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Unified uncertainty analysis by the mean value first order saddlepoint approximation

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Abstract Uncertainties exist in products or systems widely. In general, uncertainties are classified as epistemic uncertainty or aleatory uncertainty. This paper proposes a unified uncertainty analysis (UUA) method based on the mean value first order saddlepoint approximation (MVFOSPA), denoted as MVFOSPA-UUA, to estimate the systems probabilities of failure considering both epistemic and aleatory uncertainties simultaneously. In this method, the input parameters with epistemic uncertainty are modeled using interval variables while input parameters with aleatory uncertainty are modeled using probability distribution or random variables. In order to calculate the lower and upper bounds of system probabilities of failure, both the best case and the worst case scenarios of the system performance function need to be considered, and the proposed MVFOSPA-UUA method can handle these two cases easily. The proposed method is demonstrated to be more efficient, robust and in some situations more accurate than the existing methods such as uncertainty analysis based on the first order reliability method. The proposed method is demonstrated using several examples.

Keywords Reliability · Uncertainty analysis · Saddlepoint approximation · Mean value · Epistemic uncertainty · Aleatory uncertainty

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Abbreviations

UUA	Unified Uncertainty Analysis,
FORM	First Order Reliability Method,
SORM	Second Order Reliability Method,
FOSPA	First Order Saddlepoint Approximation,
PDF	Probability Density Functions,
CDF	Cumulative Distribution Functions,
MLP	Most Likelihood Point,
MVFOSPA	Mean Value First Order Saddlepoint Approximation,
MPP	Most Probable Point
FFT	Fast Fourier Transforms,
MVFOSPA-UUA	Unified Uncertainty Analysis based on the MVFOSPA,
MCS	Monte Carlo Simulation,
SPA	Saddlepoint Approximation,
CGF	Cumulant Generating Function

1 Introduction

Uncertainties widely exist in practical engineering systems. Uncertainties may be generated due to incomplete information, less sampling data, ignorance, measure errors, and inherent variations for environment. These uncertainties can be classified as epistemic or aleatory uncertainty (Kiureghian 2008; Du 2008a; Kiureqhian and Ditlevsen 2009; Zhang and Huang 2010). Epistemic uncertainty comes from ignorance or incomplete information which is usually modeled using interval variables or fuzzy variables (Du 2008a). Aleatory uncertainty arises from inherent variation which is often modeled with probability distributions. The classical reliability estimation methods, such as the first order reliability method (FORM) (Melchers 1999; Chiralaksanakul and Mahadevan 2005; Jensen 2007; Koduru and Haukaas 2010; Xiao et al. 2011) and the second order reliability method (SORM) (Hohenbichler et al. 1987; Zhao and Ono 1999; Lee and Kim 2006), are commonly used for the system reliability analysis and reliability-based design due to their good balance between accuracy and efficiency. Generally, SORM is more accurate than the FORM. However, the efficiency of the SORM is generally low because of calculating Hessian matrix (Melchers 1999). In spite of their usefulness, both the FORM and SORM are often not accurate enough in many cases, because when random variables are non-normally distributed and/or correlated, these non-normally distributed and/or correlated random variables need to be transformed into equivalent independent normal variables. Furthermore, most probable point (MPP) search is necessary for the above two methods. Sometimes the MPP search process does not converge (Huang and Du 2008). For these reasons, a new reliability analysis method which is more accurate and efficient is necessary.

Recently, the first order saddlepoint approximation (FOSPA) technique was first introduced by Du and Sudjianto (2004) for the system reliability analysis and reliability-based design optimization. The main principle of the FOSPA is to evaluate both the probability density function (PDF) and cumulative distribution function (CDF) of the system performance function by using saddlepoint approximation. In the FOSPA, system performance function is expanded as the first order Taylor series at the most likelihood point (MLP) (Du and Sudjianto 2004) which has the highest probability density on the system performance function. From the discussion and examples in Du and Sudjianto (2004), it proved that FOSPA is more accurate than the FORM. For some special cases, FOSPA is more accurate than the SORM, because when random variables are non-normally distributed and/or correlated, the SORM requires a non-normal to normal transformation. The transformation increases the nonlinearity of system performance function significantly which may lead to an increased calculation error in reliability estimation. However, FOSPA has some drawbacks such as determining of MLP (Du and Sudjianto 2004), which is an optimization process that involves time-consuming search. In some cases, there might exist even more than one MLP or none of them. To avoid such searching of MLP, Huang and Du (2008) proposed a method called mean value first order saddlepoint approximation (MVFOSPA), which chooses the expansion point at the mean values of random variables. They proved that MVFOSPA is more robust and efficient than the FOSPA because the former does not require searching the MLP. However, MVFOSPA (Huang and Du 2008) is only suitable for addressing aleatory uncertainty, which is described by precise probability distribution. It is not applicable to epistemic uncertainty.

