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Reliability of complex systems under dynamic conditions: A Bayesian multivariate degradation perspective

Weiwen Peng, Yan-Feng Li, Jinhua Mi, Le Yu, Hong-Zhong Huang *

Institute of Reliability Engineering, University of Electronic Science and Technology of China, No. 2006, Xiyuan Avenue, West Hi-Tech Zone, Chengdu, Sichuan 611731, China

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ABSTRACT

Degradation analysis is critical to reliability assessment and operational management of complex systems. Two types of assumptions are often adopted for degradation analysis: (1) single degradation indicator and (2) constant external factors. However, modern complex systems are generally characterized as multiple functional and suffered from multiple failure modes due to dynamic operating conditions. In this paper, Bayesian degradation analysis of complex systems with multiple degradation indicators under dynamic conditions is investigated. Three practical engineering-driven issues are addressed: (1) to model various combinations of degradation indicators, a generalized multivariate hybrid degradation process model is proposed, which subsumes both monotonic and non-monotonic degradation processes models as special cases, (2) to study effects of external factors, two types of dynamic covariates are incorporated jointly, which include both environmental conditions and operating profiles, and (3) to facilitate degradation based reliability analysis, a serial of Bayesian strategy is constructed, which covers parameter estimation, factor-related degradation prediction, and unit-specific remaining useful life assessment. Finally, degradation analysis of a type of heavy machine tools is presented to demonstrate the application and performance of the proposed method. A comparison of the proposed model with a traditional model is studied as well in the example.

1. Introduction

1.1. Motivation

Reliability prediction of complex systems has long been a critical issue within the field of reliability engineering [1,2]. Recently, reliability data of complex systems is evolving toward a big data situation [3]. By monitoring failure related performance indicators, reliability of complex systems can then be assessed and predicted through degradation analysis of these performance indicators [4]. Degradation based methods are increasingly introduced to lifetime analysis of complex system for facilitating reliability assessment [5–7], spare parts and preventive maintenance decision [8–10], and system health management [11,12]. Within these degradation-based methods, two general assumptions are adopted, which include the assumptions of (1) single degradation indicator and (2) constant external factors. However, modern complex systems are generally composed by multiple and mutually interactive subsystems and components [13,14], and are well characterized as multiple functional under dynamic environmental/operating conditions [15,16]. It is not uncommon for a modern complex system to possess multiple dependent performance indicators. In addition, these indicators can be significantly affected by external environmental conditions as well as operating profiles. A practical example that motivates this research on multivariate dependent degradation analysis of complex system under dynamic conditions is the reliability analysis of one type of CNC heavy machine tools.

The DL150 CNC heavy duty lathes (DL 150s) are serving as indispensable equipment in the industries of energy, transportation, aerospace, aviation and military. By summarizing operating and maintenance records, manufacturers of DL 150s have found that two types of gradually-evolving failures are critical to the reliability of these lathes: losing of machining accuracy, and accumulation of lubrication debris. Meanwhile, they also found that these types of failures vary from factories to factories due to differences of environmental conditions and loading profiles that the DL 150s endured. To achieve high availability and productivity, unit-specific condition monitoring and degradation analysis are carried out on the DL 150s by making machining accuracy and lubrication debris as performance indicators [17]. Methods for degradation analysis with multiple performance indicators under

* Corresponding author. Tel.: +86 28 6183 1252; fax: +86 28 6183 1252.
E-mail address: hzhuang@uestc.edu.cn (H.-Z. Huang).

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dynamic conditions are naturally become primary research initiatives. Moreover, degradation of machining accuracy and lubrication debris are founded to be correlated with each other. Their degradation processes are presented separately in a non-monotonic form and a monotonic form. To fully address these challenges, three engineering-driven problems are then tackled in this paper: (1) joint modeling of various types of dependent degradation indicators, (2) explicit study of environmental conditions' effects and operating profiles' influences, and (3) continual update of unit-specific reliability evaluation and factor-related degradation prediction.

1.2. Related works

Investigation on degradation-based reliability methods has received much attention. Both theoretical methods and practical approaches have been proposed. Up-to-date investigations on degradation models include the works by Guida et al. [18], Bae et al. [19], and Wang et al. [20] for stochastic process degradation models, the work by Bae et al. [21] for general path degradation models, and the researches by Kharoufeh and Cox [22] and Kharoufeh et al. [23] on Markov process degradation model and so on. Recently, Ye and Xie [24] present a review of degradation models, and they emphasize that more efforts should be focused on new types of degradation models such as inverse Gaussian process degradation models [25–27]. They also put forward that most published literatures on degradation modeling are one-dimensional, and the research on multivariate degradation modeling is far from adequate.

A few attempts have made to extend one-dimensional degradation models to the multi-dimensional domain. Wang and Coit [28] introduced the problem of degradation modeling for complex system with more than one degradation indicator and proposed a multivariate degradation model based on a multivariate normal distribution. Pan and Balakrishnan [29] introduced a bivariate degradation process model based on a bivariate Birnbaum–Saunders distribution and gamma processes. Motivated by reliability analysis of light-emitting diode, Sari et al. [30] introduced a bivariate degradation model by combining generalized linear degradation path models with copula function. Furthermore, by utilizing the flexibility of copula function to construct dependency among stochastic degradation models, Pan et al. [31] and Wang et al. [32] introduced bivariate degradation models based on Wiener process, gamma process and copula functions. Nevertheless, all these models are built on the assumption of "constant external factors", and the influences of environmental conditions and operating profiles are not considered in these models. These models can hardly be applied to fulfill the needs introduced by reliability analysis of complex systems under dynamic conditions, such as the challenges of DL 150s introduced above. Moreover, these degradation models are either based on Wiener process or gamma process, where degradation processes are characterized either in a monotonic or a non-monotonic way. In reality the degradation processes of complex systems may evolve in a hybrid way. Such as the DL 150s presented above, a combination of monotonic and non-monotonic degradation processes is desirable. Accordingly, an in-depth analysis on multivariate degradation is necessary to address the influences of dynamic environmental and operating conditions, and to characterize the time-varying nature of multiple hybrid degradations.

1.3. Overview

Based on the motivation and literature review presented above, we are motivated to explore the research topic by making three contributions as following.

1. A generalized multivariate hybrid degradation model is developed to address the modeling of complex systems exhibiting monotonic and non-monotonic degradation processes.
2. Both dynamic covariates and random effects are incorporated in the multivariate hybrid degradation model to characterize degradation processes under dynamic conditions.
3. A Bayesian inference framework is presented to perform parameter estimation, reliability assessment, and degradation prediction. Unit-specific reliability assessment and factor-related degradation prediction are investigated within the Bayesian framework.

An illustrative example is drawn from the engineering practice of DL 150s introduced in the motivation session. It is used to demonstrate Bayesian reliability analysis of complex system with the proposed multivariate hybrid degradation model. A comparison between the proposed model and existing models in literature is implemented as well. The proposed model is demonstrated a hybrid modeling technique, and being capable of incorporating different types of stochastic processes. In addition, the Bayesian framework presented in this paper can overcome the difficulty in parameter estimation, which is challenged by combing different types of marginal stochastic processes. It can also facilitate degradation inference of complex system, where unit-to-unit variability and effects of external factors are incorporated. The degradation inference is then carried out in a unit-specific and factor-related way. This is of critical importance to degradation-based optimal decision-making when systems possess multiple failure modes and their failure mechanisms are sensitive to environmental conditions and operating profiles.

The remainder of this paper is organized as follows. Section 2 introduces a generalized model for multivariate hybrid degradation processes with random effects and dynamic covariates. Section 3 presents methods for Bayesian estimation of model parameters and Bayesian inference of unit-specific remaining useful life and factor-related degradations. A numerical example is then presented in Section 4 to demonstrate the performance of the proposed method. Section 5 concludes this paper with brief discussion of future research.

2. Multivariate degradation models for complex systems

The model developed in this paper aims to characterize various combinations of dependent degradation indicators, and to study the influences of external factors on these degradation indicators for reliability analysis. This requires: (1) modeling degradation processes with different characteristics, (2) incorporating variables (covariates) of external factors into degradation models, and (3) characterizing dependence among degradation indicators for multivariate hybrid degradation model construction. In this section, these three aspects are handled progressively and ended up with a generalized multivariate hybrid degradation models.

2.1. Baseline degradation model with dynamic covariates and random effects

Suppose there is a complex system with n degradation processes, and let $\{Y_i(t), t \geq 0\}, i = 1, \ldots, n$ with $Y_i(0) = 0$ denote the ith degradation process of this complex system. We first introduce the baseline degradation model for each degradation process, and the modeling of dependence among these degradation processes is presented in the next section based on these baseline degradation model and copula function. To characterize different types of marginal degradation processes, we use a basic Wiener process model to describe degradation processes with non-monotonic paths as follows:

$$Y_i(t) = \eta_i(t) + \sigma_i B(t) \tag{1}$$

where $\eta_i(t)$ is a degradation mean function charactering approximated mean of degradation, σ_i is a volatility parameter reflecting the variability of degradation, and $B(\bullet)$ is a standard Brownian motion representing temporal uncertainty.

