Springer

KSME

Journal of Mechanical Science and Technology 28 (10) (2014) 3937~3946
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-014-0904-x

A Bayesian optimal design for degradation tests based on the inverse Gaussian process †

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(Manuscript Received January 14, 2014; Revised April 10, 2014; Accepted April 28, 2014)

Abstract

The inverse Gaussian process is recently introduced as an attractive and flexible stochastic process for degradation modeling. This process has been demonstrated as a valuable complement for models that are developed on the basis of the Wiener and gamma processes. We investigate the optimal design of the degradation tests on the basis of the inverse Gaussian process. In addition to an optimal design with pre-estimated planning values of model parameters, we also address the issue of uncertainty in the planning values by using the Bayesian method. An average pre-posterior variance of reliability is used as the optimization criterion. A trade-off between sample size and number of degradation observations is investigated in the degradation test planning. The effects of priors on the optimal designs and on the value of prior information are also investigated and quantified. The degradation test planning of a GaAs Laser device is performed to demonstrate the proposed method.

Keywords: Bayesian approach; Degradation tests; Inverse Gaussian process; Optimal design; Prior distribution

1. Introduction

Modern products and engineering systems are known for their high reliability and long service life. The manufacturers of modern products and engineering systems are receiving increasing pressure from the markets as customers/users increasingly focus on product reliability. Reliability tests based on time-to-failure observations are often hindered by observed failures [1, 2]. Classical pass-fail and lifetime tests obtain insufficient reliability data to generate a precise reliability assessment. Although these products may not fail in many situations, their characteristics degrade over time [3]. Given that the deterioration of the service life indicator of a product can be reasonably related to the failure of the product, the reliability of a product can be assessed by degradation analysis [4]. Service life indicators include oil debris for lubrication [5], crack length for gears [6], and fatigue damage for structures [7]. A degradation test is generally performed to observe the deterioration of the service life indicators of a product. The optimal design of the degradation test aims to find an optimal test plan that generates degradation data in a cost-effective manner [8]. By determining the optimum test sample, interval of observation time, and number of observations, the degradation test can generate a precise reliability assessment under limited test resources and time. Meeker and Escobar [9] developed a comprehensive framework for reliability assessment via degradation analysis. The introduction of this framework was followed by the publication of several studies on degradation modeling and analysis, such as those of Tangkuman and Yang [10], Son [11], Wang et al. [12], and Ye et al. [13]. Along with the expansion of degradation modeling, the degradation test has been demonstrated as a significant toolkit for reliability assessment, particularly for assessments that are subjected to limited sample sizes and test times.

To facilitate the design of degradation test, a lot of works that consider different degradation test perspectives have been published [2]. Tseng and Yu [8] investigated the stopping time for a degradation test by using an asymptotically equivalent MTTF estimator. Yu and Tseng [14] extended this work by simultaneously determining the sample size, inspection frequency, and termination time for a degradation test. Shi et al. [15] introduced a method for the optimal test planning of an accelerated destructive degradation test (ADDT). A general equivalence theorem was used to verify the optimality of the plans. Shi and Meeker [16] further investigated the ADDT test planning by using the Bayesian method. A large-sample approximation method was used to simplify the calculation of the Bayesian criterion. These works were conducted on the

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† This paper was presented at the ICMR2013, Emeishan, Sichuan, China, July15-18, 2013. Recommended by Guest Editor Ming J. Zuo

basis of the assumption that the degradation process followed a degradation path model. The degradation test planning for the degradation process described by the Wiener process has been recently investigated. Tang et al. [17] investigated the optimal planning of the step-stress accelerated degradation test (SSADT). The Wiener process was used to model the degradation process and to overcome the limitation of the deterministic degradation process model. By simultaneously determining the sample size, measurement frequency, and termination time, Liao and Tseng [18] proposed a method for the optimal designing of SSADT by using a Wiener process degradation model. Lim and Yum [19] recently investigated the optimal design of the accelerated degradation test (ADT) by determining the test stress levels and the proportion of test units allocated in the stress levels of the Weiner process degradation model.

The Weiner process is suitable for situations wherein the degradation process of a product is not strictly increasing. However, the degradation processes of products strictly increase in several situations, such as the crack length of gears and the fatigue damage of structures. Therefore, the optimal degradation test design for these situations must be investigated. Unfortunately, the optimal degradation test design for situations with a monotically increasing degradation process has been investigated (e.g., Tseng et al. [20] and Tsai et al. [21]). Tseng et al. [20] introduced an SSADT model by assuming that the degradation path followed a gamma process. They further proposed a method for the optimal design of SSADT. Tsai et al. [21] proposed an optimal degradation test design based on a gamma process with random effects and investigated the effect of model misspecification. Both of these studies were conducted on the basis of the assumption that the degradation process followed a gamma process. The inverse Gaussian process (IG) was recently introduced by Wang and Xu [22] and Ye and Chen [23] as an attractive and flexible stochastic process for modeling degradation processes with monotonically increasing patterns. The degradation models based on the IG process outperforms frequently used models based on the gamma process. Moreover, Tsai et al. [21] demonstrated that model misspecification can considerably affect the optimal degradation test design and lead to poor reliability assessment. Therefore, an optimal degradation test design must be developed by considering the situation wherein the IG process is more suitable than the Wiener and gamma processes for degradation modeling. Examples of such situations include the GaAs laser degradation data investigated in this paper and the fatigue crack data introduced by Wu and Ni [24]; the fatigue crack data have been investigated further by Ref. [25] by using the IG process. Accordingly, this study focuses on degradation test planning based on the IG process. This study also aims to complement the optimal degradation test planning where the Weiner and gamma processes are insufficient for degradation modeling.