Most of methods described above can not estimate the probability of failure when both epistemic and aleatory uncertainties are present in the system simultaneously. To address this problem, a unified uncertainty analysis (UUA) based on the FORM has been proposed recently by Du et al. (2005) and Du (2008a). However, the FORM-based method has its deficiencies. The main task for FORM-based method is MPP search. The MPP search is a time-consuming iterative optimization process (Du et al. 2005). Jiang et al. (2011) proposed a UUA method when both random variables and p-box variables are present in the system simultaneously, the MPP search is needed in this method. Zaman et al. (2011) developed a probabilistic approach for uncertainty representation and propagation in the system analysis. In this approach, interval variable was represented by using Johnson family distributions. Sankararaman and Mahadevan (2011) presented a likelihood-based methodology for a probabilistic representation of a stochastic quantity for which only sparse point data and/or interval data may be available. The main disadvantages of these methods are that they involve lot of computational overhead. Furthermore, Adduri and Penmetsa (2009) presented a UUA method for structural system reliability analysis under both random and interval variables. However, the computational efficiency of this method is very low as it involves the MPP search, transformation of membership functions, and convolution integral using the fast Fourier transforms (FFT).

To overcome the limitations of the above existing methods, an efficient UUA method based on MVFOSPA, denoted as MVFOSPA-UUA, is proposed to estimate the systems probabilities of failure considering both epistemic and aleatory uncertainties simultaneously. This work is an extension to the research works by Huang and Du on the MVFOSPA and FOSPA (Du and Sudjianto 2004; Huang and Du 2008). In this method, input parameters with epistemic uncertainty are modeled using interval variables while those with aleatory uncertainty are modeled using probability distributions.

This paper is organized as follows. Section 2 provides a brief background about the interval arithmetic and its operations. Section 3 proposes a UUA method under the mixture of random and interval variables in details. Three numerical examples are presented in Section 4 to demonstrate the proposed method. Section 5 presents brief discussions and conclusions.

2 Interval arithmetic and its operations

A closed bounded interval $\left[X, \bar{X}\right] = \left(X \leq X \leq \overline{X}, X \in R\right)$ is called an interval number, denoted as X^I. $\underline{X}$ and $\overline{X}$ are the lower and upper bounds on interval X^I, respectively.

The midpoint $\tilde{X}$ and radius X^r, can be calculated as follows (Chen et al. 2004)

$$\tilde{X} = \frac{\underline{X} + \overline{X}}{2}, \quad X^r = \frac{\overline{X} - \underline{X}}{2} \tag{1}$$

According to (1), interval X^I and interval variable X can be written in the following standardized forms

$$X^I = \tilde{X} + X^r \delta^I, \quad X = \tilde{X} + X^r \delta \tag{2}$$

where $\delta^I = [-1, 1]$ is the unit interval, and $\delta \in \delta^I$ is the unit interval variable. Generally, a real number y can also be expressed as an interval $[y, y]$. For interval numbers X^I and Y^I, the six basic algebraic operations are listed below (Ferson et al. 2007; Sun and Yao 2008; Kulisch 2009)

$$X^I + Y^I = \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right] \tag{3}$$

$$X^I - Y^I = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right] \tag{4}$$

$$X^I \cdot Y^I = \left[\min\left(\underline{X}\,\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\right), \max\left(\underline{X}\,\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\right)\right] \tag{5}$$

$$X^I / Y^I = \left[\left(\underline{X}, \overline{X}\right) \cdot \left(1/\overline{Y}, 1/\underline{Y}\right) \left(0 \notin Y^I\right)\right] \tag{6}$$

$$\min\left(X^I, Y^I\right) = \left[\min\left(\underline{X}, \underline{Y}\right), \min\left(\overline{X}, \overline{Y}\right)\right] \tag{7}$$

$$\max\left(X^I, Y^I\right) = \left[\max\left(\underline{X}, \underline{Y}\right), \max\left(\overline{X}, \overline{Y}\right)\right] \tag{8}$$