Under the Wiener process, degradation increments of a degradation process are independent and normally distributed as $\Delta Y_i \sim N(\Delta\eta_i(t), \sigma_i^2\Delta t)$ with $\Delta\eta_i(t) = \eta_i(t) - \eta_i(t-\Delta t)$. Let D_i denote the degradation threshold for the ith degradation process $Y_i(t)$. The first passage time of this degradation process is then defined as $T_i = \inf\{t : Y(t) \geq D_i\}$. It follows an IG distribution as $T_i \sim \mathrm{IG}\left(D_i/\eta_i, D_i^2/\sigma^2\right)$, where the $\mathrm{IG}(a,b)$ has a probability density function as follows:

$$f_{\mathrm{IG}}(y; a, b) = \left(\frac{b}{2\pi y^3}\right)^{1/2} \exp\left[-\frac{b(y-a)^2}{2a^2 y}\right], \quad y > 0 \tag{2}$$

The Wiener process model has been widely used for degradation modeling [33]. Both random effect and dynamic covariates can be easily incorporated into this model to capture unit-to-unit variability and external influences. Variations of Wiener process model have also been introduced [34,35]. These models serve as a useful basis for modeling and analysis of degradation processes subject to non-monotonic paths.

For the modeling of degradation processes with monotonic paths, both the gamma process model and the inverse Gaussian process model can be chosen as the baseline model according to characteristics of the degradation process and statistical model choosing [24]. In this paper, to simplify the description, a basic inverse Gaussian (IG) process model is chosen to demonstrate the proposed method, which is given as follows:

$$Y_i(t) \sim \mathrm{IG}\left(\Lambda_i(t), \lambda_i \Lambda_i^2(t)\right) \tag{3}$$

where $\Lambda_i(t)$ is a nonnegative and monotonically increasing function, it is a mean function of the degradation process $Y_i(t)$. λ_i is a volatility parameter, and it captures the variability of the degradation process.

Under the IG process model, degradation increments of a degradation process are independently distributed as IG distributions as follows:

$$\Delta Y_i \sim \mathrm{IG}\left(\Delta\Lambda_i(t), \lambda_i(\Delta\Lambda_i(t))^2\right), \quad \Delta\Lambda_i(t) = \Lambda_i(t) - \Lambda_i(t-\Delta t) \tag{4}$$

Given the degradation threshold D_i, the cumulative distribution function (CDF) of the first passage time T_i under the IG process model is given as follows:

$$F(t) = \Pr(T_i \leq t) = \Pr\left(Y_i(t) \geq D \mid \Lambda_i(t), \lambda_i\Lambda_i^2(t)\right)$$
$$= \Phi\left[\sqrt{\frac{\lambda_i}{D_i}}(\Lambda_i(t) - D_i)\right] - \exp(2\lambda_i\Lambda_i(t))\,\Phi\left[-\sqrt{\frac{\lambda_i}{D_i}}(\Lambda_i(t) + D_i)\right] \tag{5}$$

The IG process model is introduced recently for degradation modeling [25-27]. The incorporating of random effect and covariates into the IG process model has been studied by Ye and Chen [26] and Peng et al. [27]. These works laid a solid foundation for modeling and analysis of degradation processes with monotonic paths.

Based on the basic models for non-monotonic and monotonic degradation processes, two general degradation models with random effects and covariates are presented as follows:

$$Y_i(t) = \eta_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right) + \sigma_i B(t) \tag{6}$$

$$Y_i(t) \sim \mathrm{IG}\left(\Lambda_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right), \lambda_i\Lambda_i^2\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)\right) \tag{7}$$

where $\mathbf{X}^E$ and $\mathbf{X}^O$ separately denote variables (covariates) of environmental conditions and operating profiles. $\boldsymbol{\theta}_i^F$ and $\boldsymbol{\theta}_i^R$ separately denote parameters without random effects (fixed parameters) and parameters with random effect (random parameters). $\eta_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$ is the degradation mean function of Wiener process modified by covariates and random parameters. $\Lambda_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$ is the degradation mean function of the IG process modified by covariates and random parameters.

In the general models presented in Eqs. (6) and (7), the environmental conditions $\mathbf{X}^E$ can be actual ambient environments in which the system locates, such as ambient temperature, humidity, vibration, and others, which are assumed unit-dependent in this paper [16]. The operating profiles $\mathbf{X}^O$ represent the practical field missions for which the system endures and fulfills, such as operating speed, mission load, use rate, and so on, which are assumed mission-dependent in this paper [3]. The differentiation of these external factors in this paper is due to the consideration that environmental conditions and operating profiles may change, creating different influences on the degradation processes. Moreover, identification of dominant external factor is crucial for preventive maintenance and operation management of complex systems. These two groups of external factors are then modeled separately in this paper to present a more detailed investigation on the impacts of environmental and operating factors. It is worth mentioning that dynamic conditions for modern complex systems can be obtained through real-time monitoring and statistical reasoning using the methods by Kharoufeh and Cox [22] and Flory et al. [36].

The incorporation of these external factors $\mathbf{X}^E$ and $\mathbf{X}^O$ are fulfilled by modifying degradation rate parameter of the Wiener process model and by revising degradation mean function of the IG process model as presented above. This is originated from the consideration that external factors generally affect the failure mechanism of degradation processes as they can modify the rate of degradations. There are generally three types of modification including (1) the modification based on proportional hazard model [37], (2) the modification based on acceleration factor [38], and (3) the modification based on degradation–shock relationship [39]. In this paper, the incorporation of covariates through the acceleration factor is used. The same method has also been used and verified by Liao and Tian [16].

In addition, the parameters with random effect $\boldsymbol{\theta}_i^R$ are incorporated in the modified degradation mean function $\eta_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$ and $\Lambda_i\left(t; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$ to reflect unit-to-unit variability of systems. It is assumed that parameters $\boldsymbol{\theta}_i^R$ following specific probability distributions [40], such as normal distribution, lognormal distribution, and gamma distribution. In this paper, the gamma distribution is chosen to demonstrate the consideration of unit-to-unit variability. Parameters of these probability distributions are named as hyper-parameters and denoted as $\boldsymbol{\theta}_i^H$ in the following sections. The same methods has also been used and verified by Peng et al. [27].

2.2. Multivariate degradation model with heterogeneous degradation paths

In the following sections, these general models presented in Eqs. (6) and (7) are used to construct multivariate hybrid degradation models for complex system with n degradation indicators based on copula functions.

Let $\{Y_1(t), \ldots, Y_i(t), \ldots, Y_n(t)\},\ t \geq 0$ denote degradation processes of a complex system with n performance indicators. Each of the degradation processes is characterized by a general Wiener process model or an IG process model described above. Let $\Delta Y_{ij} = Y_i(t_j) - Y_i(t_{j-1})$, $j = 2, ..., m$ denote the degradation increments of the ith degradation process within the interval $[t_{j-1}, t_j]$. To characterize the dependence among these n degradation processes, we assume that degradation increments ΔY_{ij} of the n performance processes at the same time intervals $[t_{j-1}, t_j]$ are mutually dependent. Their dependence is characterized by a multivariate copula function as follows:

$$F(\Delta Y_{1j}, ..., \Delta Y_{ij}, ..., \Delta Y_{nj}) = \mathrm{C}\left(F_1(\Delta Y_{1j}), ..., F_i(\Delta Y_{ij}), ..., F_n(\Delta Y_{nj}); \boldsymbol{\theta}^C\right) \tag{8}$$

where $F(\Delta Y_{1j}, ..., \Delta Y_{ij}, ..., \Delta Y_{nj})$ is the joint CDF of degradation increments of the n performance indicators. $F_i(\Delta Y_{ij})$, $i = 1, ..., n$ is the CDF of degradation increment of the ith performance indicator, which is a marginal distribution of the joint CDF $F(\Delta Y_{1j}, ..., \Delta Y_{ij}, ..., \Delta Y_{nj})$. $\mathrm{C}\left(u_1, ..., u_i, ..., u_n; \boldsymbol{\theta}^C\right)$ is a n-dimensional multivariate copula with parameter $\boldsymbol{\theta}^C$.

According to Sklar' theorem [41,42], the dependence of n degradation increments given in Eq. (8) can be characterized separately from their corresponding marginal distribution functions $F_i(\Delta Y_{ij})$. It means that the characterization of the various degradation processes with different types of degradation models presented in Section 2.2.1 can be separated from the characterization of the dependence among these degradation processes presented here. Such separation makes the construction of multivariate degradation model with different marginal degradation models feasible. In addition, a n-dimensional multivariate copula used above is a multivariate distribution with uniformly distributed marginal distributions on $[0, 1]$. A pair of CDFs of the degradation increments at the same time interval is a pair of samples from the multivariate copula function. The selection of copula can then be based on the CDFs of the degradation increments through qualitative scatter plots [43] or quantitative model selection as presented in Huard et al. [44]. In general, the multivariate Gaussian copula [45], the multivariate t-copula [46], and the vine copula [47] are commonly used in various literature for multivariate modeling.

Based on the general degradation models and the copulas presented above, the degradation models for multivariate hybrid degradation process under dynamic conditions are constructed. For the multivariate degradation processes with both monotonic and non-monotonic degradations, the general Wiener process model, IG process model and the multivariate copulas are used. We further assume that there are h degradation processes that have non-monotonic degradation paths among the n degradation processes, and the remaining $n-h$ are monotonic. The multivariate degradation model for this situation is then presented as follows:

$$\begin{aligned}
&Y_i(t_j) = \sum_{l=2}^{j} \Delta Y_{il}, i = 1, ..., n,\ j = 2, ..., m,\\
&\Delta Y_{il} \sim N\left(\Delta\eta_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right), \sigma_i^2(t_l - t_{l-1})\right), \text{ for } i \in [1, h]^+,\\
&\Delta Y_{il} \sim \mathrm{IG}\left(\Delta\Lambda_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right), \lambda_i \Delta\Lambda_i^2\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)\right), \text{ for } i \in [h+1, n]^+,\\
&F(\Delta Y_{1l}, ..., \Delta Y_{il}, ..., \Delta Y_{nl}) = \mathrm{C}\left(F_1(\Delta Y_{1l}), ..., F_i(\Delta Y_{il}), ..., F_n(\Delta Y_{nl}); \boldsymbol{\theta}^C\right)
\end{aligned} \tag{9}$$

where $\Delta\eta_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right) = \eta_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right) - \eta_i\left(t_{l-1}; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$, $[\bullet, \bullet]^+$ is a integer interval, and $\Delta\Lambda_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right) = \Lambda_i\left(t_l; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right) - \Lambda_i\left(t_{l-1}; \mathbf{X}^E, \mathbf{X}^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R\right)$.

