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Saddlepoint approximation-based reliability analysis method for structural systems with parameter uncertainties

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Abstract

Due to epistemic uncertainty, precisely determining parameters of all distribution is impossible in engineering practice. In this article, a novel reliability analysis method based on the saddlepoint approximation is proposed for structural systems with parameter uncertainties. The proposed method includes four main steps: (1) sampling for random and probability-box variables, (2) approximating the cumulant generating functions for systems under the best and worst cases, (3) calculating saddlepoints for the best and worst cases, and (4) calculating the lower and upper bounds of the probability of failure. The proposed method is effective because it does not require a large sample size or solving complicated integrals. Furthermore, the proposed method provides results that have the same accuracy as the existing interval Monte Carlo simulation method, but with significantly reduced computational effort. The effectiveness of the proposed method is demonstrated with three examples that are compared against with the interval Monte Carlo simulation method.

Keywords

Epistemic uncertainty, probability-box, Monte Carlo simulation, saddlepoint approximation, parameter uncertainty

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Introduction

Uncertainty exists widely in almost all practical engineering. Uncertainty can be divided into two types: aleatory and epistemic.[1–6] Aleatory and epistemic uncertainties are usually modeled by probability theory, interval numbers, fuzzy sets theory,[7–10] and probability-boxes (p-boxes)[8] among other methods. Since the performance and reliability of products are directly affected by uncertainty, quantitative assessment of uncertainty on product performance is widely recognized as an important task in practical engineering.[11,12]

Reported methods for calculating the probability of failure can be divided into simulation methods and approximation methods. Monte Carlo simulation (MCS) is very robust because it can address many reliability issues and it is generally accurate if a sufficient number of samples are used. However, the computational efficiency of MCS is very low especially for the products with high reliability.[13] To overcome the shortcoming of the expensive computational cost of the MCS method, many approximation methods have been developed including the first-order reliability method (FORM),[14] second-order reliability method (SORM),[15] first-order saddlepoint approximation (FOSPA),[13] mean value first-order saddlepoint approximation (MV-FOSPA),[16] moment-based methods,[17] Kriging models,[18] and perturbation techniques.[19] However, these aforementioned reliability analysis methods are based on probability theory that can only address aleatory uncertainty rather than both aleatory and epistemic uncertainties simultaneously.

Oberkampf et al.[20] pointed out that the system response evaluation is a challenging problem when the

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input parameters are associated with epistemic uncertainty. This means that reliability analysis for structural systems under both aleatory and epistemic uncertainties is very difficult to deal with. Recently, a unified uncertainty analysis (UUA) method was developed by Du and colleagues[5,21] for reliability assessment and reliability-based design of structural systems. They have shown that the most probable point (MPP) search is a double optimization process with both inner and outer loops when mixture of variables exists in the system.[21] In order to avoid the MPP search and non-normal to normal transformations, a UUA method based on MV-FOSPA (denoted as MV-FOSPA-UUA) was presented by Xiao et al.,[22] wherein the performance function is linearized with the Taylor expansion. Zaman et al.[23] developed a probabilistic approach for structural systems with both aleatory and epistemic uncertainties where the interval variable is represented by a Johnson family distribution. Möller et al.[24] developed the fuzzy first-order reliability method (FFORM) for structural systems involving both aleatory and epistemic uncertainties, wherein both aleatory and epistemic uncertainties are modeled by fuzzy random variables. Wang et al.[25] proposed a reliability analysis method for structural systems with both aleatory and epistemic uncertainties using fuzzy random variables. Zhou and Mourelatos[26] proposed a reliability-based design optimization (RBDO) method based on possibility theory for structural systems with insufficient data. Furthermore, it should be noted that a set of problems involving aleatory and epistemic uncertainties were studied by the Sandia National Laboratories,[27,28] and they have shown p-boxes offer many significant advantages since they provide convenient ways to handle both aleatory and epistemic uncertainties.[8,28] Generally, the uncertain distribution parameters (such as the mean value and the standard deviation (SD)) can only be expressed as intervals rather than precisely determined under the case of limited data samples.[29] A probability distribution with interval distribution parameters can be represented using p-boxes.[30] Zhang et al.[31,32] developed an interval Monte Carlo simulation (IMCS) method for structural reliability assessment with p-box variables. Ferson and Tucker[33] proposed a sensitivity analysis method with p-box variables.

Structural reliability assessment with interval parameters is still a challenging problem and more attention should be given to handle these problems. When p-box variables exist in a system, calculation of the probability of failure needs to consider families of distributions whose parameters are within the specified intervals[30,31] and to combine the methods of interval analysis and classical probability theory. In this article, we consider both aleatory and epistemic uncertainties for the parameters when reliability analysis is performed. Based on the research by Huang and colleagues[12,34] on the SPA and SPA-based simulation methods for structural systems with aleatory uncertainty exclusively, we extend their works to propose a novel and efficient reliability method for structural systems with interval parameters. In the proposed method, probability distributions with interval parameters are modeled using p-box variables. In order to avoid considering families of distributions whose parameters are within the intervals and reduce computational burden, the simulation and SPA methods are combined to calculate the lower and upper bounds of probability of failure. Furthermore, when both p-box and random variables exist in the system simultaneously, the probability of failure is an interval rather than a precisely determined value.

This article is organized as follows. Section "SPA" provides a brief introduction to the SPA method. Section "The proposed SPA-based reliability method" proposes the UUA method for structural systems with interval parameters. Three numerical examples are presented in section "Numerical examples." Finally, section "Conclusion" presents a brief discussion and conclusions to close the article.

SPA

The SPA technique was presented by Daniels[35] for approximating the distribution of a random variable. Generally, SPA can provide an accurate estimation of a cumulative density function (CDF) in the left tail region.[22,36,37] A brief introduction of SPA is given below.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a n-dimensional random vector, the cumulant generating function (CGF) of $Y_s = g_s(\mathbf{X})$ can be expressed as

$$K(Y_s, t) = \log\left[\int_{-\infty}^{\infty} e^{y_s t} f(y_s) dy_s\right] \tag{1}$$

where $f(y_s)$ denotes the probability density function (PDF) of Y_s.