In addition to the abovementioned degradation model, the uncertainty within the planning values of the model parameters presents another critical issue in degradation test planning. Model parameters that are pre-estimated from historical data or preliminary test results are commonly used, and sensitivity analysis is performed to determine the uncertainty of these predefined model parameters. This method may lead to several iterations of the optimal procedure and an unwanted variance of the final optimal design [26]. The Bayesian method, wherein the model parameters are treated as random variables, presents a natural and alternative way for testing uncertainty. The uncertainty in the model parameters is expressed by a joint prior distribution. The optimal degradation test design is formulized as an optimization problem under a constructed Bayesian framework. The Bayesian method has been proven effective by Zhang and Meeker [26] and Yuan et al. [27] for the optimal designing of reliability tests. However, the Bayesian optimal degradation test based on the IG process has been insufficiently investigated. The effect of prior information on degradation analysis has also been proven significant by Wang and Zhang [27] and Liao and Tian [29]. Liu and Tang [30] and Shi and Meeker [31] emphasized the importance of incorporating prior information for optimal degradation test planning by using the Bayesian method. However, quantifying the value of such information must be investigated further, particularly for test designs subjected to limited test resources. Classical examples include the degradation testing of secondary batteries in spacecraft investigated by Jin et al. [32] and the manufacturing equipment investigated by Kharoufeh et al. [33]. Accordingly, the Bayesian optimal degradation test based on the IG process must be investigated. The Bayesian optimal designs for non-informative and informative priors must be compared with each other, and the incorporation of prior information for the degradation test must be investigated further.

Given these issues, this study proposes a Bayesian method for designing an optimal degradation test design on the basis of the IG process. Implementing the Bayesian optimal design method, comparing different prior distributions, and quantifying the value of prior information are all investigated by using an illustrative example. An average pre-posterior variance of reliability is used as the optimization criterion. A trade-off between sample size and number of degradation observations under limited test resource is studied. A step-by-step stochastic optimization procedure based on large-sample approximation and parametric surface smoothing techniques is presented to solve the Bayesian optimal design problem.

The rest of the paper is organized as follows. Sec. 2 describes the degradation model based on the IG process. Sec. 2 presents the basic assumptions for the construction of degradation test planning. Sec. 3 describes the Bayesian optimal degradation test design and discusses the deduction of the precision criterion, the derivation of prior information, and the solution for the Bayesian optimal design problem. Sec. 4 illustrates the proposed Bayesian optimal degradation test design via a classical example. Sec. 5 concludes the paper and highlights potential topics for future works.

2. Model assumption

Let $\{Y(t);t>0\}$ denote the degradation process of a product with $Y(0)=0$. We assume that the degradation process follows an IG process with mean function $\Lambda(t)$ and scale parameters η. $Y(t)$ has independent increments, with each increment following an IG distribution as presented in Refs. [22] and [23].

$$\Delta Y(t)\sim \mathrm{IG}\left(\Delta\Lambda(t),\eta\left[\Delta\Lambda(t)\right]^2\right), \tag{1}$$

where $\Delta Y(t)=Y(t+\Delta t)-Y(t)$, $\Delta\Lambda(t)=\Lambda(t+\Delta t)-\Lambda(t)$, and $\mathrm{IG}(\Delta\Lambda(t),\eta[\Delta\Lambda(t)]^2)$ denote an IG distribution with mean $\Delta\Lambda(t)$ and variance $\Delta\Lambda(t)/\eta$.

We assume that $\Lambda(t)=t^q, q>0$ with $\Lambda(0)=0$, as shown in Ye and Chen [23]. The degradation process is then modeled as $Y(t)\sim \mathrm{IG}(\Lambda(t),\eta[\Lambda(t)]^2)$. A physical interpretation is ascribed to the mean function $\Lambda(t)=t^q$, that is, the mean of the degradation process is described by a monotonically non-decreasing function. In real applications, a physical model can be incorporated through a proper choice of the degradation mean function $\Lambda(t)$. Moreover, prior information can be integrated through a well-defined derivation of prior distributions for the parameters in this degradation mean function.

The product fails when its degradation path $Y(t)$ reaches a predefined threshold level Y_D. Therefore, the lifetime T of a product is defined as follows:

$$T=\inf\left\{t\mid Y(t)\geq Y_D\right\}. \tag{2}$$

On the basis of the monotonic increments of the IG process, product reliability is defined as follows [22]:

$$\begin{aligned} R(t\mid q,\eta) &= P(T>t\mid q,\eta)=P\left(Y(t)<Y_D\mid q,\eta\right)\\ &=\Phi\left[\sqrt{\frac{\eta}{Y_D}}\left(Y_D-t^q\right)\right]+\exp\left(2\eta t^q\right)\Phi\left[-\sqrt{\frac{\eta}{Y_D}}\left(Y_D+t^q\right)\right]. \end{aligned} \tag{3}$$

The reliability of a product over a predefined mission time is often used to assess the probability of a product to fulfill a mission successfully. The decision-making process must consider the usage of the product in real engineering practice. Reliability is also used as an important measure for manufacturers to determine the warranty period of a product [34]. We use the variance of product reliability for a predefined mission time as a precision criterion. Let $\boldsymbol{\theta}=\{q,\eta\}$ denote the vector of model parameters. On the basis of Eq. (3), the reliability for a predefined mission time t_m is obtained as $R_m(\boldsymbol{\theta}) = R(t_m|\boldsymbol{\theta})$. Let $\mathrm{Var}(R_m(\boldsymbol{\theta}))$ denote the variance of product reliability for a predefined mission time. An efficient degradation test must be designed in such a way that the product reliability for a specific mission time can be estimated precisely.