Within this model, the degradation processes $Y_i(t_j)$ are characterized through their degradation increments ΔY_{il} on observed intervals $[t_{l-1}, t_l]$. The marginal distributions of these degradation increments are modeled based on the general Wiener process model and IG process model separately presented in Eqs. (6) and (7). The dependence among degradation processes are characterized by the multivariate copula through the CDFs of degradation increments. It's worth noting that this model can also include the multivariate degradation model for purely monotonic or non-monotonic degradation processes as special situations. By separately making h equal to 0 or n, this model presented in Eq. (9) become a multivariate degradation model for complex systems with only monotonic or non-monotonic degradation processes.

The failure time T of a complex system with n degradation processes is defined as the time point that any of its n degradation processes $Y_1(t), ..., Y_i(t), ..., Y_n(t)$ crosses their respective degradation thresholds $D_1, ..., D_i, ..., D_n$ [28,31]. Based on the first passage time defined for the Weiner process and the IG process model, the CDF of the failure time T with the multivariate degradation processes characterized by Eq. (9) is obtained as follows:

$$F(t) = 1 - \Pr\left\{\sup_{s \le t} Y_1(s) < D_1, ..., \sup_{s \le t} Y_i(s) < D_i, ..., \sup_{s \le t} Y_n(s) < D_n\right\} \tag{10}$$

If degradation processes of a complex system are observed up to the time point t_j, the remaining useful life T_{RUL} of this system is then given as

$$T_{RUL} = \inf\left\{ s : \begin{array}{c} Y_1(t_j + s) \ge D_1 \\ \text{or} \\ \vdots \\ Y_i(t_j + s) \ge D_i \\ \text{or} \\ \vdots \\ Y_n(t_j + s) \ge D_n \end{array} \left| \begin{array}{c} \mathbf{y}_{1,1:j}, Y_1(t_j) < D_1 \\ \vdots \\ \mathbf{y}_{i,1:j}, Y_i(t_j) < D_i \\ \vdots \\ \mathbf{y}_{n,1:j}, Y_n(t_j) < D_n \end{array} \right.\right\} \tag{11}$$

where $\mathbf{y}_{i,1:j}$ denote the degradation observations of the ith degradation process $Y_i(t)$ up to the time point t_j.

3. Bayesian degradation analysis

A Bayesian method is presented in this section to facilitate the reliability analysis of complex system with the multivariate hybrid degradation model presented above. Specifically, a Bayesian parameter estimation method, a unit-specific RUL estimation method and a factor-related degradation prediction method is presented.

3.1. Parameter estimation

Suppose one type of complex system is observed with a sample size N. Let $\mathbf{y}_{ik} = \{y_i(t_{k1}), ..., y_i(t_{kj}), ..., y_i(t_{km_k})\}$ denote m_k sequential degradation observations of the ith degradation process for the kth sample, where $i = 1, ..., n$ and $k = 1, ..., N$. When the multivariate degradation model presented in Eq. (9) is chosen, a joint likelihood function for the degradation observations $\mathbf{y}_{1:n,1:N} = \{\mathbf{y}_{1,1:N}, ..., \mathbf{y}_{i,1:N}, ..., \mathbf{y}_{n,1:N}\}$ given the environmental variable $\mathbf{X}_{1:N}^E$ and operating variable $\mathbf{X}_{1:N}^O$ is obtained as follows:

$$L\left(\mathbf{y}_{1:n,1:N} \middle| \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O, \boldsymbol{\theta}^F, \boldsymbol{\theta}^R, \boldsymbol{\theta}^C\right) = \prod_{k=1}^{N} \prod_{j=2}^{m_k} \left[c\left(F_1\left(\Delta y_{1kj}\right), ..., F_i\left(\Delta y_{ikj}\right), ..., F_n\left(\Delta y_{nkj}\right)\right) \prod_{i=1}^{n} f_i\left(\Delta y_{ikj}\right) \right] \tag{12}$$

where $\Delta y_{ikj} = y_i(t_{kj}) - y_i(t_{k,j-1})$ is the observation of degradation increment of the ith degradation process at the time interval $[t_{k,j-1}, t_{kj}]$ for the kth sample. $F_i\left(\Delta y_{ikj}\right)$ and $f_i\left(\Delta y_{ikj}\right)$ are separately the CDF and PDF of Δy_{ikj} with the probability distribution given in Eq. (9), i.e. the normal distributions and the IG distributions. c$(u_1, \ldots, u_i, \ldots, u_n)$ is PDF of the copula function $C(u_1, \ldots, u_i, \ldots, u_n)$ given in Eq. (9), where $c(u_1, \ldots, u_i, \ldots, u_n) = \partial^n C / \partial u_1 \ldots \partial u_n$.

To facilitate parameter estimation based on the joint likelihood function presented above, a two-step Bayesian method is presented in the following section. This two-step Bayesian method is intended to mitigate computation difficulty originated from simultaneously handling the PDFs of degradation increments $f_i(\Delta y_{ikj})$ as well as the PDF of a copula function $c\left(F_1\left(\Delta y_{1kj}\right), \ldots, F_i\left(\Delta y_{ikj}\right), \ldots, F_n\left(\Delta y_{nkj}\right)\right)$ where CDFs of degradation increments are used as inputs. The key idea of this two-step method is to utilize the characteristic of the proposed multivariate degradation model. It is based on the fact that marginal distributions of degradation increments are either normal distributions or IG distributions, and these marginal distributions are separable from their dependence structure which is characterized by the copula function. In addition, there is no sharing parameter among these marginal distributions of degradation increments and copula function. Accordingly, it is possible to estimate the parameters of marginal distributions firstly, and then carry out the parameter estimation for copula function [49].

The first step is to estimate parameters of marginal distributions for degradation processes $Y_i\left(t; \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H\right)$. Given the observations $\mathbf{y}_{i,1:N}$ and the variables of environmental condition $\mathbf{X}_{1:N}^E$ and the variables of operating mission $\mathbf{X}_{1:N}^O$, the Bayesian estimation of model parameters for the degradation process $Y_i\left(t; \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H\right)$ is given as follows:

$$
\begin{aligned}
&p\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H \middle| \mathbf{y}_{i,1:N}, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right) \propto \pi\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^H\right) \times \prod_{k=1}^{N} g\left(\boldsymbol{\theta}_{ik}^R \middle| \boldsymbol{\theta}_i^H\right) \prod_{j=2}^{m_k} f_i\left(\Delta y_{ikj} \middle| \mathbf{X}_k^E, \mathbf{X}_k^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\right) \\
&= \begin{cases} \pi\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^H\right) \times \prod_{k=1}^{N} g\left(\boldsymbol{\theta}_{ik}^R \middle| \boldsymbol{\theta}_i^H\right) \prod_{j=2}^{m_k} \phi\left(\dfrac{\Delta y_{ikj} - \Delta \eta_i(t_{kj})}{\sigma_i \sqrt{t_{kj} - t_{k,j-1}}}\right), & \text{for} \quad i \in [1, h]^+ \\ \pi\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^H\right) \times \prod_{k=1}^{N} g\left(\boldsymbol{\theta}_{ik}^R \middle| \boldsymbol{\theta}_i^H\right) \prod_{j=2}^{m_k} f_{\mathrm{IG}}\left(\Delta y_{ikj}; \Delta \Lambda_i(t_{kj}), \lambda_i (\Delta \Lambda_i(t_{kj}))^2\right), & \text{for} \quad i \in [h+1, n]^+ \end{cases}
\end{aligned}
\tag{13}
$$

where $\Delta \eta_i(t_{kj})$ and $\Delta \Lambda_i(t_{kj})$ indicate $\Delta \eta_i\left(t_{kj}; \mathbf{X}_k^E, \mathbf{X}_k^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\right)$ and $\Delta \Lambda_i\left(t_{kj}; \mathbf{X}_k^E, \mathbf{X}_k^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\right)$ separately, $p\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H | \mathbf{y}_{i,1:N}, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right)$ is the joint posterior distribution of model parameters for degradation processes $Y_i\left(t; \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H\right)$, $\pi\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^H\right)$ is the joint prior distribution of model parameters $\boldsymbol{\theta}_i^F$ and $\boldsymbol{\theta}_i^H$, $g\left(\boldsymbol{\theta}_{ik}^R | \boldsymbol{\theta}_i^H\right)$ is the probability distribution assumed for random parameter $\boldsymbol{\theta}_{ik}^R$ associated with the kth sample, $\phi(\cdot)$ is the PDF of a standard normal distribution and $f_{\mathrm{IG}}(a, b)$ is the PDF of an IG distribution as given in Eq. (2).

It should be mentioned that there are generally two types of prior distributions used for Bayesian degradation analysis, i.e., the non-informative priors and the informative priors. The non-informative priors are used for situations that no pre-knowledge is available for degradation analysis, where a diffuse probability distribution such as a uniform distribution within a relative large interval is used. The informative priors are used for situations where continually updating of degradation observations are available, and estimation results of previous degradation data $\mathbf{y}^{(M-1)}$ are used as prior distributions for the degradation analysis of present degradation data $\mathbf{y}^{(M)}$. A detail discussion and demonstration of the Bayesian analysis with these prior distributions is referred to [27] and an illustrative example presented in Section 4.