Any function defined on real values can be extended to intervals in a straightforward way. The extension to intervals of a function f defined on the real number, for intervals $\mathbf{X}^I = \left(X_1^I, X_2^I, \cdots, X_n^I\right)$, is (Ferson et al. 2007)

$$f\left(\mathbf{X}^I\right) = \left[f\left(X_1, X_2, \cdots, X_n\right), X_1 \in X_1^I, X_2 \in X_2^I, \cdots, X_n \in X_n^I\right] \tag{9}$$

3 Unified uncertainty analysis under the mixture of random and interval variables

Huang and Du proposed the methods of the MVFOSPA and FOSPA for system reliability analysis recently. In Huang and Du (2008), they showed that the MVFOSPA is useful due to its high efficiency and accuracy. However, MVFOSPA is only suitable for the system which only has random variables. It can not analyze system with both interval and random variables. In this section, we propose a unified uncertainty analysis method based on MVFOSPA (referred to as MVFOSPA-UUA). The MVFOSPA-UUA can analyze systems with both random and interval variables. Thus the proposed method can be used to solve the problem when both epistemic uncertainty and aleatory uncertainty exist in system simultaneously.

3.1 Approximate performance function by the first order Taylor series under mixed variables

From the practical viewpoint, when interval variables exist in a system, we may not need the exact ranges of the function $f(\mathbf{X^I})$. In order to determine the approximate ranges of the function $f(\mathbf{X^I})$, a feasible method is expanding the function $f(\mathbf{X}^I)$ with Taylor series at the midpoints $\tilde{\mathbf{X}} = \left(\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n\right)$ and keep only linear terms in the expansion. The expansion process can be expressed as

$$\begin{aligned} f\left(\mathbf{X}^I\right) &\approx f_L\left(\mathbf{X}^I\right) = f\left(\tilde{\mathbf{x}}\right) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} \left(X_i^I - \tilde{x}_i\right) \\ &= f\left(\tilde{\mathbf{x}}\right) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r \delta^I \\ &= f\left(\tilde{\mathbf{x}}\right) + \left(\sum_{i=1}^{n} \left|\left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r\right|\right) \delta^I \end{aligned} \tag{10}$$

From (10), the approximate ranges of the function $f(\mathbf{X^I})$ can be expressed as

$$\begin{aligned} f\left(\mathbf{X}^I\right) &\in \left\{\left[f\left(\tilde{\mathbf{x}}\right), f\left(\tilde{\mathbf{x}}\right)\right] + \left[-\left(\sum_{i=1}^{n} \left|\left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r\right|\right), \left(\sum_{i=1}^{n} \left|\left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r\right|\right)\right]\right\} \\ &= \left[f\left(\tilde{\mathbf{x}}\right) - \left(\sum_{i=1}^{n} \left|\left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r\right|\right), f\left(\tilde{\mathbf{x}}\right) + \left(\sum_{i=1}^{n} \left|\left.\frac{\partial f}{\partial X_i}\right|_{\tilde{\mathbf{X}}} X_i^r\right|\right)\right] \end{aligned} \tag{11}$$

Let $X(R)$ be the set of all real random variables in a probability space (Ω, A, P), $\mathbf{X}^R = (X_1, X_2 \cdots, X_i)$ is a random vector of $X(R)$. A random interval vector $\mathbf{X}^{IR}$ can be expressed as $\mathbf{X}^{IR} = \left(X_1, X_2 \cdots, X_i, X_{i+1}^I, X_{i+2}^I, \cdots, X_n^I\right)$. The mean values of random vector $\mathbf{X}^R$ can be expressed as

$$E\left(\mathbf{X}^R\right) = \left[E\left(X_1\right), E\left(X_2\right), \cdots, E\left(X_i\right)\right] \tag{12}$$

For the function $f(\mathbf{X}^{IR})$, we can expand function $f(\mathbf{X}^{IR})$ with Taylor series at the mean values $E(X_j)(j = 1, 2, \cdots, i)$ for random variables, at the midpoints $\tilde{X}_j(j = i + 1, i + 2, \cdots, n)$ for interval variables, and keep only linear terms in the expansion. Let $\mathbf{X}_*^{IR} = \left[E(X_1), \cdots, E(X_i), \tilde{X}_{i+1}, \cdots, \tilde{X}_n\right]$, the expansion becomes

$$f\left(\mathbf{X}^{IR}\right) \approx f_L\left(\mathbf{X}^{IR}\right) = f\left(\mathbf{x}_*^{IR}\right) + \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} \left[X_j - E\left(X_j\right)\right] + \sum_{j=i+1}^{n} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} \times \left(X_j^I - \tilde{X}_j\right) \tag{13}$$