Daniels[35] developed the SPA formula to $f(y_s)$ as

$$f_{Y_s}(y_s) = \left\{\frac{1}{2\pi \left.\frac{d^2[K(Y_s,t)]}{dt^2}\right|_{t_s^*}}\right\}^{\frac{1}{2}} e^{\left[K\left(Y_s, t_s^*\right) - t_s^* y_s\right]} \tag{2}$$

where $d^2[K(\bullet)]/dt^2$ denotes the second derivative of the CGF with respect to t, and t_s^* is the saddlepoint that is the solution of the equation[13]

$$\frac{d[K(Y_s, t)]}{dt} = 0 \tag{3}$$

where $d[K(Y_s, t)]/dt$ is the first derivative of the CGF with respect to t.

Using the approximation models proposed by Lugannani and Rice,[38] the probability of failure P_f can be expressed as

$$P_f = P_r[g_s(\mathbf{X}) < 0] = \Phi(w) + \phi(w)\left(\frac{1}{w} - \frac{1}{v}\right) \tag{4}$$

where $\Phi(\bullet)$ and $\phi(\bullet)$ are the CDF and the PDF of the standard normal random variable, respectively; w and v are defined as $w = sign(t)\{2[0 - K(Y_s, t)]\}^{1/2}\big|_{t_s^*}$, $v = t\{d^2[K(Y_s, t)]/dt^2\}^{1/2}\big|_{t_s^*}$, where

$$sign(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$ stands for the sign function

From equations (1), (3), and (4), we know that the key application of SPA is to determine the CGF of Y when the distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is given. It should be noted that the direct evaluation of the integration in equation (1) is extremely difficult due to the unknown PDF of $f(y_s)$ and the complicated performance function $Y_s = g_s(\mathbf{X})$. In order to obtain the approximated CGF of $Y_s = g_s(\mathbf{X})$, the power expansion method is adopted which is given by[12]

$$K(Y_s, t) \approx k_1 t + \frac{k_2 t^2}{2!} + \frac{k_3 t^3}{3!} + \frac{k_4 t^4}{4!} \tag{5}$$

where k_i is the ith cumulant of Y_s.

The values of the first four cumulants $k_1, k_2, \ldots, k_4$ can be determined by using MCS which are given by[37]

$$\begin{cases} k_1 = \dfrac{S_1}{N} \\ k_2 = \dfrac{NS_2 - S_1^2}{N(N-1)} \\ k_3 = \dfrac{2S_1^3 - 3NS_1S_2 + N^2S_3}{N(N-1)(N-2)} \\ k_4 = \dfrac{-6S_1^4 + 12NS_1^2S_2 - 3N(N-1)S_2^2 - 4N(N+1)S_1S_3 + N^2(N+1)S_4}{N(N-1)(N-2)(N-3)} \end{cases} \tag{6}$$

where $S_i = \sum_{j=1}^{N} (y_s^j)^i (i = 1, 2, 3, 4)$, N is the sample size, and $y_s^1, y_s^2, \ldots, y_s^N$ are sample values of Y_s.

SPA has many advantages since it not only yields extremely accurate probability estimations but also provides straightforward approximation to both CDF and PDF without performing any complicated integration.[13,33] SPA has widespread applications in statistics.[38–41]

The proposed SPA-based reliability method

General procedure

Consider k p-box variables $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k$ (random variables with interval distribution parameters) and $(n-k)$ random variables $X_{k+1}, X_{k+2}, \ldots, X_n$; these variables can be expressed as $\mathbf{V} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k, X_{k+1}, X_{k+2}, \ldots, X_n)$, supposed that $\mathbf{V}$ is the input of a system g. In this article, $g(\mathbf{V}) = 0$ and $Y = g(\mathbf{V})$ are, respectively, called the limit-state function and the performance function. In order to calculate the lower and upper bounds of the probability of failure, the proposed method includes (1) a simulation stage and (2) an analytical evaluation stage. In the simulation stage, random and p-box variables are sampled following their corresponding distributions. Then the elements are mapped through the performance function using an optimization approach in order to find the lower and upper bounds of performance function, the CGFs of the system under the best and the worst cases therefore can be determined by MCS. In the analytical evaluation stage, the saddlepoints for the best and worst cases are calculated, and the lower and upper bounds of the probability of failure of the system are determined. The procedure of the proposed method is shown in Figure 1, and the details of each step are presented in the following subsections.

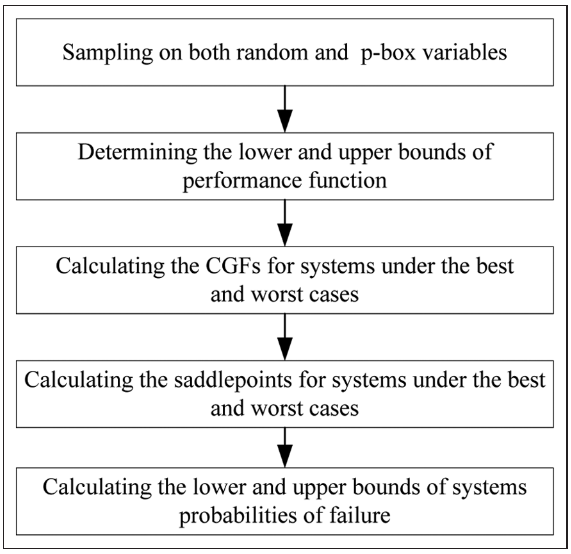

Figure 1. Procedure of the proposed method.
CGF: cumulant generating function.

Sampling on random and p-box variables

Consider a random variable X_i with its corresponding CDF F_{X_i}. The jth sample of X_i can be generated by using inverse transformation method as

$$x_i^j = F_{X_i}^{-1}\left(u_i^j\right) \tag{7}$$

where $F_{X_i}^{-1}$ is the inverse function of F_{X_i}, and u_i^j is a uniformly distributed number in the interval $[0, 1]$.