In addition, since there is no analytical solution of the joint posterior distribution presented in Eq. (13), Markov Chain Monte Carlo (MCMC) simulation method is used to generate posterior samples of model parameters from this joint posterior distribution. Posterior samples and statistical summarizations of these samples, such as the point estimations of model parameters $\hat{\boldsymbol{\theta}}_i^F$, $\hat{\boldsymbol{\theta}}_{i,1:N}^R$ and $\hat{\boldsymbol{\theta}}_i^H$ are further used for parameter estimation and degradation analysis. For detail information about Bayesian estimation of model parameters of Wiener degradation process and the IG degradation models, please refer to [31,27].

The second step is to choose the copula function and to estimate its model parameters. As mentioned above, the estimations of model parameters for the marginal distributions are used to calculate the CDFs of their corresponding degradation increments as $F_i\left(\Delta y_{ikj} \middle| \hat{\boldsymbol{\theta}}_i^F, \hat{\boldsymbol{\theta}}_{ik}^R, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right)$. Each group of CDFs of degradation increments of the n degradation processes at the time interval $\left[t_{k,j-1}, t_{kj}\right]$ is a sample from the copula function $C\left(u_1, \ldots, u_i, \ldots, u_n; \boldsymbol{\theta}^C\right)$. The selection of copula function and the estimation of model parameters of the selected copula function are based on these CDFs of the degradation increments.

As mentioned above, a qualitative and a quantitative way can be implemented to select the copula function. In detail, the qualitative way is based on the qualitative analysis of the scatter plots of CDFs of the degradation increments. Different patterns of scatter plots indicate different types of dependency among the degradation processes. For a particular pattern, a specific copula function is suitable for the dependence modeling based on the characteristics of the copula function [43]. For instance, the Gaussian copula is suitable for the pattern of scatter plot with lower-lower and upper-upper tail dependence, which is chosen for the dependence modeling in the illustrative example. The quantitative way is through the quantitative analysis of the CDFs of the degradation increments of the degradation processes. Because the pairs of CDFs of the degradation increments are samples from the copula function. Classical method like the Akaike Information Criteria (AIC) and the goodness-of-fit tests can be implemented on the samples, which are CDFs of the degradation increments, to select the best fitted multivariate distribution, which is the multivariate copula function in this paper. A brief review of quantitative selection of copula function and a Bayesian copula selection method was presented in Huard et al. [44].

As long as the copula function is selected, its parameters $\boldsymbol{\theta}^C$ are then estimated based on these samples. Using Bayesian method, the estimation of model parameters $\boldsymbol{\theta}^C$ is given as follows:

$$
p\left(\boldsymbol{\theta}^C \middle| \mathbf{y}_{1:n,1:N}, \hat{\boldsymbol{\theta}}_{1:n}^F, \hat{\boldsymbol{\theta}}_{1:n,1:N}^R, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right) \propto \pi\left(\boldsymbol{\theta}^C\right)
$$

$$\times \prod_{k=1}^{N}\prod_{j=2}^{m_k} c\left(\hat{u}_{1kj},...,\hat{u}_{ikj},...,\hat{u}_{nkj}\middle|\boldsymbol{\theta}^C\right)$$

$$\hat{u}_{ikj} = F_i\left(\Delta y_{ikj}\middle|\hat{\boldsymbol{\theta}}_i^F, \hat{\boldsymbol{\theta}}_{ik}^R, \mathbf{X}_k^E, \mathbf{X}_k^O\right)$$

$$c\left(\hat{u}_{1kj},...,\hat{u}_{ikj},...,\hat{u}_{nkj}|\boldsymbol{\theta}^C\right) = \frac{\partial C^n\left(\hat{u}_{1kj},...,\hat{u}_{ikj},...,\hat{u}_{nkj}|\boldsymbol{\theta}^C\right)}{\partial\hat{u}_{1kj},...,\partial\hat{u}_{nkj}} \tag{14}$$

where $\pi\left(\boldsymbol{\theta}^C\right)$ and $p\left(\boldsymbol{\theta}^C|\mathbf{y}_{1:n,1:N}, \hat{\boldsymbol{\theta}}_{1:n}^F, \hat{\boldsymbol{\theta}}_{1:n,1:N}^R, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right)$ are separately the prior distribution and posterior distribution of model parameters $\boldsymbol{\theta}^C$.

The MCMC method is used to generate posterior samples from the posterior distribution presented in Eq. (14). Point estimation $\hat{\boldsymbol{\theta}}^C$ and interval estimation are then summarized from these generated posterior samples. Both estimations and posterior samples are further used in the posterior degradation analysis presented below.

3.2. Simulation based posterior degradation analysis

The failure time distribution and RUL of a complex system with multiple degradation processes characterized by the proposed model are separately given in Eqs. (10) and (11). Since there is no analytical solution for these two indices, simulation based methods are employed in this section to facilitate posterior degradation analysis. The key aspect of simulation based methods lies in the predicting of degradation observations for a sequential future observation time points, which are obtained based on the posterior samples of model parameters and the availability of external factors $\mathbf{X}^E$ and $\mathbf{X}^O$. By comparing the predicted degradation observations with the failure thresholds, failure time distribution and RUL of the complex system are obtained.

Given a complex system is observed up to the time point t_{m_k}, the degradation observation at a future observation point t_{m_k+p} is given as follows:

$$y_i(t_{k,m_k+p}) = y_i(t_{km_k}) + \sum_{l=m_k+1}^{m_k+p}\Delta y_{ikl}, l = m_k+1,...,m_k+p$$

$$\Delta y_{ikl} = \int_{\boldsymbol{\theta}_i^F,\boldsymbol{\theta}_{i,k}^R} F_i^{-1}\left(u_{ikl}\middle|\mathbf{X}_k^E, \mathbf{X}_k^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\right) p\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\middle|\mathbf{y}_i^{1:N}, \mathbf{X}_k^E, \mathbf{X}_k^O\right) d\boldsymbol{\theta}_i^F d\boldsymbol{\theta}_{ik}^R$$

$$F_C(u_{ikl},...,u_{ikl},...,u_{ikl}) = \int_{\boldsymbol{\theta}_C} C\left(u_{ikl},...,u_{ikl},...,u_{ikl}\middle|\boldsymbol{\theta}^C\right) p \left(\boldsymbol{\theta}^C\middle|\mathbf{y}_i^{1:N}, \hat{\boldsymbol{\theta}}_i^F, \hat{\boldsymbol{\theta}}_{ik}^R, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right) d\boldsymbol{\theta}^C \tag{15}$$

where $y_i(t_{k,m_k+p})$ is the degradation observation of the ith degradation process for the kth complex system at the observation time point t_{k,m_k+p}, $F_i^{-1}\left(u_{ikl}|\mathbf{X}_k^E, \mathbf{X}_k^O, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{ik}^R\right)$ is the inverse CDF of the degradation increment Δy_{ikl}, where the CDF of Δy_{ikl} is given in Eq. (9) with either normal distribution or IG distribution, F_C $(u_{ikl},...,u_{ikl},...,u_{ikl})$ is a joint CDF of a group of random variables with uniformly distributed marginal distributions on $[0,1]$.

By generating degradation predictions on a series of future observation points, and then comparing these degradation predictions with their corresponding failure thresholds, the failure time points can be obtained. Then the failure time distribution and RUL of a complex system can be summarized from these failure time points. Accordingly, the estimations of reliability and the assessment of RUL are then given as follows:

$$F(t) = 1 - \Pr\left\{\sup_{t_{m_k+p}\le t} y_1(t_{k,m_k+p}) < D_1, ..., \sup_{t_{m_k+p}\le t} y_i(t_{k,m_k+p}) < D_i, ..., \sup_{t_{m_k+p}\le t} y_n(t_{k,m_k+p}) < D_n\right\} \tag{16}$$

$$T_{RUL} = t_{m_k+p_{fail}} - t_{m_k}, p_{fail} = \inf\left\{p: \begin{array}{c} y_1(t_{k,m_k+p}) \ge D_1 \\ or \\ \vdots \\ y_i(t_{k,m_k+p}) \ge D_i \\ or \\ \vdots \\ y_n(t_{k,m_k+p}) \ge D_n \end{array}\middle| \begin{array}{c} \mathbf{y}_{1k}, y_1(t_{km_k}) < D_1 \\ \vdots \\ \mathbf{y}_{ik}, y_i(t_{km_k}) < D_i \\ \vdots \\ \mathbf{y}_{nk}, y_n(t_{km_k}) < D_n \end{array}\right\} \tag{17}$$

where $y_i(t_{k,m_k+p})$ is the prediction of degradation observation at future observation time point t_{m_k+p} as given in Eq. (15)

The calculations of Eqs. (15)–(17) are implemented through simulation based integration using posterior samples of model parameters $\boldsymbol{\theta}_i^F$, $\boldsymbol{\theta}_{i,1:N}^R$, $\boldsymbol{\theta}^C$. Since the effect of external factors and unit-to-unit variability are considered, the simulation based posterior degradation analysis is characterized as factor-related and unit-specific. Namely, the degradation analysis results are highly correlated with the environmental condition variables $\mathbf{X}^E$ and the mission operating variables $\mathbf{X}^O$. In addition, degradation analysis results are obtained for individual samples within a population. The analysis results may demonstrate unit-to-unit variability for a group of complex systems. The specific procedure for factor-correlated and unit-specific posterior analysis is given as follow.

Step 1: set $k=1$ to start the simulation based degradation analysis.

