The former indicates that the system is more unreliable with a greater midpoint value of p_1. From Tables 2 and 3, we can conclude that the interval variable b_1 is a key variable, which more attention should be paid to in the design stage. The relationship between the non-probabilistic index and interval variable b_1 under the same coefficients of variation is given in Figure 2.

Example 2: harmonic drive

A harmonic drive, shown in Figure 3, is widely used in the solar array drive mechanism and the antenna pointing mechanism because of its high carrying capacity, light weight, small size, etc.[17]

The performance function for its life estimation is

$$G(T_h,N_V,T,K,m)=\frac{75\times 10^5}{N_V}\left(\frac{T_h}{KT}\right)^3-8760\times m$$

Table 1. Details of interval variables.

Interval variables	p_1 (kN)	p_2 (kN)	b_1(m)	b_2(m)	m_{cr} (kN.m)
Lower bound	4.4	1.7	1.8	4.5	32
Upper bound	5.6	2.3	2.2	5.5	40

Table 2. Non-probabilistic reliability sensitivity for midpoint.

Sensitivity type	$\tilde{\eta}$	$\frac{\partial\tilde{\eta}}{\partial\bar{p}_1}$	$\frac{\partial\tilde{\eta}}{\partial\bar{p}_2}$	$\frac{\partial\tilde{\eta}}{\partial\bar{b}_1}$	$\frac{\partial\tilde{\eta}}{\partial\bar{b}_2}$	$\frac{\partial\tilde{\eta}}{\partial\bar{m}_{cr}}$
Proposed method	1.9576	0.1044	−0.3230	−0.4432	−0.3132	0.1210
Without considering dependency	1.8080	−0.2714	−0.6786	−0.6994	−0.2922	0.1149

Table 3. Non-probabilistic reliability sensitivity for radius.

Sensitivity type	$\frac{\partial\tilde{\eta}}{\partial p_1^r}$	$\frac{\partial\tilde{\eta}}{\partial p_2^r}$	$\frac{\partial\tilde{\eta}}{\partial b_1^r}$	$\frac{\partial\tilde{\eta}}{\partial b_2^r}$	$\frac{\partial\tilde{\eta}}{\partial m_{cr}^r}$
Proposed method	−0.1043	−0.8513	−0.9528	−0.5102	−0.2370
Without considering dependency	−0.4385	−1.0959	−1.1074	−0.4500	−0.2078

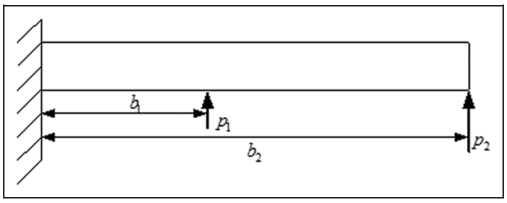

Figure 1. A cantilever.

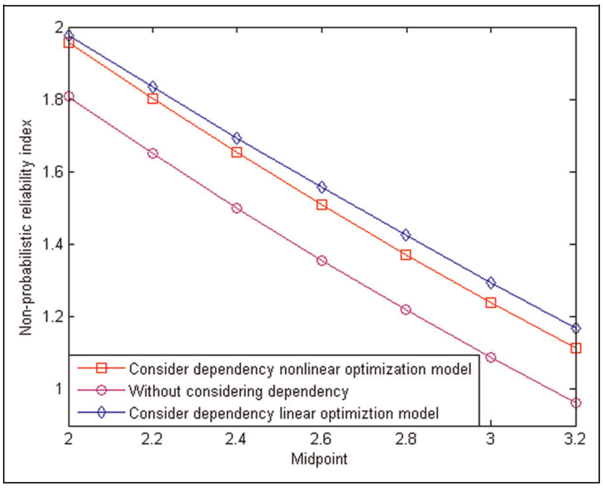

Figure 2. Relationship between the non-probabilistic index and interval variable b_1.

where m is the number of years; and T_h, N_v, K, and T are the rated output torque, input speed, condition factor, and nominal output torque, respectively. When $G > 0$, the system is considered to be safe. When $G < 0$,

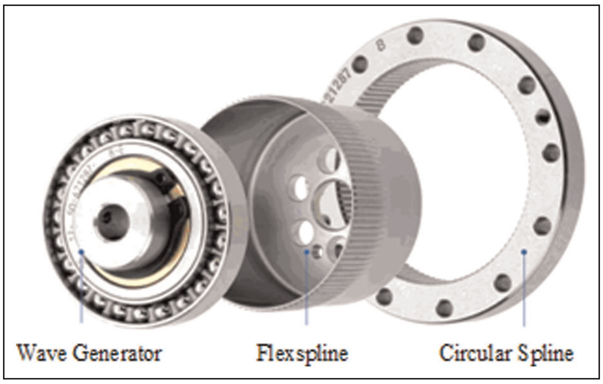

Figure 3. Harmonic drive.

the system falls in the failure domain. The dependency of the interval variables T, T_h, and K are determined by the inequality constraint $\frac{T}{T_h} \geqslant 4K$. The details of interval variables are given in Table 4.