The N samples of $\mathbf{X}$ generated by MCS can be expressed as

$$\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right) \tag{8}$$

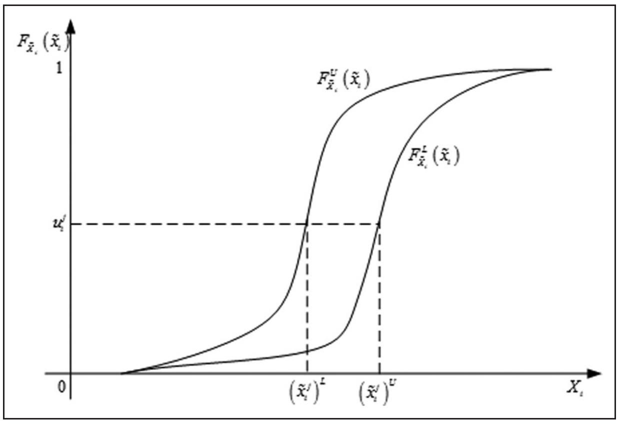

Figure 2. Sampling for p-box variables.

where $\mathbf{x}^j$ is the jth sample of $\mathbf{X}$, and $\mathbf{x}^j = (x^j_{k+1}, x^j_{k+2}, \ldots, x^j_n)$.

For a p-box variable $\tilde{X}_i$, the sampling procedure using inverse transformation method is shown in Figure 2. The lower and upper bounds of CDFs of $\tilde{X}_i$ are denoted by $F^L_{\tilde{X}_i}$ and $F^U_{\tilde{X}_i}$, respectively. For each u^j_i, the lower and upper bounds of the jth sample can be, respectively, given by

$$(\tilde{x}^j_i)^L = \left(F^U_{\tilde{X}_i}\right)^{-1}(u^j_i) \tag{9}$$

$$(\tilde{x}^j_i)^U = \left(F^L_{\tilde{X}_i}\right)^{-1}(u^j_i) \tag{10}$$

The N samples of $\tilde{\mathbf{X}}$ can be denoted as

$$(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2, \ldots, \tilde{\mathbf{x}}^N) \tag{11}$$

where $\tilde{\mathbf{x}}^j$ is the jth sample of $\tilde{\mathbf{X}}$, and $\tilde{\mathbf{x}}^j = \{[(\tilde{x}^j_1)^L, (\tilde{x}^j_1)^U], \ldots, [(\tilde{x}^j_k)^L, (\tilde{x}^j_k)^U]\}$. The mathematical justification of this sampling can be found in Alvarez.[42]

Since the Latin Hypercube sampling (LHS) method generally requires fewer samples than the direct MCS with the same accuracy and provides an efficient way for sampling the entire range of each variable, the samples generated using LHS is adopted in the article.[34]

After N samples $\mathbf{V}$ are obtained, those are mapped through g in order to find their corresponding images, which are intervals and samples of Y. The jth corresponding value of the performance function $Y = g(\tilde{\mathbf{X}}, \mathbf{X})$ is given by

$$\begin{aligned} y^j &= g(\tilde{\mathbf{x}}^j, \mathbf{x}^j) \\ &= g\left\{\left[(\tilde{x}^j_1)^L, (\tilde{x}^j_1)^U\right], \ldots, \left[(\tilde{x}^j_k)^L, (\tilde{x}^j_k)^U\right], x^j_{k+1}, \ldots, x^j_n\right\} \end{aligned} \tag{12}$$

where $\tilde{\mathbf{x}}^j$ and $\mathbf{x}^j$ are the jth sample for $\tilde{\mathbf{X}}$ and $\mathbf{X}$, respectively, $\tilde{\mathbf{x}}^j = \{[(\tilde{x}^j_1)^L, (\tilde{x}^j_1)^U], \ldots, [(\tilde{x}^j_k)^L, (\tilde{x}^j_k)^U]\}$, and $\mathbf{x}^j = (x^j_{k+1}, x^j_{k+2}, \ldots, x^j_n)$.

Generally, the more samples are generated, the more accurate the results will be. In this article, consider the balance between accuracy and efficiency, the sample size N is provided to be 500–1000.

Determining the lower and upper bounds of the performance function

Since $[(\tilde{x}^j_i)^L, (\tilde{x}^j_i)^U] (i = 1, 2, \ldots, k)$ are intervals, from equation (12) we know that the jth value of the performance function y^j is also an interval. The lower and upper bounds of y^j can be, respectively, computed by equation (13)

$$\begin{cases} (y^j)^L = \min g(\tilde{\mathbf{X}}, \mathbf{x}^j) \\ (y^j)^U = \max g(\tilde{\mathbf{X}}, \mathbf{x}^j) \\ s.t. \\ (\tilde{x}^j_1)^L \leqslant \tilde{X}_1 \leqslant (\tilde{x}^j_1)^U \\ (\tilde{x}^j_2)^L \leqslant \tilde{X}_2 \leqslant (\tilde{x}^j_2)^U \\ \quad \vdots \qquad \vdots \qquad \vdots \\ (\tilde{x}^j_k)^L \leqslant \tilde{X}_k \leqslant (\tilde{x}^j_k)^U \end{cases} \tag{13}$$

It should be noted that many constrained global optimization algorithms and software can be used to solve the above two optimization models easily. From equations (12) and (13), the lower and upper bounds of the corresponding N samples of the performance function can be, respectively, expressed as

$$\left\{(y^1)^L, (y^2)^L, \ldots, (y^N)^L\right\} \tag{14}$$

and

$$\left\{(y^1)^U, (y^2)^U, \ldots, (y^N)^U\right\} \tag{15}$$

Generally, the performance function usually cannot be expressed using an explicit function in engineering practices. In this case, the finite element analysis (FEA) method could be adopted, and the finite element equation for linear structural systems can be given by

$$\mathbf{K}_s\mathbf{U} = \mathbf{f} \tag{16}$$

where $\mathbf{K}_s$, $\mathbf{U}$, and $\mathbf{f}$ are the stiffness matrix, displacement, and load vectors, respectively.