It is aimed to make the degradation analysis start from the first sample of the N samples and to make the following analysis results unit-specific.

Step 2: for the kth sample of a complex system, generate a group of posterior samples $\left(\tilde{\boldsymbol{\theta}}_i^F, \tilde{\boldsymbol{\theta}}_{ik}^R, \tilde{\boldsymbol{\theta}}_i^H, \tilde{\boldsymbol{\theta}}^C\right)$ from their joint posterior distributions $p\left(\boldsymbol{\theta}_i^F, \boldsymbol{\theta}_{i,1:N}^R, \boldsymbol{\theta}_i^H|\mathbf{y}_i^{1:N}, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right)$ and $p\left(\boldsymbol{\theta}^C|\mathbf{y}_{1:n}^{1:N}, \hat{\boldsymbol{\theta}}_{1:n}^F, \hat{\boldsymbol{\theta}}_{1:n,1:N}^R, \mathbf{X}_{1:N}^E, \mathbf{X}_{1:N}^O\right)$ with $i=1,...,n$.

These posterior samples are associated with degradation models of all the n degradation processes, and the MCMC method mentioned above are used to facilitate the generation of posterior samples.

Step 3: set $l=m_k+1$ to make predictions of future degradation observations progressively from the latest observations as $\{y_1(t_{km_k}),...,y_i(t_{km_k}),...,y_n(t_{km_k})\}$.

Step 4: generate a set of uniformly distributed random variables $\{u_{ikl},...,u_{ikl},...,u_{ikl}\}$ from the copula function $C\left(u_{ikl},...,u_{ikl},...,u_{ikl}\middle|\tilde{\boldsymbol{\theta}}^C\right)$ based on the posterior sample $\tilde{\boldsymbol{\theta}}^C$.

Step 5: obtain degradation increments Δy_{ikl} from the inverse CDFs of degradation increments as $\Delta y_{ikl} = F_i^{-1}\left(u_{ikl}\middle|\mathbf{X}^E, \mathbf{X}^O, \tilde{\boldsymbol{\theta}}_i^F, \tilde{\boldsymbol{\theta}}_{ik}^R\right)$ with $i=1,...,n$.

The CDFs of degradation increments are given in Eq. (9). The posterior samples $\tilde{\boldsymbol{\theta}}_i^F, \tilde{\boldsymbol{\theta}}_{ik}^R$, the random variables generated from the copula function u_{ikl}, the observation time interval $[t_{k,l-1}, t_{kl}]$, and the variables of environment condition and operating profiles $\mathbf{X}^E$, $\mathbf{X}^O$ are used as inputs for the inverse CDFs to generate degradation increments. Different inputs of $\mathbf{X}^E, \mathbf{X}^O$ usually result in different degradation increments and consequently lead to factor-related degradation predictions.

Step 6: calculate predictions of degradation observations $y_i(t_{kl}) = y_i(t_{k,l-1}) + \Delta y_{ikl}$ with $i=1,...,n$ for the kth sample of a complex system under external factors $\mathbf{X}^E, \mathbf{X}^O$.

Step 7: compare the predicted degradation observations $y_i(t_{kl})$ with the corresponding degradation thresholds D_i for all the n degradation processes. If there is any one of the n degradation predictions crosses its degradation threshold, let $T = t_{kl}$ and $RUL = t_{kl} - t_{km_k}$.

Step 8: set $l = l+1$ and then repeat Steps 4 to 8 until $T = t_{kl}$ and $l > m_k + p$ to obtain the failure time of the complex system and the degradation predictions of the n degradation processes at an interested time point t_{m_k+p}.

Step 9: set $k = k+1$ and then repeat Steps 2 to 8 until $k > N$ to obtain the failure time points and degradation predictions for the N samples of the complex system.

Step 10: repeat Steps 1 to 9 M times to obtain multiple sets of samples of the degradation predictions, failure time points, and RULs with a sample size M. These samples are then further used to derive statistical summarizations of these indices such as the point estimation/prediction, interval estimation/prediction and approximated kernel densities as well.

When new degradation data are available, the continual updating of the degradation analysis is then implemented through the Bayesian strategy and simulation based degradation analysis presented above. It lies in the utilizing of the Bayesian estimation of model parameters presented in Section 4.4.1. The parameter estimations based on the previous degradation data are then used as prior distributions of model parameters in Eq. (13). These priors are then updated by newly observed degradation observations through this Bayesian estimation method in order to obtain newly updated posterior distributions of model parameters. These newly updated posterior distributions are then used to substitute the posterior distributions used in the Step 2 for simulation based degradation analysis. Finally, the unit-specific reliability analysis and factor-related degradation predictions are updated continually according to the available of newly observed degradation observations.

4. Illustrative example

This section presents an illustrative example originated from the condition monitoring and degradation analysis of DL150 CNC heavy duty lathes. The losing of machining accuracy and accumulation of lubrication debris are recognized as two main interdependent failure modes. Condition monitoring have been implemented to collect unit-specific vibration signal amplitudes for losing of machining accuracy, metallic debris numbers in lubrication oil, and the environmental conditions and operating profiles experienced by the DL 150s. A comprehensive study of these degradation processes under dynamic conditions is then implemented as follows.

4.1. Degradation indicators and external factors

Five DL150 CNC heavy duty lathes have been monitored with ten missions fulfilled. The amplitudes of vibration signals and numbers of metallic debris are monitored as degradation indicators. The environmental conditions and operating profiles are collected as external factors. The observations of these two degradation indicators as well as their corresponding external factors are separately presented in Figs. 1 and 2. It is worth noting that the environmental condition remains constant for individual heavy duty lathe, while it varies among the population. The operating profiles are dynamic for each heavy duty lathe with different loads and duration lengths.

Let $y_{ih}(t_{kj})$ with $i = 1, 2$, $k = 1, \ldots, 5$, $h = 1, \ldots, 10$, and $j = 1, \ldots, m_h$ denote the degradation observation of the ith degradation indicator of the kth heavy duty lathe at the observation time point t_{kj}, where this heavy duty lathe is under the hth mission load and the t_{kj} is the jth observation points within the m_h observation time of the hth mission. Let X_{kh}^E and X_k^O separately denote the environmental condition variable and the operating mission variable. In addition, the degradation thresholds of these two degradation indicators are separately denoted as $D_1 = 80$ and $D_2 = 40$.

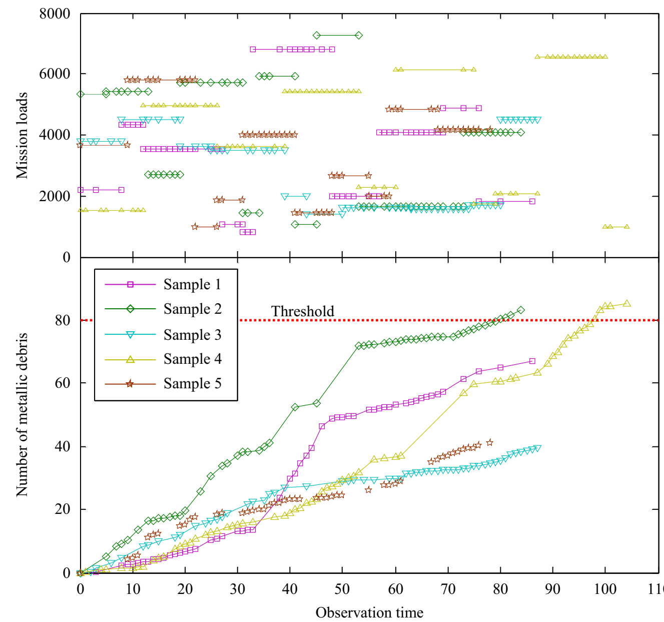

Fig. 1. Mission loads and number of metallic debris observed for DL150 CNC heavy duty lathes.

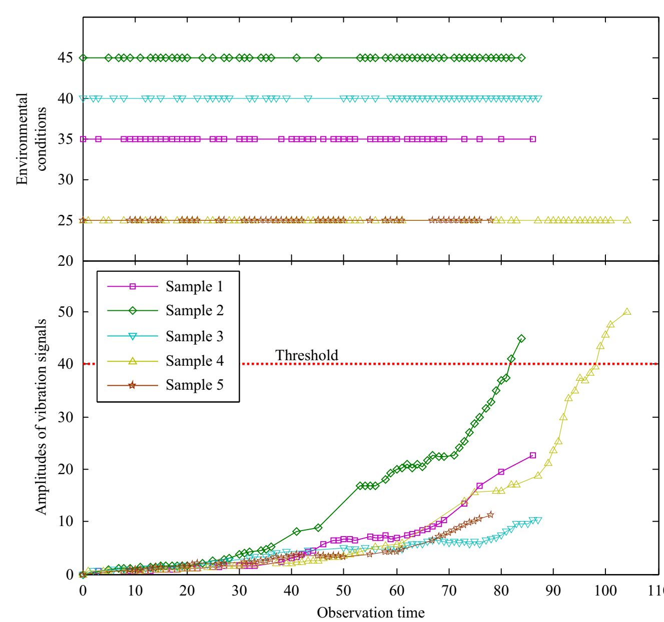

Fig. 2. Environmental conditions and amplitudes of vibration signals for DL150 CNC heavy duty lathes.

4.2. Degradation modeling and parameter estimation

Given the degradation observations presented above, a further depiction of the corresponding degradation increments of these degradation processes is carried out. A pictorial description of the correlation between degradation increments of these two degradation processes is presented in Fig. 3, where both nominal and logarithmic amplitudes of vibration signals are used. From the scatter plots of degradation increments, we can conclude that the number of metallic debris increases monotonically, whereas the amplitudes of vibration signals evolve non-monotonically.