Suppose that n samples are randomly selected for the degradation test. The degradation observations of each sample are measured every f units of time. The maximum number of observations for each sample is denoted as m. Let $y_i(t_j)$ denote the degradation observations of the ith sample at observation time point t_j, where $t_j = f \bullet j$ with $j = 0,\ldots, m$ and $t_0 = 0$. This paper pre-defines the measurement frequency f, which is generally determined according to the characteristics of the test equipment that performs the degradation test. The optimal test plan specifies the sample size n and the measurement number m. For a given test plan $D = \{n, m\}$ with the degradation observations $\boldsymbol{Y}_i=\{y_i(t_j)\}_{j=1}^m$ where $i = 1,\ldots, n$, the joint posterior distribution of the model parameters can be obtained as $f(\boldsymbol{\theta}|\ \boldsymbol{Y}, D)$ by using the Bayesian method. The variance of product reliability for a predefined mission time is obtained as $\mathrm{Var}(R_m(\boldsymbol{\theta})|\ \boldsymbol{Y}, D)$. By averaging $\mathrm{Var}(R_m(\boldsymbol{\theta})|\ \boldsymbol{Y}, D)$ over the degradation observations $\boldsymbol{Y}$, the average value of the variance of product reliability for a predefined mission time is obtained as $\mathrm{AVar}(R_m(\boldsymbol{\theta})|\ D)$. The value of such variance is used as the optimization criterion in this paper to describe the relationship between the test plan and estimation precision. Sec. 3 presents additional details on the derivation and calculation of these parameters. The cost of the test is determined by the test plan $D = \{n, m\}$. The total cost comprises the costs of samples and measurements and is computed as follows:

$$TC(n,m)=C_{sa}n+C_{me}nm, \tag{4}$$

where C_{sa} is the unit cost of each test sample and C_{me} is the unit cost of each degradation observation.

The optimal design of degradation tests is formulated as the following optimization problem:

Minimize

$$\mathrm{AVar}\left(R_m(q,\eta)\mid n,m\right)$$

subject to

$$\begin{aligned} &TC(n,m)\leq C_D\\ &N_L\leq n\leq N_U\ ,\\ &M_L\leq m\leq M_U \end{aligned}$$

where $\mathrm{AVar}(R_m(q, \eta)|n, m)$ denotes the average value of the variance of product reliability for a specific mission time, C_D denotes the predefined budget for performing the degradation test, (N_L, N_U) denotes the predefined bounds of the sample size, and (M_L, M_U) denotes the number of observations.

3. Bayesian optimal test plan

To handle the abovementioned optimization problem, the framework of the Bayesian optimal test plan comprises the following parts:

(1) The precision criterion for the Bayesian optimal planning;

(2) The prior distribution for handling uncertainty in the

planning values;

(3) The optimization algorithm for solving the Bayesian optimization problem.

3.1 Precision criterion

For a given test plan $D = \{n, m\}$ with the degradation observations $\boldsymbol{Y}_i = \{y_i(t_j)\}_{j=1}^{m}$ where $i = 1, \ldots, n$, the joint posterior distribution of the model parameters is obtained by using Bayes' theorem:

$$p(\boldsymbol{\theta} \mid \boldsymbol{Y}, D) = \frac{\pi(\boldsymbol{\theta}) L(\boldsymbol{Y} \mid \boldsymbol{\theta}, D)}{f_Y(\boldsymbol{Y} \mid D)}, \tag{5}$$

where $\pi(\boldsymbol{\theta})$ is the joint prior distribution of model parameters $\boldsymbol{\theta}$, $L(\boldsymbol{Y} \mid \boldsymbol{\theta}, D)$ is the likelihood function of degradation observations $\boldsymbol{Y}$ under a specific test plan D, and $f_{\boldsymbol{Y}}(\boldsymbol{Y}| D)$ is the pre-posterior marginal distribution of degradation observations $\boldsymbol{Y}$:

$$f_Y(\boldsymbol{Y} \mid D) = \int_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta}) L(\boldsymbol{Y} \mid \boldsymbol{\theta}, D) \mathrm{d}\boldsymbol{\theta} . \tag{6}$$

On the basis of the joint posterior distribution of the model parameters, the reliability of the product at a predefined mission time is obtained by using Eq. (3). The posterior variance of $R_m(\boldsymbol{\theta})$ is given as follows:

$$\mathrm{Var}\left(R_m(\boldsymbol{\theta}) \mid \boldsymbol{Y}, D\right) = \mathrm{Var}\left(R(t_m \mid \boldsymbol{\theta}) \times p(\boldsymbol{\theta} \mid \boldsymbol{Y}, D)\right). \tag{7}$$

Given that Eq. (7) has no analytical expression, the variance of $R_m(\boldsymbol{\theta})$ is obtained using a sample-based simulation method. By generating the posterior samples of the model parameters from the joint posterior distribution, the value of $R_m(\tilde{\theta})$ at a specific sample $\tilde{\theta}$ is obtained by using Eq. (3). The samples of $R_m(\boldsymbol{\theta})$ are obtained, and the variance is summarized from these samples.