When all system parameters are interval numbers, equation (16) can be expressed as

$$\mathbf{K}^I_s\mathbf{U}^I = \mathbf{f}^I \tag{17}$$

where $\mathbf{K}^I_s = [(\mathbf{K}_s)^L, (\mathbf{K}_s)^U]$, $\mathbf{U}^I = [(\mathbf{U})^L, (\mathbf{U})^U]$, and $\mathbf{f}^I = [(\mathbf{f})^L, (\mathbf{f})^U]$, respectively.

Perturbation technique or interval iterative algorithms[43,44] can be adopted to solve equation (17). Let $\tilde{\mathbf{x}}^j$ and $\mathbf{x}^j$ be, respectively, the jth samples for $\tilde{\mathbf{X}}$ and $\mathbf{X}$, from equation (17), the jth finite element equation can be expressed as

$$(\mathbf{k}^I_s)^j(\tilde{\mathbf{x}}^j, \mathbf{x}^j)(\mathbf{u}^I)^j = (\mathbf{f}^I)^j \tag{18}$$

where

$$(\mathbf{k}_s^I)^j = \begin{bmatrix} (k_{s11}^I)^j, (k_{s12}^I)^j, \ldots, (k_{1m}^I)^j \\ (k_{s21}^I)^j, (k_{s22}^I)^j, \ldots, (k_{2m}^I)^j \\ \vdots \quad \vdots \quad \vdots \\ (k_{sm1}^I)^j, (k_{sm2}^I)^j, \ldots, (k_{mm}^I)^j \end{bmatrix} \tag{19}$$

and

$$(\mathbf{f}^I)^j = \left\{(f_1^I)^j, (f_2^I)^j, (f_3^I)^j, \ldots, (f_m^I)^j\right\}^T \tag{20}$$

respectively.

Since $\mathbf{x}^j$ is the jth sample for random vector $\mathbf{X}$, it should be noted that some elements in the matrix $\mathbf{k}_s^I$ are precisely determined rather than interval numbers, that is, $k_{sij}^I = k_{sij}$. According to equation (18), the jth interval displacement vector can be expressed as

$$\begin{aligned}(\mathbf{u}^I)^j &= \left\{(u_1^I)^j, (u_2^I)^j, \ldots, (u_m^I)^j\right\}^T \\ &= \left\{\left[(u_1^L)^j, (u_1^U)^j\right], \ldots, \left[(u_m^L)^j, (u_m^U)^j\right]\right\}^T\end{aligned} \tag{21}$$

For illustration purposes, the performance function can be expressed as

$$Y = U_r - a \tag{22}$$

where U_r is the displacement of the rth node, and a is a constant.

According to equation (22), the lower and upper bounds of the corresponding N samples of the performance function based on the FEA can be, respectively, expressed as

$$\left\{(y^1)^L, (y^2)^L, \ldots, (y^N)^L\right\} \tag{23}$$

and

$$\left\{(y^1)^U, (y^2)^U, \ldots, (y^N)^U\right\} \tag{24}$$

where $(y^i)^L = (u_r^i)^L - a$, and $(y^i)^U = (u_r^i)^U - a (i = 1, 2, \ldots, N)$.

Approximating the CGF for systems under the best and the worst cases

When p-box variables exist in a system, there is a family of CGFs for the system, and the probability of failure is an interval rather than a number. In order to reduce computational burden and avoid enumerating all distributions whose parameters are within the intervals, the lower and upper bounds of the probability of failure are calculated. In this situation, the system under both the best and the worst cases should be considered separately, and the corresponding best case CGF and the worst case CGF of the system can be determined. Let the number of samples be denoted by N; from equations (5), (6), (14), and (15), the best case CGF and the worst case CGF can be given by

$$K^{best}(Y, t) = k_1^{best} t + \frac{k_2^{best} t^2}{2!} + \frac{k_3^{best} t^3}{3!} + \frac{k_4^{best} t^4}{4!} \tag{25}$$

and

$$K^{worst}(Y, t) = k_1^{worst} t + \frac{k_2^{worst} t^2}{2!} + \frac{k_3^{worst} t^3}{3!} + \frac{k_4^{worst} t^4}{4!} \tag{26}$$

respectively, where

$$\begin{cases} k_1^{best} = \dfrac{S_1^{best}}{N} \\ k_2^{best} = \dfrac{N S_2^{best} - \left(S_1^{best}\right)^2}{N(N-1)} \\ k_3^{best} = \dfrac{2\left(S_1^{best}\right)^3 - 3N\left(S_1^{best}\right)\left(S_2^{best}\right) + N^2\left(S_3^{best}\right)}{N(N-1)(N-2)} \\ k_4^{best} = \dfrac{\left\{-6\left(S_1^{best}\right)^4 + 12N\left(S_1^{best}\right)^2\left(S_2^{best}\right) - 3N(N-1)\left(S_2^{best}\right)^2 - 4N(N+1)\left(S_1^{best}\right)\left(S_3^{best}\right) + N^2(N+1)\left(S_4^{best}\right)\right\}}{N(N-1)(N-2)(N-3)} \end{cases} \tag{27}$$

$$S_i^{best} = \sum_{j=1}^{N} \left[(y^j)^U\right]^i (i = 1, 2, 3, 4) \tag{28}$$

$$\begin{cases} k_1^{worst} = \dfrac{S_1^{worst}}{N} \\ k_2^{worst} = \dfrac{N S_2^{worst} - \left(S_1^{worst}\right)^2}{N(N-1)} \\ k_3^{worst} = \dfrac{2\left(S_1^{worst}\right)^3 - 3N\left(S_1^{worst}\right)\left(S_2^{worst}\right) + N^2\left(S_3^{worst}\right)}{N(N-1)(N-2)} \\ k_4^{worst} = \dfrac{\left\{-6\left(S_1^{worst}\right)^4 + 12N\left(S_1^{worst}\right)^2\left(S_2^{worst}\right) - 3N(N-1)\left(S_2^{worst}\right)^2 - 4N(N+1)\left(S_1^{worst}\right)\left(S_3^{worst}\right) + N^2(N+1)\left(S_4^{worst}\right)\right\}}{N(N-1)(N-2)(N-3)} \end{cases} \tag{29}$$

Table 1. Methods for selecting t_s^{best}.