From (13), the range of $f_L(\mathbf{X}^{IR})$ can be expressed as

$$f_L\left(\mathbf{X}^{IR}\right) \in \left\{ \begin{array}{c} \left[f\left(\mathbf{x}_*^{IR}\right), f\left(\mathbf{x}_*^{IR}\right)\right] \\ + \left[-\left(\sum_{j=i+1}^{n} \left|\left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j^r\right|\right), \right. \\ \left.\left(\sum_{j=i+1}^{n} \left|\left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j^r\right|\right)\right] \\ + \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} \left[X_j - E\left(X_j\right)\right] \end{array} \right\} \tag{14}$$

From (14), we have

$$f_L\left(\mathbf{X}^{IR}\right) \in \left[\begin{array}{c} f\left(\mathbf{x}_*^{IR}\right) - \left(\sum_{j=i+1}^{n} \left|\left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j^r\right|\right) \\ - \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} E\left(X_j\right) + \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j, \\ f\left(\mathbf{x}_*^{IR}\right) + \left(\sum_{j=i+1}^{n} \left|\left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j^r\right|\right) \\ - \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} E\left(X_j\right) + \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} X_j \end{array} \right] \tag{15}$$

For simplification, the function $f_L(\mathbf{X}^{IR})$ in (13) can also be rewritten as

$$f_L\left(\mathbf{X}^{IR}\right) = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_i X_i + a_{i+1} X_{i+1}^r \delta_{i+1}^I + \cdots + a_n X_n^r \delta_n^I \tag{16}$$

where $a_0 = f\left(\mathbf{x}_*^{IR}\right) - \sum_{j=1}^{i} \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} E\left(X_j\right)$, and $a_j = \left.\frac{\partial f}{\partial X_j}\right|_{\mathbf{x}_*^{IR}} (j = 1, 2, \cdots, n)$.

For discussion convenience, assume $a_i > 0 (i = 1, 2, \cdots, n)$. From (15) and (16), the global minimum value $f_L^{\min}$ and maximum value $f_L^{\max}$ of f_L over interval variables become

$$f_L^{\min}\left(\mathbf{X}^{IR}\right) = a_0 - |a_{i+1}| X_{i+1}^r - \cdots - |a_n| X_n^r + a_1 X_1 + a_2 X_2 + \cdots + a_i X_i \tag{17}$$

and

$$f_L^{\max}\left(\mathbf{X}^{IR}\right) = a_0 + |a_{i+1}| X_{i+1}^r + \cdots + |a_n| X_n^r + a_1 X_1 + a_2 X_2 + \cdots + a_i X_i, \tag{18}$$

respectively.

3.2 First order saddlepoint approximation

Daniels (1954) introduced the saddlepoint approximation (SPA) technique for the approximate distribution of random variables. SPA can provide an accurate estimation of cumulative distribution function (CDF) in a tail area (Wood et al. 1993; Huang and Du 2006; Gillespie and Renshaw 2007; Du 2008b). The FOSPA technique was first introduced by Du and Sudjianto (2004) in the domains of system reliability analysis and reliability-based design optimization recently. Brief introductions about both the SPA and MVFOSPA are given below.

Assume a performance function is denoted by $f(\mathbf{X})$. In the MVFOSPA (Huang and Du 2008), $f(\mathbf{X})$ is linearized in the original random space. The expansion point is at the mean values of random variables rather than the MLP (Huang and Du 2008). The linearized function can be expressed as

$$f\left(\mathbf{X}\right) \approx f_L\left(\mathbf{X}\right) = f\left(\mathbf{x}^*\right) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_{\mathbf{X}^*} \left[X_i - E\left(X_i\right)\right] \tag{19}$$

where $\mathbf{X}^*$ and $E(X_i)$ are $\mathbf{X}^* = [E(X_1), E(X_2), \cdots, E(X_n)]$, and the mean value of the ith random variable, respectively.

Let the cumulant generating function (CGF) of X_i is denoted as $K(X_i, t)$. There are two useful properties of CGF described as follows (Huang and Du 2008):

Property 1 If $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are independent random variables and their corresponding CGFs are

$K(X_i, t)$ $(i = 1, 2, \cdots, n)$, then the CGF of $Y = \sum_{i=1}^{n} X_i$ is given by

$$K(Y, t) = \sum_{i=1}^{n} K(X_i, t) \tag{20}$$

Property 2 If X is a random variable and its CGF is $K(X, t)$, then the CGF of $Y = aX + b$ (a and b are constants) is given by

$$K(Y, t) = K(X, at) + bt \tag{21}$$

For example, if X follows exponential distribution with CGF $K(X, t) = -\ln(1 - t/\alpha)$, then the CGF of Y is $K(Y, t) = K(X, at) + bt = -\ln(1 - at/\alpha) + bt$.