The posterior variance $\mathrm{Var}(R_m(\boldsymbol{\theta})|\boldsymbol{Y}, D)$ depends on the degradation observations $\boldsymbol{Y}$. To describe the relationship between the test plans and estimation precision, the average value of the pre-posterior variance of $R_m(\boldsymbol{\theta})$ is used as the Bayesian planning criterion. Such value is computed as the average of the posterior variance $\mathrm{Var}(R_m(\boldsymbol{\theta})|\boldsymbol{Y}, D)$ over the degradation observations $\boldsymbol{Y}$:

$$\begin{aligned} &\mathrm{AVar}\left(R_m(\boldsymbol{\theta}) \mid D\right) = \\ &\quad \int_{\mathbf{Y}|D} \mathrm{Var}\left(R(t_m \mid \boldsymbol{\theta}) \bullet p(\boldsymbol{\theta} \mid \mathbf{Y}, D)\right) f_Y(\mathbf{Y} \mid D) \mathrm{d}\mathbf{Y} \end{aligned} \tag{8}$$

3.2 Prior distribution

In the Bayesian optimal reliability test design, we use prior distribution to describe the uncertainty in the model parameters (planning values) [31] to complement the classical design of the reliability test with fixed model parameters. The prior distribution is also used to incorporate prior information in the analysis of degradation observations. Zhang and Meeker [26] and Tang and Liu [30] demonstrated that prior distributions can exert considerable effects on the optimal test plan.

The joint prior distribution $\pi(\boldsymbol{\theta})$ for parameters $\boldsymbol{\theta} = \{q, \eta\}$ is specified in this paper on the basis of historical data and expert testimonies. The normal, gamma, and lognormal distribution for the prior distribution generated from historical data are specified through a statistical analysis of historical data. The implementation of Bayesian analysis on the historical data can produce a joint posterior distribution for these parameters. This joint posterior distribution is used as the prior distribution for the test design of new products.

The prior distributions for parameters $\boldsymbol{\theta} = \{q, \eta\}$ cannot be easily specified for the prior distribution generated from expert testimonies. Following the method of Yuan et al. [27], we indirectly collect the testimonies of experts and then specify the prior distribution for these parameters. Given that the degradation process $Y(t)$ follows $IG(t^q, \eta t^{2q})$ with mean t^q and variance t^q/η, we obtain the expert testimonies on the degradation mean M_t^E and variance V_t^E at specific time points t. The subjective information is quantified and presented as the joint probability distribution of degradation mean M_t^E and variance V_t^E as $f_{M,V}(M_t^E, V_t^E) = f_M(M_t^E) \bullet f_V(V_t^E)$, where the probability encoding method is generally used [35]. The joint probability distribution of the model parameters is obtained by utilizing the relationships of parameters $\boldsymbol{\theta} = \{q, \eta\}$ with the degradation mean M_t^E and variance V_t^E, which are expressed as $M_t^E = t^q$ and $V_t^E = t^q/\eta$. Such a distribution is implemented by the multivariate transformation of random variables from M_t^E and V_t^E into q and η. Mathematically, the prior distribution obtained through this derivation process is given as $\pi(q, \eta) = f_{M,V}(q, \eta) \bullet |J|$, where J is the Jacobian matrix of the function relationships $M_t^E = t^q$ and $V_t^E = t^q/\eta$. Prior derivation is important for the Bayesian design of reliability tests, particularly for situations that involve the uncertainty of the model parameters and the subjective knowledge of the failure mechanism. Refer to Refs. [36-38] for additional details on subjective information quantification and prior distribution derivation.

3.3 Optimization algorithm

To perform the test planning, an optimization algorithm is introduced to solve the abovementioned optimization problem. Given the complexity of the objective function (Eq. (8)), an analytical expression is unavailable for the solution of the optimization problem. Monte Carlo simulation and large-sample normal approximation are generally used to find the optimal plan. The Monte Carlo method heavily depends on huge numbers of iterations and requires an extremely long computation time [39]. The large-sample normal approximation depends on a reasonably large-sample size, and the accuracy of approximation is difficult to verify. We introduce a stochastic optimization method with parametric smoothing technique to solve the Bayesian optimal design problem for the IG process. This method has been used by Liu and Tang [30] for ADT designs and by Yuan et al. [27] with the non-parametric kernel smoothing technique for the step-stress accelerated life test designs. The procedures of the algorithm are summarized as follows:

Step 1: Select N designs of the test plan that spread over the design space (i.e., $D_i = \{n_i, m_i\}$, $i = 1,\dots, N$).

Step 2: For each design D_i, $i = 1,\dots, N$, calculate the Bayesian planning criterion AVar($R_m(\boldsymbol{\theta})| D_i$) by using Monte Carlo integration.

2(a) Draw R sets of degradation observations independently (i.e., $\boldsymbol{Y}_{i,k}$, $k = 1,\dots, R$). These sets are implemented by generating a set of model parameters $\boldsymbol{\theta}_{i,k}$ from the prior distribution $\pi(\boldsymbol{\theta})$, and then generating degradation observations $\boldsymbol{Y}_{i,k} = \{y_{i,k}(t_j)\}_{j=1}^{m}$ with the test plan $D_i = \{n_i, m_i\}$ from the IG process model shown in Eq. (1).