Case	t_s^L, t_s^U
$k_4^{best} < 0$	$t_s^L = \dfrac{k_3^{best} - \sqrt{(k_3^{best})^2 - 2k_2^{best}k_4^{best}}}{-k_4^{best}}$ $t_s^U = \dfrac{k_3^{best} + \sqrt{(k_3^{best})^2 - 2k_2^{best}k_4^{best}}}{-k_4^{best}}$
$k_4^{best} \geqslant 0$ and $(k_3^{best})^2 - 2k_2^{best}k_4^{best} \leqslant 0$	$t_s^L = \Re$ $t_s^U = \Re$
$k_4^{best} \geqslant 0$ and $(k_3^{best})^2 - 2k_2^{best}k_4^{best} > 0$ and $k_3^{best} > 0$	$t_s^L = \dfrac{-k_3^{best} + \sqrt{(k_3^{best})^2 - 2k_2^{best}k_4^{best}}}{(k_4^{best})}$ $t_s^U = \infty$
$k_4^{best} \geqslant 0$ and $(k_3^{best})^2 - 2k_2^{best}k_4^{best} > 0$ and $k_3^{best} < 0$	$t_\alpha^L = -\infty$ $t_s^U = \dfrac{-k_3^{best} - \sqrt{(k_3^{best})^2 - 2k_2^{best}k_4^{best}}}{k_4^{best}}$

and

$$S_i^{worst} = \sum_{j=1}^{N} \left[\left(y^j\right)^L \right]^i (i = 1, 2, 3, 4) \tag{30}$$

respectively.

Calculating saddlepoints for the best and worst cases

According to equation (3), and CGFs for the best and worst cases in equations (25) and (26), the corresponding saddlepoints t_s^{best} and t_s^{worst} for the best and worst cases can be determined by solving the following equations

$$\frac{d\left[K^{best}(Y,t)\right]}{dt} = k_1^{best} + k_2^{best}t + \frac{k_3^{best}t^2}{2} + \frac{k_4^{best}t^3}{6} = 0 \tag{31}$$

and

$$\frac{d[K^{worst}(Y,t)]}{dt} = k_1^{worst} + k_2^{worst}t + \frac{k_3^{worst}t^2}{2} + \frac{k_4^{worst}t^3}{6} = 0 \tag{32}$$

respectively.

It should be noted that there are three solutions for each of equations (31) and (32). Yuen et al.[37] described how to select a proper saddlepoint in details for system with precisely determined distribution parameters. However, when p-box variables exist in a system, the number of saddlepoints is infinite, and the probability of failure is an interval. In order to reduce computational burden and calculate the bounds of probability of failure, the corresponding saddlepoints t_s^{best} and t_s^{worst} for the best and the worst cases are required. Based on the method for selecting saddlepoints presented by Yuen et al.,[37] the methods for selecting the t_s^{best} and t_s^{worst} are, respectively, given in Tables 1 and 2; $\Re$ denotes real number, and $t_s^{best}, t_s^{worst} \in [t_s^L, t_s^U]$.

Calculating the lower and upper bounds of the probability of failure

From equation (4), once saddlepoints for the best and worst cases are, respectively, determined, the use of SPA becomes straightforward. According to equations (4), (31), and (32), the lower and upper bounds of system probability of failure can be calculated by

$$P_f^L = \Phi\left(w^{best}\right) + \phi\left(w^{best}\right)\left(\frac{1}{w^{best}} - \frac{1}{v^{best}}\right) \tag{33}$$

and

$$P_f^U = \Phi(w^{worst}) + \phi(w^{worst})\left(\frac{1}{w^{worst}} - \frac{1}{v^{worst}}\right) \tag{34}$$

respectively, where

$$w^{best} = sign\left(t_s^{best}\right)\left\{2\left[-K^{best}\left(Y, t_s^{best}\right)\right]\right\}^{0.5} \tag{35}$$

$$v^{best} = t_s^{best}\left\{\frac{d^2\left[K^{best}(Y,t)\right]}{dt^2}\Bigg|_{t_s^{best}}\right\}^{0.5} \tag{36}$$

$$w^{worst} = sign\left(t_s^{worst}\right)\left\{2\left[-K^{worst}\left(Y, t_s^{worst}\right)\right]\right\}^{0.5} \tag{37}$$

$$v^{worst} = t_s^{worst}\left\{\frac{d^2[K^{worst}(Y,t)]}{dt^2}\Bigg|_{t_s^{worst}}\right\}^{0.5} \tag{38}$$

$$\frac{d^2\left[K^{best}(Y,t)\right]}{dt^2}\Bigg|_{t_s^{best}} = k_2^{best} + k_3^{best}t_s^{best} + \frac{k_4^{best}\left(t_s^{best}\right)^2}{2} \tag{39}$$

and

Table 2. Methods for selecting t_s^{worst}.

Cases	t_s^L, t_s^U
$k_4^{worst} < 0$	$t_s^L = \dfrac{k_3^{worst} - \sqrt{(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst}}}{(-k_4^{worst})}$ $t_s^U = \dfrac{k_3^{worst} + \sqrt{(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst}}}{(-k_4^{worst})}$
$k_4^{worst} \geqslant 0$ and	$t_s^L = \Re$
$(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst} \leqslant 0$	$t_s^U = \Re$
$k_4^{worst} \geqslant 0$ and	$t_s^L = \dfrac{-k_3^{worst} + \sqrt{(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst}}}{(k_4^{worst})}$
$(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst} > 0$ and $k_3^{worst} > 0$	$t_s^U = \infty$
$k_4^{worst} \geqslant 0$ and	$t_s^L = -\infty$
$(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst} > 0$ and $k_3^{worst} < 0$	$t_s^U = \dfrac{-k_3^{worst} - \sqrt{(k_3^{worst})^2 - 2k_2^{worst}k_4^{worst}}}{k_4^{worst}}$

$$\left.\frac{d^2[K^{worst}(Y,t)]}{dt^2}\right|_{t_s^{worst}} = k_2^{worst} + k_3^{worst}t_s^{worst} + \frac{k_4^{worst}\left(t_s^{worst}\right)^2}{2} \tag{40}$$

respectively.