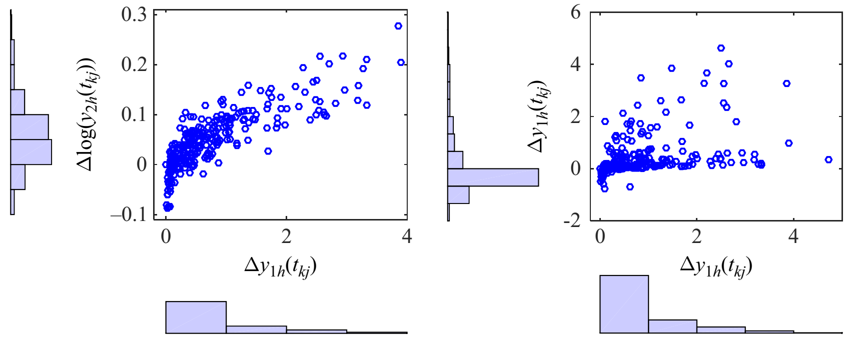

Fig. 3. Scatter plots of degradation increments of the number of metallic debris and the nominal (logarithmic) amplitudes of vibration signals.

Table 1
Estimations of model parameters and priors used in Bayesian estimation.

Parameter	Posterior		Posterior percentiles		Prior distribution
	Mean	SD	2.5%	97.5%	
β_1	0.1759	0.01609	0.1464	0.2098	Uniform(0, 10)
λ_1	2.199	0.3122	1.628	2.853	Uniform(0, 10)
β_1^O	0.4247	0.02019	0.385	0.4645	Uniform(0, 10)
δ_1	−73.42	19.31	−98.69	−31.51	Uniform(−100, 100)
γ_1	34.06	24.45	1.601	90.2	Uniform(0, 100)
θ_2	−0.5394	0.02138	−0.581	−0.4974	Uniform(−10, 10)
β_2	0.008443	0.002704	0.003204	0.01383	Uniform(0, 10)
σ_2	0.04571	0.002319	0.04146	0.05057	Uniform(0, 10)
β_2^O	0.01249	0.0007	0.01104	0.01393	Uniform(0, 10)
δ_2	−71.77	22.92	−99.12	−25.5	Uniform(−100, 100)
γ_2	20.73	19.15	0.7573	74.5	Uniform(0, 100)

Moreover, a strong correlation between the increments of number of metallic debris and logarithmic amplitudes of vibration signals is found as well.

The multivariate hybrid degradation model presented in Eq. (9) is used to characterize the degradation processes of heavy duty lathes. In detail, an IG process model with linear mean degradation function is used for the number of metallic debris $Y_1(t)$ as one of the marginal degradation processes. An exponential degradation model is used for the amplitudes of vibration signals $Y_2(t)$ as the other marginal degradation process [48], where a logarithm transformation of $Y_2(t)$ can be carry out to facilitate the calculation. By incorporating the covariates of the external factors and the random effect of the unit-to-unit variability, the degradation model is given as follows:

$$
\begin{aligned}
&y_{1h}(t_{kj}) \sim \mathrm{IG}\left(\Lambda_{1h}(t_{kj}), \lambda_1 (\Lambda_{1h}(t_{kj}))^2\right), \quad k=1,...,5, \quad h=1,...,10,\\
&\log\left(y_{2h}(t_{kj})\right) \sim \mathrm{N}(\mu_{2h}(t_{kj}), {\sigma_2}^2 t_{kj}), \quad j=1,...,m_h,\\
&F(\Delta y_{1h}(t_{kj}), \Delta \log\left(y_{2h}(t_{kj})\right)) = C\left(F_1(\Delta y_{1h}(t_{kj})), F_2(\Delta \log\left(y_{2h}(t_{kj})\right)); \boldsymbol{\theta}^C\right),\\
&\Lambda_{1h}(t_{kj}) = \exp\left(\beta_1^O \frac{X_{kh}^O - X_0^O}{X_0^O}\right)\left(\beta_1 + \beta_{1k}^E \log\left(X_k^E\right)\right) t_{kj},\\
&\beta_{1k}^E \sim \mathrm{Gamma}(\delta_1, \gamma_1),\\
&\mu_{2h}(t_{kj}) = \theta_2 + \left(\beta_2 + \beta_2^O \frac{X_{kh}^O - X_0^O}{X_0^O} + \beta_{2k}^E \log\left(X_k^E\right)\right) t_{kj},\\
&\beta_{2k}^E \sim \mathrm{Gamma}(\delta_2, \gamma_2)
\end{aligned}
\tag{18}
$$

where X_{kh}^O is the mission load endured by the kth sample during the hth mission. X_0^O is the minimal mission load generally endured by heavy duty lathes. X_k^E is the environmental condition experienced by the kth sample for all missions.

Following the methods of covariate incorporation presented in Section 2.2.1, the incorporation of mission load is implemented based on an exponential acceleration relationship for the degradation process of $Y_1(t)$, and an additive acceleration relationship for the degradation process of $Y_2(t)$. The incorporation of environment condition is implemented through an additive acceleration relationship for both degradation processes. In addition, stochastic coefficients are associated with the environment conditions to describe unit-to-unit variability among the heavy duty lathe fleet. To make the values of external factors compatible to the model, a proportional comparison of mission loads and a logarithm of environment conditions are used in the model. It is worth noting that the incorporating of external factors should be carried out according to specific applications with general methods discussed above in Section 2.2.1.

By utilizing the two-step Bayesian estimation method presented in Section 3, model parameters for marginal degradation processes are estimated firstly. According to Eq. (13), Bayesian estimations of these parameters are presented as follow.

$$
\begin{aligned}
&p\left(\boldsymbol{\theta}_1^F, \boldsymbol{\theta}_{1,1:5}^R, \boldsymbol{\theta}_1^H \middle| \mathbf{y}_{1,1:5}, \mathbf{X}_{1:5}^E, \mathbf{X}_{1:5}^O\right) \propto\\
&\quad \pi\left(\beta_1, \lambda_1, \beta_1^O, \delta_1, \gamma_1\right) \times \prod_{k=1}^{5} g\left(\beta_{1k}^E \middle| \delta_1, \gamma_1\right) \prod_{h=1}^{8} \prod_{j=2}^{m_h} f_{\mathrm{IG}}(\Delta y_{1h}(t_{kj});\\
&\quad \Delta\Lambda_{1h}(t_{kj}), \lambda_1 (\Delta\Lambda_{1h}(t_{kj}))^2)\\
&\quad p\left(\boldsymbol{\theta}_2^F, \boldsymbol{\theta}_{2,1:5}^R, \boldsymbol{\theta}_2^H \middle| \mathbf{y}_{2,1:5}, \mathbf{X}_{1:5}^E, \mathbf{X}_{1:5}^O\right) \propto\\
&\quad \pi\left(\theta_2, \beta_2, \sigma_2, \beta_2^O, \delta_2, \gamma_2\right) \times \prod_{k=1}^{5} g\left(\beta_{2k}^E \middle| \delta_2, \gamma_2\right)\\
&\quad \prod_{h=2}^{8} \prod_{j=2}^{m_h} \phi\left(\frac{\Delta \log\left(y_{2h}(t_{kj})\right) - \Delta\mu_{2h}(t_{kj})}{\sigma_2 \sqrt{t_{kj} - t_{k,j-1}}}\right)\\
&\quad \times \prod_{k=1}^{5} g\left(\beta_{2k}^E \middle| \delta_2, \gamma_2\right) \phi\left(\frac{\log\left(y_{21}(t_{k2})\right) - \mu_{21}(t_{k2})}{\sigma_2 \sqrt{t_{k2}}}\right)
\end{aligned}
$$

$$\prod_{j=3}^{m_1} \phi\left(\frac{\Delta \log\left(y_{21}(t_{kj})\right) - \Delta\mu_{21}(t_{kj})}{\sigma_2\sqrt{t_{kj}-t_{k,j-1}}}\right) \tag{19}$$

where $\boldsymbol{\theta}_1^F = \left\{\beta_1, \lambda_1, \beta_1^O\right\}$, $\boldsymbol{\theta}_{1,1:5}^R = \left\{\beta_{11}^E, \beta_{12}^E, ..., \beta_{15}^E\right\}$, and $\boldsymbol{\theta}_1^H = \{\delta_1, \gamma_1\}$ are separately fixed parameters, random parameters and hyper-parameters for the degradation model of number of metallic debris, $\boldsymbol{\theta}_2^F = \left\{\theta_2, \beta_2, \sigma_2, \beta_2^O\right\}$, $\boldsymbol{\theta}_{2,1:5}^R = \left\{\beta_{21}^E, \beta_{22}^E, ..., \beta_{25}^E\right\}$, and $\boldsymbol{\theta}_2^H = \{\delta_2, \gamma_2\}$ are the corresponding parameters for the degradation model of amplitude of vibration signals, $\mathbf{y}_{1,1:5}$, $\mathbf{y}_{2,1:5}$, $\mathbf{X}_{1:5}^E$ and $\mathbf{X}_{1:5}^O$ are separately the degradation observations of the two degradation processes and the information of environmental conditions and operating profiles, $g\left(\beta_{ik}^E \mid \delta_i, \gamma_i\right), i = 1, 2$ is the PDF of a gamma distribution with δ_i, γ_i separately the shape parameter and rate parameter, $\pi(\bullet)$ is the joint prior distribution for model parameters. In this study, non-informative uniform prior distributions with diffuse intervals of corresponding parameters are used firstly.