2(b) For each set of degradation observations $\boldsymbol{Y}_{i,k}$, $k = 1,\dots, R$, calculate the posterior distribution of the model parameters by using the Bayesian method. The distribution is implemented by sampling the posterior samples of the model parameters from the posterior distribution given in Eq. (5) by using the Markov chain Monte Carlo method (MCMC).

2(c) For each group of posterior samples of the model parameters, calculate the variance of product reliability at a predefined mission time by using the posterior sample-based simulation method. The variance is implemented by substituting the posterior samples into Eq. (3) to obtain the posterior samples of $R(t_m| \boldsymbol{\theta})$. The variance of product reliability for the degradation observations is summarized from the variance of these samples.

2(d) Calculate the mean value of Var($R(t_m|\boldsymbol{\theta})\bullet p(\boldsymbol{\theta}|\boldsymbol{Y}, D_i)$) for the R sets of degradation observations to obtain AVar($R_m(\boldsymbol{\theta})|D_i$) under the test plan D_i.

Step 3: Fit a smooth surface by using the method of kernels to the R pairs of points, which include test plans D_i and their corresponding AVar($R_m(\boldsymbol{\theta})| D_i$). To facilitate the calculation of the optimal point on the basis of the smooth surface, a parametric polynomial regression model with maximum degrees of up to five is used in this research. For the two decision variables $D = \{n, m\}$ considered in this paper, the smooth curve is computed as follows:

$$\begin{aligned}\mathrm{AVar}\left(R_m(\boldsymbol{\theta}) \mid n,m\right) &= p_{00} + p_{11}nm + \sum_{k=1}^{2} p_{k,3-k} n^k m^{3-k} \\ &+ \sum_{l=1}^{3} p_{l,4-l} n^l m^{4-l} + \sum_{i=1}^{4} p_{i0} n^i + \sum_{j=1}^{4} p_{0j} m^j\end{aligned}, \quad (9)$$

where ps denotes the parameters of the smooth curve.

Step 4: Find the optimal design D^* by using the smooth surface obtained above.

Table 1. Prior information of model parameters that are generated from the GaAs Laser degradation data.

Parameter	Mean	SD	2.5%	97.5%	Fitted PDF
q	1.404	0.01834	1.368	1.440	Lognormal (0.3390, 0.0131)
η	9.966	1.046	8.017	12.11	Gamma (90.6531, 0.1099)

4. Illustrative example

This study uses an optimal GaAs laser degradation test design and the dataset of Meeker and Escobar [9] to demonstrate the proposed method. The degradation test design for the dataset with the gamma process model has been previously investigated by Tsai et al. [21]. In this section, we investigate this dataset with the IG process model by using the proposed method. Wang and Xu [22] and Ye and Chen [23] found that the IG model is suitable for this dataset compared with the gamma process model.

The light output of a GaAs Laser device will degrade over time. To maintain a constant light output, the operating current gradually increases as a complement of the inherent degradation. The GaAs Laser degradation dataset describes an increase in operating current over time for 15 GaAs laser devices. This dataset is presented as (Y_i, T_i), $i = 1,\dots, n$, with the sample size n equals 15. The observation time points T_i = {0.25, 0.5, 0.75, …, 4} thousand hours are the same for all samples with $f_i = 0.25$ thousand hours and $m_i = 16$. The degradation observations Y_i measure the increase in operating current in terms of percentage. The GaAs Laser device fails when the operating current reaches the predefined threshold of Y_D = 6%.

To perform the Bayesian optimal degradation test on the GaAs Laser device, the abovementioned degradation data are encoded as the prior distributions of the model parameters. The Bayesian analysis results of the degradation data are presented in Table 1. In the following degradation test design, the distributions of the model parameters that are shown in Table 1 are treated as true model parameters with uncertainty. The generation of model parameters in Step 2(a) is sampled from this prior distribution.

The predefined variables for the degradation test planning are given as follows. The measurement interval (frequency) is assumed as $f = 0.05$ thousand hours. The predefined mission time of interest is given as $t_m = 5$ thousand hours. The cost configuration is assumed $C_{sa} = 30$ and $C_{me} = 2$. The constraints of the sample size and the number of observations are given as $2 \le n \le 50$ and $5 \le m \le 50$.

On the basis of prior distribution and the predefined setting of the degradation test, we implement the optimization procedure presented in Sec. 3.3. The smooth curve (Fig. 1) is obtained by performing Steps 1 to 3 presented above.

The dots denote the pairs of points including the test plans $D_i = \{n_i, m_i\}$ and their corresponding AVar($R_m(\boldsymbol{\theta})| D_i$). The smooth curve denotes the fitted result of the polynomial re-

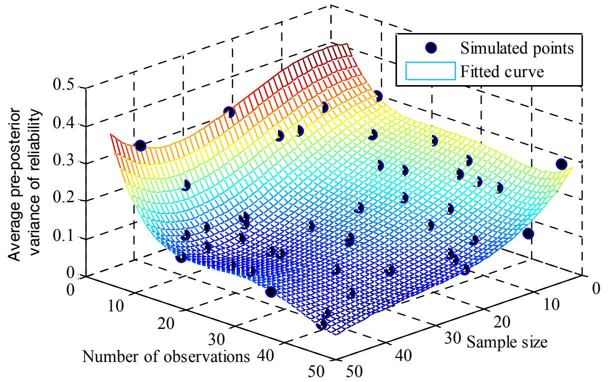

Fig. 1. Fitted smooth surface for the simulated plans with non-informative priors.