Numerical examples

In this section, three examples are used to demonstrate the accuracy as well as the efficiency of the proposed method. The first example is an aero-turbine with the nonlinear limit-state function, the second example is a beam with the highly nonlinear limit-state function, and the third example is a truss system with an implicit limit-state function. A comparative study is also provided against IMCS.[31] The results using IMCS with large samples are used as the reference when the comparison of accuracy and efficiency is made.

Example 1—an aero-turbine

An aero-turbine, shown in Figure 3, the performance function of fracture criterion is determined by[43]

$$Y = \sigma_s S - \frac{C\omega^2}{2\pi} - 2\rho\omega^2 J$$

where σ_s is the ultimate strength (measured in MPa), S is the cross-sectional area (in m^2), C is a constant, ω is the rotating speed (in rad/s), ρ is the mass density (in kg/m^3), and J is the cross-sectional moment of inertia (in m^4). Details of both random and p-box variables are given in Table 3.

The CDFs of σ_s and C are shown in Figures 4 and 5, respectively.

In this example, traditional reliability methods, such as FORM, SORM, and MCS, cannot be used to calculate the probability of failure directly because p-box variables exist in the system. The calculation of the probability of failure needs to consider families of distributions whose parameters are within the intervals. Generally, IMCS with large sample sizes can be used for reliability analysis of structural systems with p-box variables. However, the evaluation of the lower and upper bounds of the performance function under each sample is an iterative optimization process, which means that the computational burden by using IMCS for estimating the lower and upper bounds of the probability of failure is extremely high. The lower and upper bounds of the system probability of failure $P_r[Y \leqslant y]$ calculated by IMCS and the proposed method are given in Table 4 and Figure 6, respectively. It shows that the CDFs using IMCS and the proposed method are almost identical to each other over the entire distribution range. This example shows that the computational efficiency of the proposed method is higher than IMCS since it only requires 600 samples while the

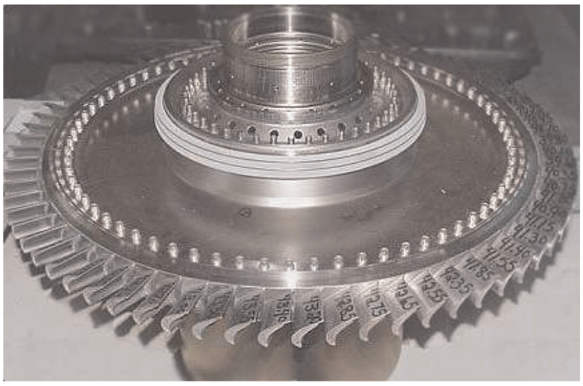

Figure 3. Aero-turbine.

Table 3. Details of both random and p-box variables.

Variable	Mean value	Standard deviation	Distribution type
σ_s	$[1100, 1130]$	155.0360	Normal
S	6.2048×10^{-3}	1.2161×10^{-4}	Normal
C	5.6682	$[0.0220, 0.0250]$	Normal
ω	1293.2890	50.6969	Normal
ρ	8240	484.5120	Normal
J	1.214657×10^{-4}	7.1422×10^{-6}	Normal

Table 4. Probabilities of failure calculated by IMCS and the proposed method.

y		-1×10^6	0	1×10^6	Sample sizes
IMCS	P_f^L	0.0018	0.1420	0.4459	100,000
	P_f^U	0.0035	0.1895	0.5149	
Proposed method	P_f^L	0.0020	0.1430	0.4417	600
	P_f^U	0.0032	0.1852	0.5122	

IMCS: interval Monte Carlo simulation.

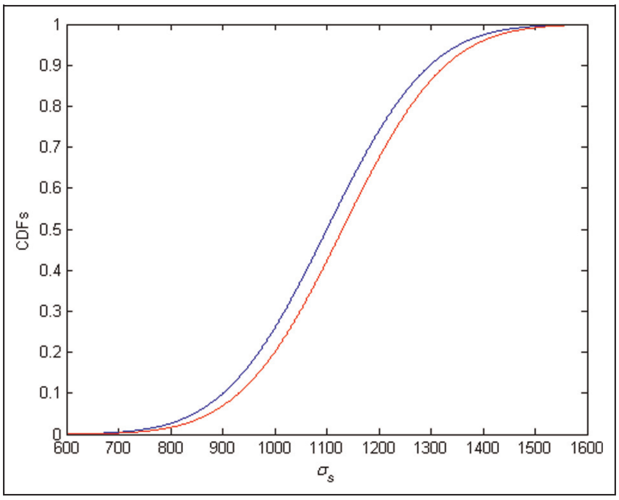

Figure 4. The interval CDFs of σ_s.
CDF: cumulative density function.

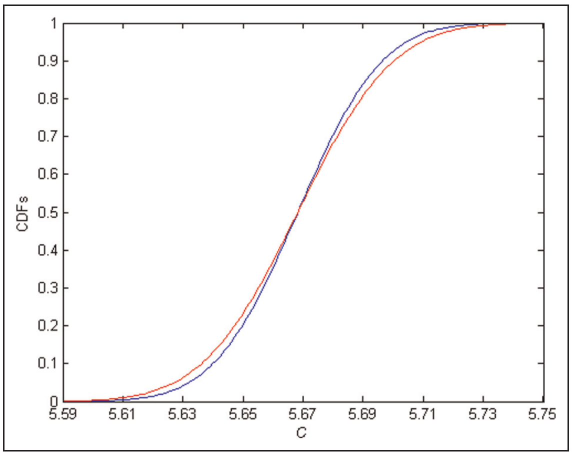

Figure 5. The interval CDFs of C.
CDF: cumulative density function.