Non-probabilistic reliability index $\tilde{\eta}$ calculated by considering dependency and without considering dependency under different years are given in Table 5 and Figure 4, respectively. Non-probabilistic reliability sensitivity under the consideration of dependency is given in Table 6.

Table 4. Details of interval variables.

Interval variables	T_h(N m)	N_v(r/min)	K	T(N m)
Lower bound	380	0.1	1.1	1800
Upper bound	420	0.12	1.3	2000

Table 5. Non-probabilistic index $\tilde{\eta}$ under different running years.

Running year	10	11	12	13	14	15
Proposed nonlinear optimization model	1.4611	1.4235	1.3860	1.3484	1.3108	1.2733
Proposed linear optimization model	1.1872	1.1479	1.1085	1.0692	1.0299	0.9906
Without considering dependency	1.4130	1.3793	1.3457	1.3120	1.2784	1.2448

Table 6. Non-probabilistic reliability sensitivity.

Sensitivity type	$\frac{\partial\tilde{\eta}}{\partial\bar{N}_v}$	$\frac{\partial\tilde{\eta}}{\partial\bar{T}_h}$	$\frac{\partial\tilde{\eta}}{\partial\bar{K}}$	$\frac{\partial\tilde{\eta}}{\partial\bar{T}}$	$\frac{\partial\tilde{\eta}}{\partial N_v^r}$	$\frac{\partial\tilde{\eta}}{\partial T_h^r}$	$\frac{\partial\tilde{\eta}}{\partial K^r}$	$\frac{\partial\tilde{\eta}}{\partial T^r}$
Proposed method	−1.8606	0.0081	1.1916	−0.0015	−15.0939	−0.0091	−5.9000	−0.0016
Without considering dependency	−1.8708	0.0031	−0.5394	−0.0004	−12.9870	−0.0112	−3.6179	−0.0023

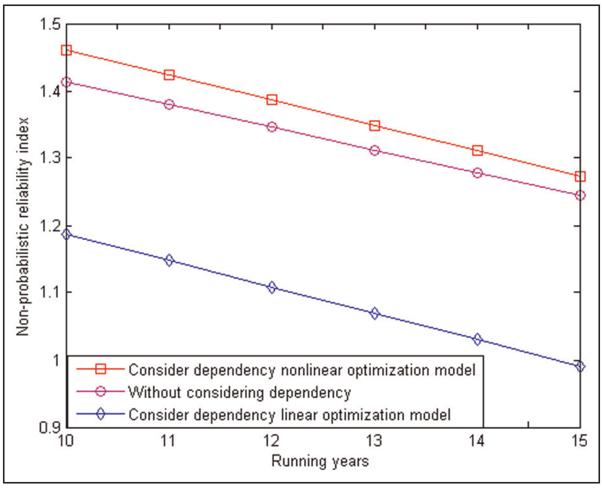

Figure 4. Relationship between non-probabilistic reliability index and running years.

From Table 6, we can see that the results without considering dependency are quite different from that with considering dependency. Input speed N_v is a key interval variable that more attention should be paid to in the design stage.

Conclusions

Owing to insufficient data and imprecise information, the impact of various uncertainties in engineering practices should be considered, especially in the early design stage of products. In this case, the traditional probability-based reliability method cannot be used because it needs a large amount of data to determine probabilistic distributions of random parameters. Fortunately, interval-based non-probabilistic reliability methods are appropriate to deal with this case. In this article, the existing non-probabilistic reliability index model and non-probabilistic reliability sensitivity method are extended, and a novel non-probabilistic reliability sensitivity method for structural systems, with interval variables whose dependence is determined by constraints, is proposed. The dependency of interval variables is determined by inequality and equality constraints. The results of the two examples show that the proposed method is effective because it provides a means of reliability sensitivity analysis under the case of severe uncertainty. Generally, the proposed method is more general than the traditional non-probabilistic sensitivity method because it considers the dependency of interval variables. The numerical examples indicate the results of non-probabilistic reliability sensitivity, calculated without considering dependency, are quite different from that calculated with considering dependency.

It is should be noted that the results calculated by using the proposed method are not the precise solutions. The main error comes from the finite difference method. Comparing some global approaches and improving the computational accuracy will be considered in our future works.

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