gression model given in Eq. (8) to the pairs of points. The curve describes the relationship between the average pre-posterior variance of reliability AVar(R_m ($\boldsymbol{\theta}$)| D_i) and the decision variables {n_i, m_i}. In detail, the curve is generated from 46 pairs of test plans and their pre-posterior variance of reliability. The 46 designs of test plans are uniformly picked in Step 1 within the constraints of design variables. The corresponding pre-posterior variances of reliability are obtained in Steps 2 and 3 by using the Bayesian method with non-informative prior distribution. The non-informative priors for the model parameters are incorporated in Step 2(b) to make the estimation result depend on the degradation data. The priors are given as $q \sim \text{Uniform}(0,5)$ and $\eta \sim \text{Uniform}(0,100)$.

The optimal degradation test designs are obtained in Step 4. The results are presented in Table 2.

The fitted smooth curve and optimal test plans indicate that a large-sample size and number of observations improve the reliability estimation accuracy. However, the sample size and number of observations must be balanced when the test resources are limited. In this situation, the optimization results indicate that large numbers of observations are more preferred than a larger sample size. The same indication is observed by comparing the degree of inclination between the upper right part and upper left part of the smooth curve in Fig. 1. The optimal test plans presented in Table 2 also demonstrate this conclusion. When the test cost is extremely limited, such as a budget of C_D = {500, 400, 100}, a large number of observations close to the boundary may be generated on the basis of the optimal results presented in Table 2. Fig. 2 presents a detailed description of the relationship between the design variables and precision criterion. The solid line indicates the relationship between the number of observations and precision criterion with a fixed sample size of $n = 2$. The dash line indicates the relationship between the sample size and precision criterion with a fixed number of observations of $m = 5$. By comparing the slopes of these curves, we find that the effect of increasing the number of observations greatly overcomes the effect of increasing the sample sizes.

Table 2. Optimal test plans with non-informative priors for the model parameter under different constraints of test costs.

Test constrains			Optimal test plans			$AVar(R_M(\boldsymbol{\theta})\mid D)$
C_D	(N_L,N_U)	(M_L,M_U)	n	m	$TC(n,m)$	
100	(2, 50)	(5, 50)	2	10	100	0.3272
200	(2, 50)	(5, 50)	2	35	200	0.2722
300	(2, 50)	(5, 50)	2	43	232	0.2594
400	(2, 50)	(5, 50)	3	43	348	0.2468
500	(2, 50)	(5, 50)	4	44	472	0.2349
1000	(2, 50)	(5, 50)	8	45	960	0.1931
1500	(2, 50)	(5, 50)	13	42	1482	0.1595
2000	(2, 50)	(5, 50)	31	17	1984	0.1213
2500	(2, 50)	(5, 50)	37	18	2442	0.1000
3000	(2, 50)	(5, 50)	40	20	2800	0.0958
3500	(2, 50)	(5, 50)	40	20	2800	0.0958
4000	(2, 50)	(5, 50)	33	45	3960	0.0947
4500	(2, 50)	(5, 50)	36	47	4464	0.0911
5000	(2, 50)	(5, 50)	40	47	4960	0.0875
5500	(2, 50)	(5, 50)	44	47	5456	0.0833
6000	(2, 50)	(5, 50)	46	50	5980	0.0767
6500	(2, 50)	(5, 50)	50	50	6500	0.0638

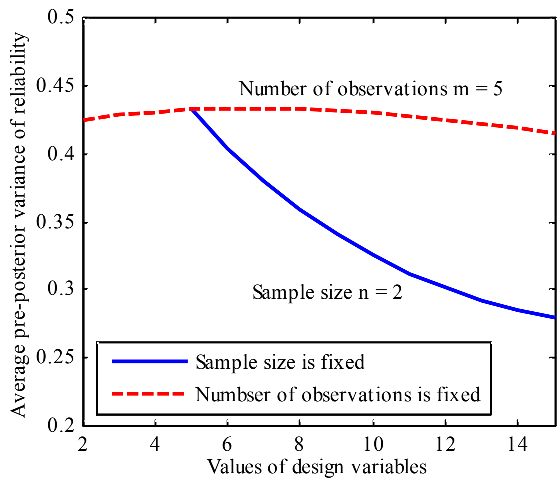

Fig. 2. Fitted smooth curve between the design variable (n or m) and precision criterion $\text{AVar}(R_M(\boldsymbol{\theta})\mid D)$.

5. Value of prior information

In the Bayesian optimal test design, prior distributions are used to describe the uncertainty of the planning values (model parameters). Prior distributions are performed by generating the degradation data presented in Step 2(a). The prior information of the model parameters is incorporated in the analysis of the degradation data, which is implemented by the Bayesian analysis of degradation data in Step 2(b) and is formulated in Eq. (5). The Bayesian optimal test design provides great flexibility for considering parameter uncertainty and incorporating prior information. The prior distribution adopted in the Bayes-

Table 3. Prior distributions adopted in the Bayesian optimal test design.