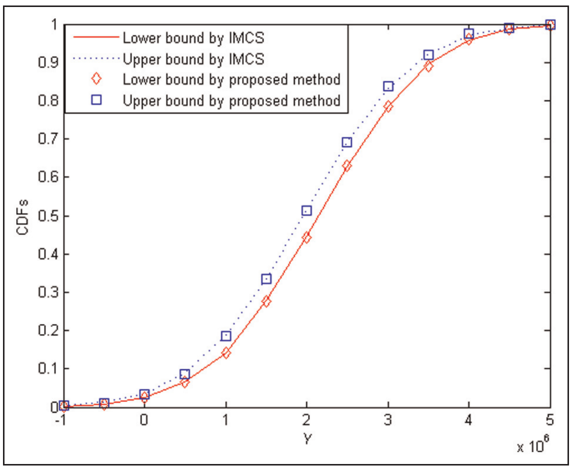

Figure 6. The CDFs of Y calculated by IMCS and the proposed method.
IMCS: interval Monte Carlo simulation; CDF: cumulative density function.

sample size of IMCS is 100,000 while they achieve almost the same accuracy.

Example 2—a beam

A beam, as shown in Figure 7, is adopted to demonstrate the efficiency and accuracy of the proposed method. The performance function is given by[12]

$$Y = f(P, L, a, S, d, b_f, t_w, t_f) = \sigma_{\max} - S$$

where

$$\sigma_{\max} = \frac{Pa(L-a)d}{2LI}$$

and

$$I = \frac{b_f d^3 - (b_f - t_w)(d - 2t_f)^3}{12}$$

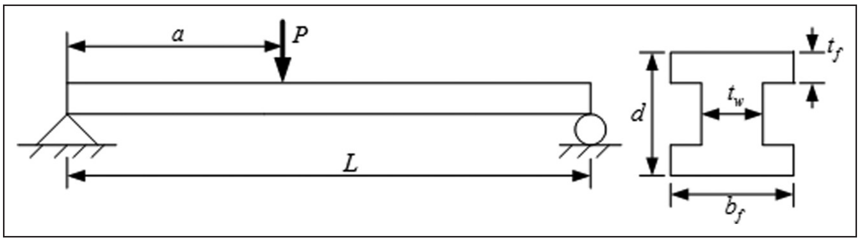

Figure 7. A beam.

Table 5. Details of both random and p-box variables.

Variable	Parameter 1	Parameter 2	Distribution type
P	[6070, 6100]	200	Normal
L	120	6	Normal
a	72	6	Normal
S	170,000	4760	Normal
d	2.3	1/24	Normal
t_w	0.1	0.22	Uniform
t_f	0.2	0.32	Uniform
b_f	[2.25, 2.35]	1/24	Normal

The details of both random and p-box variables are given in Table 5.

In Table 5, parameter 1 and parameter 2 denote the mean value and the SD for the normal distribution, respectively. While for the uniform distribution, parameter 1 and parameter 2 denote the lower and the upper bounds, respectively.

In this example, the sample sizes of the proposed method and IMCS are 500 and 100,000, respectively. The CDFs of P and b_f are shown in Figures 8 and 9, respectively.

The lower and upper bounds of the system probability of failure $P_r[Y \leqslant y]$ calculated by IMCS and the proposed method are given in Table 6 and Figure 10, respectively. It is noted that the CDFs are almost identical to each other over the entire distribution range. The main advantages of the proposed method are that it not only yields very accurate probability estimation but also can be used to calculate the lower and upper bounds of the probability of failure directly by finding saddlepoints for the best and worst cases without complicated integrations. Furthermore, the proposed method is robust because it can be used for reliability analysis of structural systems with parameter uncertainties. This example also shows that the computational burden of the proposed method is significantly lower than that of the IMCS while maintaining almost the same accuracy.

Example 3—truss system

A truss system with six members,[44] shown in Figure 11, is used to demonstrate the effectiveness of the proposed method.

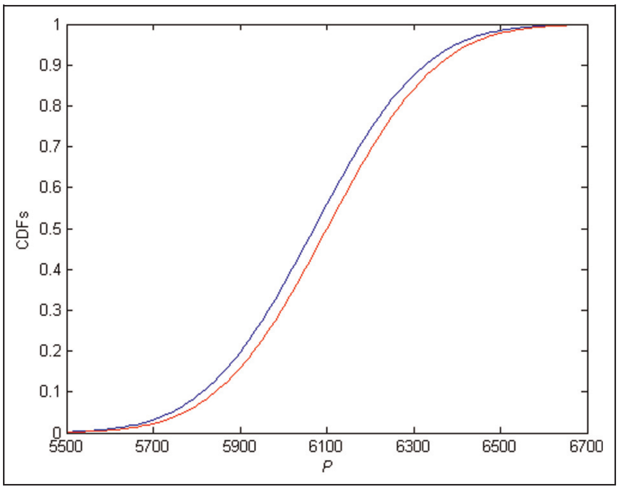

Figure 8. The interval CDFs of P.
CDF: cumulative density function.

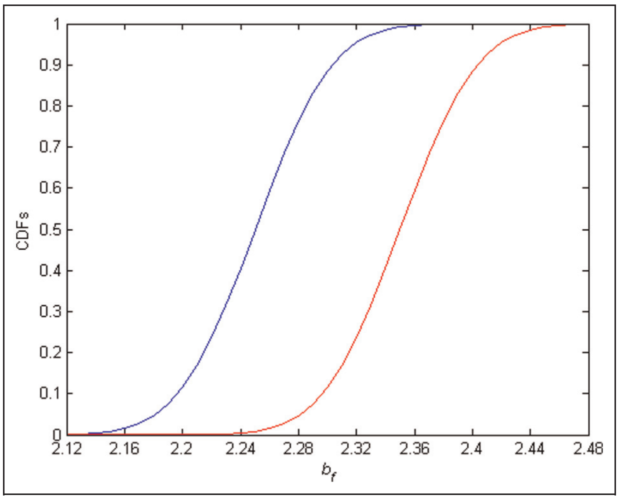

Figure 9. The interval CDFs of b_f.
CDF: cumulative density function.