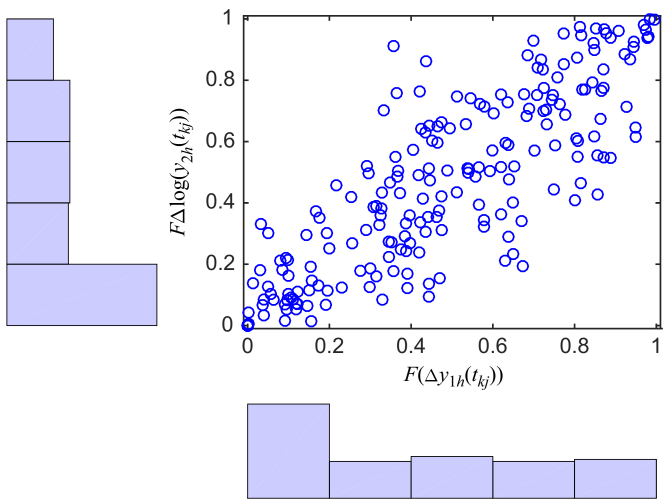

Fig. 4. Scatter plots of CDFs of degradation increments of the number of metallic debris and the logarithmic amplitudes of vibration signals.

As the continually available of degradation observations, the joint posterior distribution $p(\bullet)$ is used as the prior distribution for Bayesian estimation of newly observed degradation observations.

Posterior samples of model parameters from the joint posterior distribution presented in Eq. (19) are generated using the MCMC. The implementation of the MCMC is through a software package OpenBUGS [50]. Statistical summarizations for the parameters are presented in Table 1, where the corresponding intervals of uniform distributions used as prior distributions are presented as well.

Based on the estimations of model parameter presented in Table 1, the CDFs of degradation increments of the degradation processes are obtained as $(F_1(\Delta y_{1h}(t_{kj})), F_2(\Delta \log(y_{2h}(t_{kj}))))$. The correlation between these two degradation processes are then further depicted using these CDFs and given in Fig. 4.

A lower-lower and upper-upper tail dependence is demonstrated through Fig. 4, and the Gaussian copula is chosen to characterize this dependence. The copula function in Eq. (18) is further specified as follows:

$$\begin{aligned} F(\Delta y_{1h}(t_{kj}), \Delta \log(y_{2h}(t_{kj}))) &= C\left(F_1(\Delta y_{1h}(t_{kj})), F_2(\Delta \log(y_{2h}(t_{kj}))); \boldsymbol{\theta}^C\right) \\ &= \int_{-\infty}^{\Phi^{-1}(F_1(\Delta y_{1h}(t_{kj})))} \int_{-\infty}^{\Phi^{-1}(F_2(\Delta \log(y_{2h}(t_{kj}))))} \frac{1}{2\pi\sqrt{1-\alpha^2}} \exp\left\{-\frac{x^2 - 2\alpha xy + y^2}{2(1-\alpha^2)}\right\} dx dy \end{aligned} \tag{20}$$

By carrying out the second step of the two-step Bayesian estimation method presented in Section 3.3.1, the model parameter $\boldsymbol{\theta}^C = \alpha$ of copula function is estimated. The CDFs of degradation

Table 2
Estimations of model parameter α.

Parameter	Posterior		Posterior percentiles	
	Mean	SD	2.5%	97.5%
α	0.9417	0.0060	0.9289	0.9523

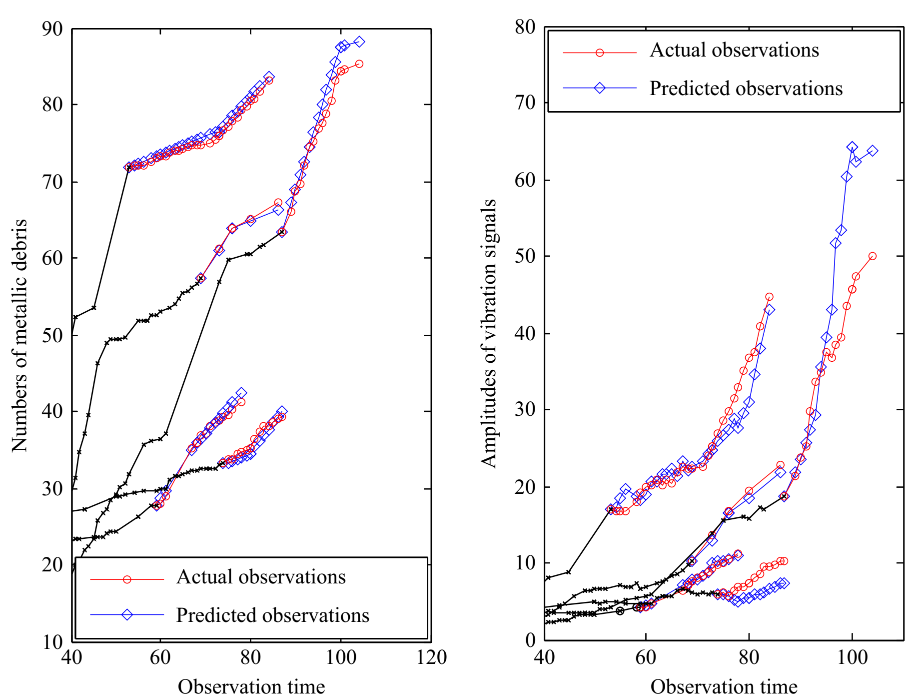

Fig. 5. The predicted vs. the actual degradation observations for the latest two missions.

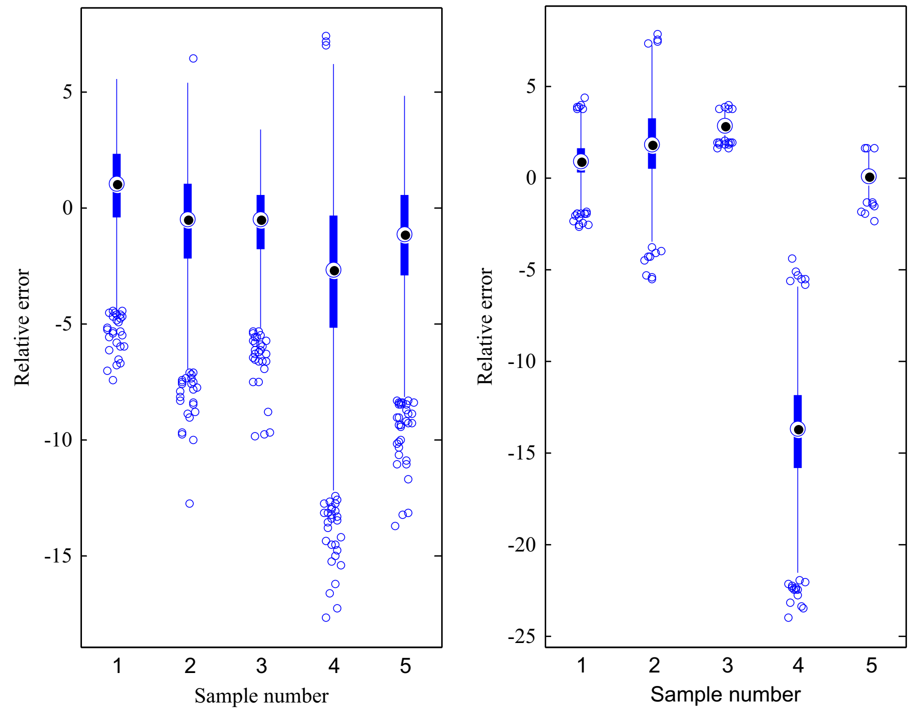

Fig. 6. Boxplots of relative errors of degradation predictions at the latest observation points.

Table 3
Estimations of model parameter for the simplified model.

Parameter	Posterior		Posterior percentiles		Prior distribution
	Mean	SD	2.5%	97.5%	
β_1	0.746	0.07977	0.6111	0.92	Uniform(0, 10)
λ_1	0.3718	0.08462	0.2235	0.5505	Uniform(0, 10)
θ_2	−0.5544	0.03359	−0.6207	−0.4886	Uniform(−10, 10)
β_2	0.04675	0.002377	0.04207	0.05146	Uniform(0, 10)
σ_2	0.07206	0.00366	0.06537	0.07969	Uniform(0, 10)
α	0.9356	0.006686	0.9212	0.9474	Uniform(−1, 1)

increments are used as samples from the Gaussian copula, a uniform distribution within the interval $[-1,1]$ is used as a non-informative prior distribution for α. The posterior distribution is given as follow.

$$p\left(\alpha \middle| \mathbf{y}_1^{1:5}, \mathbf{y}_2^{1:5}, \hat{\boldsymbol{\theta}}_{1:2}^F, \hat{\boldsymbol{\theta}}_{1:2,1:5}^R, \mathbf{X}_{1:5}^E, \mathbf{X}_{1:5}^o\right) \propto \pi(\alpha)$$
$$\times \prod_{k=1}^{5} \prod_{h=1}^{8} \prod_{j=2}^{m_h} \frac{1}{\sqrt{1-\alpha^2}} \exp\left\{-\frac{a_{khj}^2 - 2\alpha a_{khj} b_{khj} + b_{khj}^2}{2(1-\alpha^2)}\right\},$$
$$a_{khj} = \Phi^{-1}\left(F_1\left(\Delta y_{1h}(t_{kj}) \middle| \hat{\boldsymbol{\theta}}_1^F, \hat{\boldsymbol{\theta}}_{1,1:5}^R, \mathbf{X}_{1:5}^E, \mathbf{X}_{1:5}^o\right)\right),$$
$$b_{khj} = \Phi^{-1}\left(F_2\left(\Delta \log(y_{2h}(t_{kj})) \middle| \hat{\boldsymbol{\theta}}_2^F, \hat{\boldsymbol{\theta}}_{2,1:5}^R, \mathbf{X}_{1:5}^E, \mathbf{X}_{1:5}^o\right)\right) \quad (21)$$

The MCMC and OpenBUGS are used to generate samples from the posterior distribution of α. Statistical summarizations of posterior samples of parameter α are presented in Table 2.