Type of priors	Prior distributions
Diffused priors for non-informative situation (Type I)	$\begin{cases} q \sim \text{Uniform}(0,5) \\ \eta \sim \text{Uniform}(0,100) \end{cases}$
Priors obtained from historical data (Type II)	$\begin{cases} q \sim \text{Lognormal}(0.3390, 0.0131) \\ \eta \sim \text{Gamma}(90.6531, 0.1099) \end{cases}$
Priors obtained from testimonies of experts (Type III)	$\begin{cases} q \sim \text{Lognormal}(0.35, 0.01) \\ \eta \sim \text{Gamma}(10,1) \end{cases}$

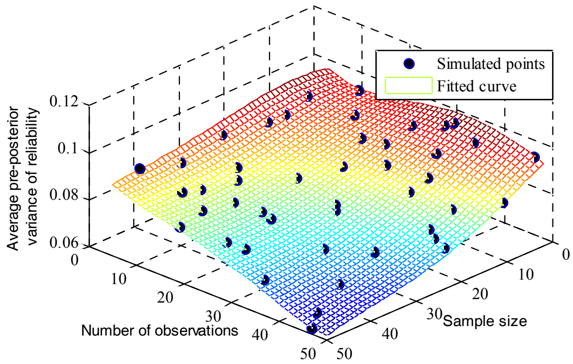

Fig. 3. Fitted smooth curve for the Bayesian optimal test design with informative priors generated from historical data.

ian test design can consequently exert a considerable effect on the optimal test plan.

To study the effect of prior distribution on the optimal test design, we use the prior distribution generated from historical data to describe the uncertainty in the model parameters (planning values). The non-informative prior distributions, the priors obtained from historical data, and the priors generated from expert testimonies are separately used in the analysis of degradation observation to investigate the value of prior information for the optimal test design. These prior distributions are presented in Table 3.

The fitted smooth curves and corresponding optimal test plans are obtained by applying the procedures in Sec. 3.3 with the priors in Table 3. The fitted smooth curve for the non-informative prior is presented in Fig. 1. The fitted smooth curves for the other priors are separately presented in Figs. 3 and 4. Similar to Fig. 1, the dots refer to the pairs of points that include the test plans $D_i = \{n_i, m_i\}$ and their corresponding AVar(R_m ($\boldsymbol{\theta}$)| D_i). The smooth curves denote the fitted results of the polynomial regression model given in Eq. (8) to the pairs of points. The optimal test plans are obtained on the basis of the fitted smooth curves.

By comparing these fitted smooth curves, we find that the test plans with informative priors can generate results that are more precise than those obtained under non-informative priors. This result is attributed to the effect of incorporating prior information through prior distributions. The effect of priors becomes obvious for test plans with limited resources, such as

Table 4. Comparison of the optimal test plans with different priors under a constrained sample size ($2 \le n \le 50$) and number of observations ($5 \le m \le 50$).

Prior type	Optimal test results				Constraint
	n	m	$TC(n,m)$	$\text{AVar}(R_M(\boldsymbol{\theta}) \mid D)$	C_D
I	2	10	100	0.3272	100
II	2	10	100	0.0962	
III	2	10	100	0.0929	
I	2	43	232	0.2594	300
II	2	50	260	0.0950	
III	5	15	300	0.0916	
I	4	44	472	0.2349	500
II	3	50	390	0.0939	
III	8	16	496	0.0910	
I	40	20	2800	0.0958	3000
II	23	50	2990	0.0745	
III	24	47	2976	0.0761	

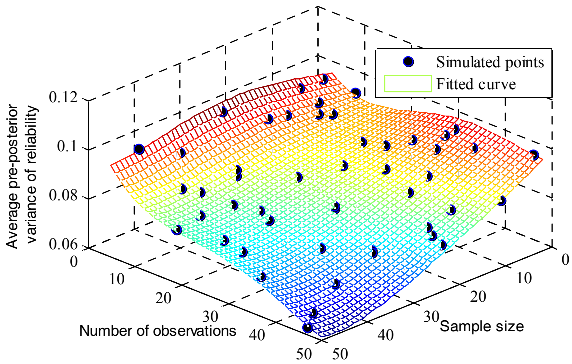

Fig. 4. Fitted smooth curve for the Bayesian optimal test design with informative priors generated from expert testimonies.

plans with a sample size of $2 \le n \le 5$ and a number of observations of $5 \le m \le 10$. Table 4 shows a detailed comparison of optimal test plans under limited test resources with different priors. When the test has a limited budget, such as C_D = {100, 300, 500}, the optimal test plans obtained under informative priors can arrive at more precise reliability estimations than plans that are generated under non-informative priors. Moreover, the optimal test plans generated under informative priors with a test cost of 100 can lead to reliability estimation results that are as precise as those of the optimal test plans generated under non-informative priors with a test cost of 3000. Table 4 highlights these findings by underlining the corresponding results.

A further study has indicated that the prior information of parameter q is more effective than the prior information of parameter η for the Bayesian optimal degradation test design developed on the basis of IG process. If these prior distributions of the model parameters are wrongly specified, an anomalous smooth curve will be generated, thus consequently

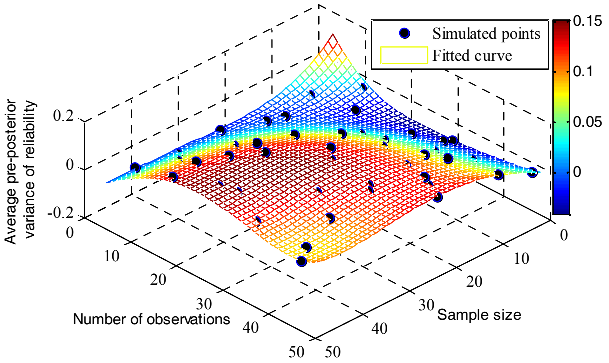

Fig. 5. Fitted smooth curve for the Bayesian optimal test design with the wrongly informative prior of parameter q.