The performance function of the truss system is defined as

$$Y = 1.09 - U_{3x}$$

where U_{3x} denotes the displacement of node 3 in x-axis, and it is measured in millimeters.

Table 6. Probabilities of failure calculated by IMCS and the proposed method.

y		-2×10^4	0	2×10^4	Sample sizes
IMCS	P_f^L	0.4189	0.7553	0.9396	100,000
	P_f^U	0.5472	0.8478	0.9752	
Proposed method	P_f^L	0.4110	0.7539	0.9367	500
	P_f^U	0.5438	0.8489	0.9769	

IMCS: interval Monte Carlo simulation.

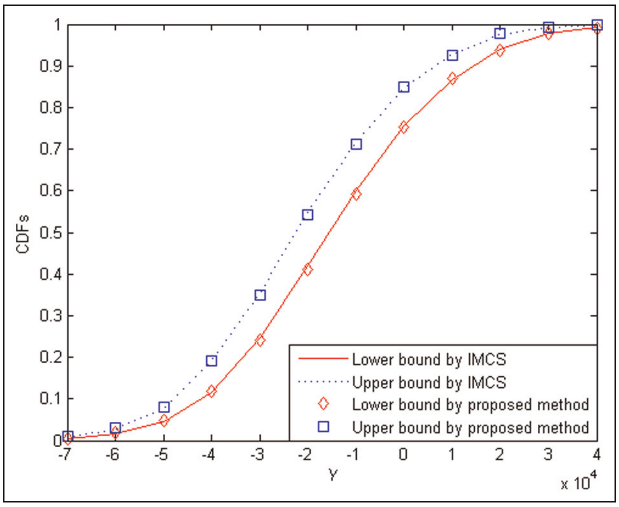

Figure 10. The CDFs of Y calculated by IMCS and proposed method.
CDF: cumulative density function; IMCS: interval Monte Carlo simulation.

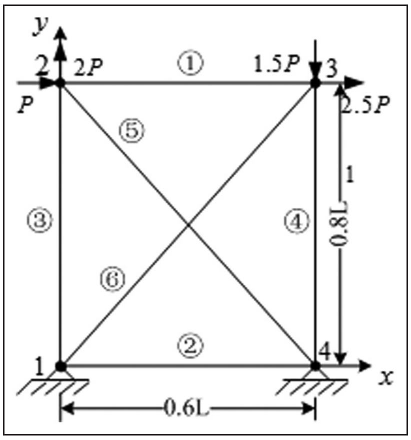

Figure 11. Truss system with six members.

$\mathbf{f} = (P, 2P, 2.5P, -2.5P)^T$ is the applied load vector (measured in kN), $E = 2.1 \times 10^8$ is the elastic modulus (in kN/m^2), $L = 1.0$ is the length (in m), and $A_i(i = 1, 2, \ldots, 6)$ are cross-sectional areas (in m^2). The details of both random and p-box variables are given in Table 7.

In this example, we consider the case that the performance function is an implicit function. In order to calculate the U_{3x}, FEA is used, and the lower and upper bounds of the system's probability of failure calculated

Table 7. Details of both random and p-box variables.

Variable	Mean value	Standard deviation	Distribution type
P	$[20, 20.5]$	2	Normal
$A_1 \sim A_4$	1.0×10^{-3}	1.0×10^{-4}	Normal
A_5, A_6	$[1.0\times10^{-3}, 1.1\times10^{-3}]$	1.0×10^{-4}	Normal

Table 8. Probabilities of failure calculated by IMCS and the proposed method.

Probability of failure	Proposed method	IMCS
P_f^L	0.0015	0.0013
P_f^U	0.7392	0.7301
Sample sizes	600	15,000

IMCS: interval Monte Carlo simulation.

by using the proposed method and IMCS are given in Table 8.

Since FEA is time costly, the results calculated by the FEA-based IMCS method with 15,000 samples are used as reference results for accuracy and efficiency comparison. This example shows that the proposed method is also applicable to the situation where the limit-state function is an implicit function. Furthermore, the example also shows that a small change in the distribution parameters may lead to a large change in the reliability results. This means that the reliability analysis results for structural systems by using precisely determined distribution parameters may not be correct under the case of less data or imprecise information. This shows the risk of assuming constant values for the parameters of the distributions.

Conclusion

Based on MCS and SPA, a novel reliability analysis method is proposed for structural systems with uncertain distribution parameters. The proposed method uses a mixture of random and p-box variables rather than only random variables to model both types of uncertainties that exist in practical engineering widely. The main advantages of the proposed method are that it can yield both accurate probability estimation and the lower and upper bounds of the probability of failure directly without complicated integrations. The proposed method provides an approach to making the trade-off between accuracy and efficiency since it only requires small sample sizes. The results of three examples show that the computational efficiency of the proposed method is higher than IMCS because of small sample sizes while compared with IMCS. The numerical examples indicate that the proposed method is robust because it is also applicable to the situation where the limit-state function is an implicit function. Furthermore, the proposed method is an extended method for dealing with the reliability analysis under interval distribution parameters, and consequently has more wide application in engineering practices.

It should be noted that the lower and upper bounds of the probability of failure calculated by the proposed method are approximate solutions rather than exact solutions, and the error of results calculated using the proposed method for small failure probability problem may be larger. The more samples we use, the more accurate the results will be. Integration of the proposed method in the framework of RBDO problems under interval parameters will be considered in our future work.

Declaration of conflicting interests

The authors declare that there is no conflict of interest.

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