4.3. Posterior analysis and model comparison

The parameter estimation presented above is based on degradation observations of the first eight missions. Degradation observations of the latest two missions are reserved for model verification and comparison. Based on posterior samples of model parameters generated from the joint posterior distributions presented in Eqs. (19) and (21), degradation observations of the latest two missions are obtained following the simulation based degradation inference described in Section 3.2. The predicted degradation observations and the actual observations are presented in Fig. 5.

A good capability for factor-related degradation prediction is visually verified through Fig. 5. The predicted degradations presented in Fig. 5 are the mean values of predicted degradation samples generated by the simulation based method presented in Section 3.2. To give a more comprehensive depiction of the predicted degradation observation by taking account the variances, boxplots of the relative errors of the degradation observations at the latest observation points are presented in Fig. 6. The relative error is defined as the error between the actual observation and the predicted observation over the actual observation, which is given in a percentage form.

From both the means and the boxplots of relative errors of predicted observation presented above, the capability of the proposed method for degradation prediction can be verified. To give a further demonstration of the capability of the proposed method for factor-related degradation prediction, we perform the degradation analysis of the heavy duty lathes with a bivariate degradation model (simplified model) under the “constant external factor” assumption. In this model, the external factors are simplified and the degradation predictions are not related to the external factors that the heavy duty lathes experienced. This simplified model is similar to the model presented in [30-32] and given as follows:.

$$y_{1h}(t_{kj}) \sim \mathrm{IG}\left(\beta_1 t_{kj}, \lambda_1 (\beta_1 t_{kj})^2\right), \quad k=1,\dots,5, \quad h=1,\dots,10,$$
$$\log(y_{2h}(t_{kj})) \sim \mathrm{N}(\theta_2 + \beta_2 t_{kj}, \sigma_2), \quad j=1,\dots,m_h,$$
$$F(\Delta y_{1h}(t_{kj}), \Delta \log(y_{2h}(t_{kj}))) = C(F_1(\Delta y_{1h}(t_{kj})), F_2(\Delta \log(y_{2h}(t_{kj})))) \quad (22)$$

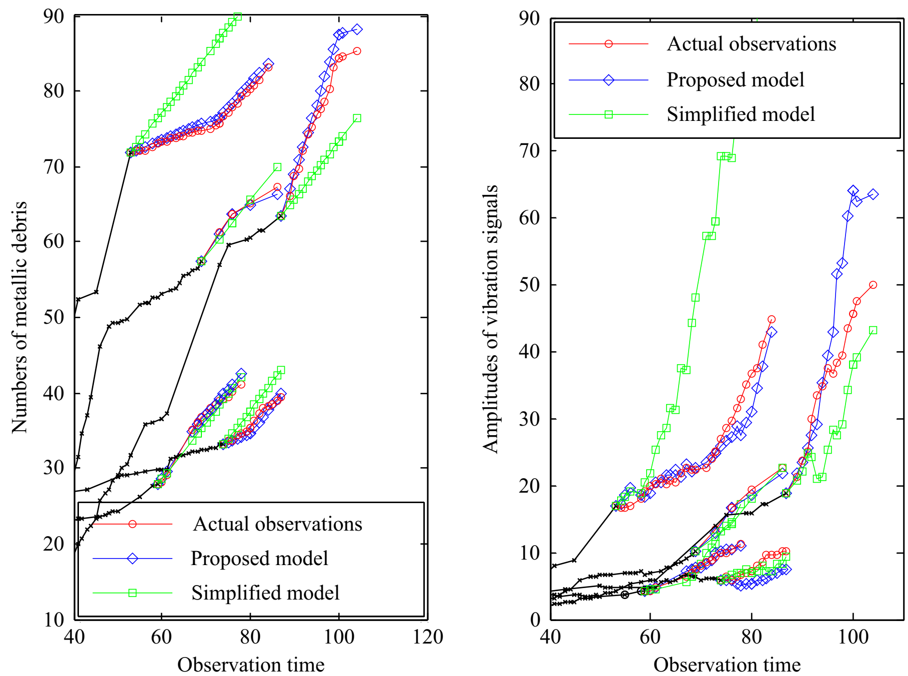

Fig. 7. Comparison of degradation predictions of the proposed model and the simplified model.

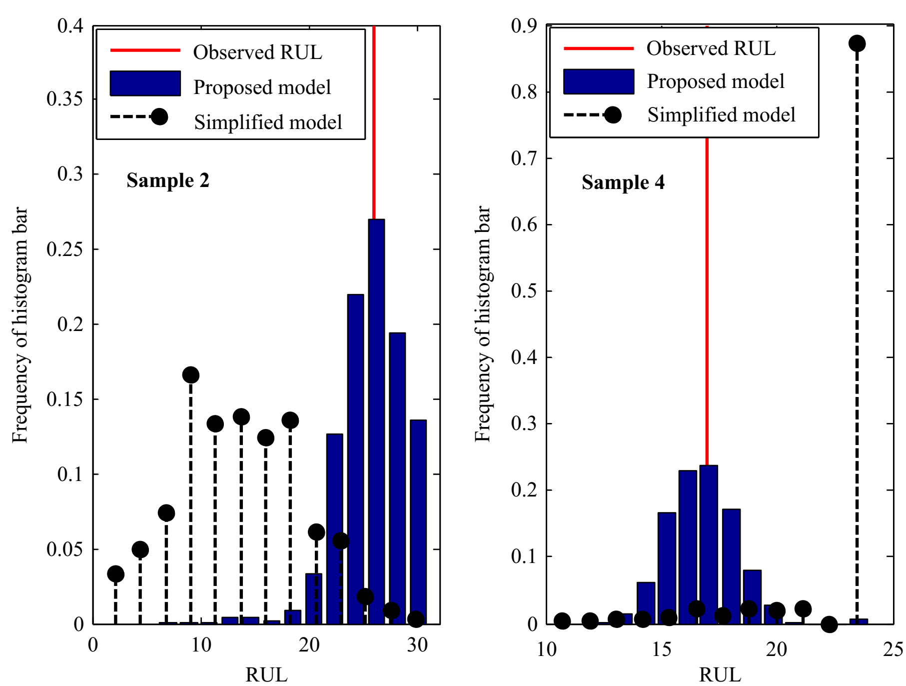

Fig. 8. Comparison of RUL assessment of the proposed model and the simplified model.

In this model, the effects of external factors and random effect are omitted under the assumption of "constant external factor", however the dependence between degradation processes are remained. The two-step Bayesian estimation method introduced in this paper are then applied to this simplified model with degradation data presented in Figs. 1 and 2. The results of parameter estimation are presented in Table 3.

Degradation predictions for the latest two missions are obtained for this model and presented in Fig. 7. A comparison of prediction results of the simplified model with the ones generated by the proposed model in this paper is also demonstrated in Fig. 7. Large deviations between the actual observations and the predictions generated by the simplified model are observed. This is mainly due to the oversimplification of the model by omitting the external factors including the mission loading and environmental conditions.

A further demonstration of the proposed method for unit-specific RUL assessment is implemented through the sample 2 and sample 4 of DL 150s. The failure time of these two samples are observed as shown in Figs. 1 and 2. Assume the time points interested here are the last observation time of the eighth mission. The RULs for these two samples are then obtained based on the degradation predictions of the latest two missions obtained above and the simulation based method presented in Section 3.2. The results of RUL assessment are presented in Fig. 8. Comparisons of the RULs obtained by the proposed method with the ones obtained by the simplified model as well as real observed RULs are presented in this figure. It shows that a higher precision of unit-specific RUL assessment is achieved by the proposed method compared with the simplified method. The capability of the proposed method for unit-specific RUL assessment is then verified. Moreover, by leveraging the factor-related degradation prediction and the unit-specific RUL assessment, further investigation on the optimal decision-making for the DL 150s can then be implemented.

5. Conclusion

In this paper two general assumptions used for degradation modeling of complex systems, i.e., "single degradation indicator" and "constant external factors" are eliminated by introducing a generalized multivariate hybrid degradation model. The new model has the capability of modeling various types of dependent degradation processes including monotonic, non-monotonic, and hybrid degradation processes. With this model, the effect of external factors such as variable environmental conditions and dynamic operating profiles, which is generally omitted by tradition models yet imperative for complex systems, can be incorporated and analyzed coherently. Moreover, a two-step Bayesian parameter estimation method and a Bayesian posterior sample based degradation analysis are presented to facilitate the multivariate degradation analysis with the proposed model. The parameter estimation method can overcome the computational difficulty introduced by multiple degradation models. It can be further utilized to carry out a simulation based degradation analysis including unit-specified RUL assessment and factor-related degradation prediction. The applicability of the proposed method is demonstrated on a group of heavy duty lathes operating in different environmental and operation conditions. A comparison of the proposed method with a simplified method is also provided based on the degradation data from the heavy duty lathes. It shows that the proposed method achieves a significant improvement on degradation analysis over the simplified method.

Some interesting questions remain open and deserve further investigation in the future. For instance, the incorporation of external shocks and time-varying external factors on the degradation processes is of interest to expand the proposed model for more extensive applications. In addition, developing optimization models for operational management and preventive maintenance based on the degradation analysis is also worth of further investigation.

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