leading to fallacious optimal test plans. For instance, the anomalous smooth curve presented in Fig. 5 is obtained with prior $q \sim \text{Lognormal}(1.0303, 0.0652)$ and $\eta \sim \text{Gamma}(10,1)$, where the mean of q in this prior distribution is double of the real mean of q. The abnormality of the smooth curve is presented as the anomalous increase of posterior variance with the increase of sample size and number of observations. This anomaly indicates a contradiction between the prior information and degradation observations. However, the incorrect incorporation of prior distribution can be prevented by checking the smooth curve or by following a well-defined procedure for prior derivation, such as the procedure presented in Sec. 3.2.

Accordingly, the value of prior information is determined by the optimal test planning under a limited test budget. This value is also highlighted from the perspective of test cost under equivalent test plans. The informative priors have a high worth within the Bayesian optimal degradation test design if they are correctly derived and incorporated.

6. Conclusion

This paper introduces a Bayesian approach for the development of an optimal degradation test design on the basis of the IG process. The degradation process is modeled by using an IG process. The uncertainty within the planning values of the model parameters is addressed by adopting a Bayesian optimal test design. Such uncertainty is formulized as an optimization problem that aims to minimize the average pre-posterior variance of reliability at a predefined mission time. A trade-off between the sample size and number of observations is investigated. A stochastic optimization with a step-by-step procedure is developed to solve the Bayesian optimization problem. The Bayesian approach and procedure are illustrated through a classic example. The proposed method is a valuable complement of degradation test planning for the situations wherein the Wiener and gamma processes are insufficient for degradation modeling. The Bayesian optimal test design for the handling of uncertainty within the planning values of the model parameters is important for degradation test planning with uncertainty. The effect of prior distributions on the optimal test design is investigated. The incorporation of priors for the optimal degradation test design presents significant potential. Such incorporation is particularly effective for situations with limited test resources.

Interesting results are obtained from the degradation test planning of GaAs lasers. A large-sample size and number of observations improve the precision of the reliability estimation of GaAs lasers. However, when the test resources are limited, a large number of observations is preferred than a large-sample size for the degradation test of GaAs lasers. The prior distributions also exert an indispensable effect on the optimal design. Test plans with informative priors can arrive at more precise results than those obtained under non-informative priors. Well-derived and correctly incorporated prior distributions can greatly increase the precision of reliability estimation.

However, several points are worth investigating further. The Bayesian approach is developed for the degradation test on the basis of the IG process. This model can be extended to the degradation test planning on the basis of the IG process with random effects. The other types of Bayesian optimal degradation test designs under heterogeneous operating environments also warrant further investigation.

Acknowledgment

This research is partially supported by the National Natural Science Foundation of China (Contract No. 11272082). Comments and suggestions from the reviewers and the editor are highly appreciated.

Nomenclature

$Y(t)$: Degradation process of a product
$\Lambda(t)$: Mean function of the IG process
η : Scale parameter of the IG process
$\Delta Y(t)$: The degradation increment
T : Lifetime of a product
$R(t \mid q, \eta)$: Reliability function of a product
t_m : A predefined mission time
$\boldsymbol{\theta} = \{q, \eta\}$: Model parameters of a degradation process
$R_m(\boldsymbol{\theta})$: Reliability of a product at time t_m
$\text{Var}(R_m(\boldsymbol{\theta}))$: Variance of reliability at time t_m
n : Sample size
f : Measurement frequency
m : Maximum number of observations
t : Observation time point
D : A test plan
$f(\boldsymbol{\theta} \mid \boldsymbol{Y}, D)$: Posterior distribution of model parameters conditionally on a test plan and degradation observations
$\text{Var}(R_m(\boldsymbol{\theta}) \mid \boldsymbol{Y}, D)$: Variance of reliability at time t_m conditionally on a test plan and degradation observations
$\text{Var}(R_m(\boldsymbol{\theta}) \mid D)$: Variance of reliability at time t_m conditionally

on a test plan
$TC(n,m)$: Total cost of a degradation test
C_{sa} : Unit cost of a test sample
C_{me} : Unit cost of a degradation observation
C_D : Predefined budget for the degradation test
N_L : Lower bound of the sample size
N_U : Upper bound of the sample size
M_L : Lower bound of the number of observations
M_U : Upper bound of the number of observations
$\pi(\boldsymbol{\theta})$: Prior distribution of the model parameters
$L(\boldsymbol{Y} \mid \boldsymbol{\theta}, D)$: Likelihood function of the degradation observations under a test plan
$p(\boldsymbol{\theta} \mid \boldsymbol{Y}, D)$: Posterior distribution of the model parameters conditionally on a test plan and degradation observations
$f_Y(\boldsymbol{Y}|D)$: Pre-posterior marginal distribution of the degradation observations $\boldsymbol{Y}$
M_t^E : Mean value of a degradation process
V_t^E : Variance value of a degradation process
J : Jacobian matrix

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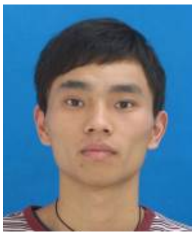

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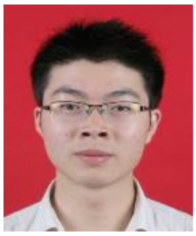

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