According to the two properties in (20) and (21), the CGF of $f_L(\mathbf{X})$ is given by

$$K_f(\mathbf{X}, t) = \left[f(\mathbf{x}^*) - \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_{\mathbf{x}^*} E(X_i) \right] t + \sum_{i=1}^{n} K\left(X_i, \left.\frac{\partial f}{\partial X_i}\right|_{\mathbf{x}^*} t \right) \tag{22}$$

The saddlepoint t^* (Du 2008b) can be determined by solving the following equation

$$\frac{d\left[K_f(\mathbf{X}, t)\right]}{dt} - y = 0 \tag{23}$$

After acquiring the saddlepoint t^*, according to Lugannani and Rice (1980) or Barndorff-Nielsen (1986), the probability of failure, P_f, can be approximated by

$$P_f = P\left[f_L(\mathbf{X}) \le y\right] = \Phi(w) + \phi(w)\left(\frac{1}{w} - \frac{1}{v}\right) \tag{24}$$

or

$$P_f = P\left[f_L(\mathbf{X}) \le y\right] = \Phi\left[w + \frac{1}{w}\log\frac{v}{w}\right] \tag{25}$$

where $w = \left.\operatorname{sign}(t)\left\{2\left[ty - K_f(\mathbf{X}, t)\right]\right\}^{\frac{1}{2}}\right|_{t^*}$, $v = \left.t\left[\frac{d^2[K_f(\mathbf{X},t)]}{dt^2}\right]^{\frac{1}{2}}\right|_{t^*}$, $\Phi()$ and $\phi()$ are the CDF and PDF of standard normal distribution, respectively. The function sign() is a sign function and its definition is $\operatorname{sign}(t) = \begin{cases} 1, & if\, t > 0 \\ 0, & if\, t = 0 \\ -1, & if\, t < 0 \end{cases}$. The CGF of some general distributions are given in Table 1 (Du and Sudjianto 2004; Huang and Du 2008).

3.3 Unified uncertainty analysis based on MVFOSPA

When both interval variables and random variables are present in the system, the system performance function can be denoted as $f(\mathbf{X}^{IR})$, $\mathbf{X}^{IR} = (X_1, X_2 \cdots, X_i, X_{i+1}^I, X_{i+2}^I, \cdots, X_n^I)$. As discussed in Section 3.1, we can expand the function $f(\mathbf{X}^{IR})$ by Taylor series at the mean values $E(X_j)$ $(j = 1, 2, \cdots, i)$ for random variables and the mid-points $\tilde{X}_j$ $(j = i+1, i+2, \cdots, n)$ for interval variables. Only linear terms are kept in the expansion. The global minimum value $f_L^{\min}$ and maximum value $f_L^{\max}$ of f_L over interval variables can be expressed in (17) and (18).

According to (16)–(18) and (20)–(22), the corresponding CGFs of f_L, $f_L^{\min}$ and $f_L^{\max}$ can be expressed as

$$K_{f_L}\left(\mathbf{X}^{IR}, t\right) = \left(a_0 + a_{i+1}X_{i+1}^r\delta_{i+1}^I + \cdots + a_n X_n^r \delta_n^I\right) t + \sum_{j=1}^{i} K\left(X_j, a_j t\right) \tag{26}$$

and

$$K_{f_L^{\min}}\left(\mathbf{X}^{IR}, t\right) = \left(a_0 - |a_{i+1}|\, X_{i+1}^r - \cdots - |a_n|\, X_n^r\right) t + \sum_{j=1}^{i} K\left(X_j, a_j t\right) \tag{27}$$

Table 1 CGF of some distributions

Distribution	PDF	CGF
Normal	$f(x) = \left(1\big/\sqrt{2\pi}\sigma\right)\exp\left[(x-\mu)^2/2\sigma^2\right]$	$K(t) = \mu t + \frac{1}{2}\sigma^2 t^2$
Uniform	$f(x) = 1\big/(b-a)$	$K(t) = \ln\left(e^{bt} - e^{at}\right) - \ln(b-a) - \ln(t)$
Exponential	$f(x) = \alpha\exp(-\alpha x)$	$K(t) = -\ln(1 - t/\alpha)$
Gumbel	$f(x) = (1/\sigma)\exp\left[(x-\mu)/\sigma\right]$ $\exp\left\{-\exp\left[(x-\mu)/\sigma\right]\right\}$	$K(t) = \mu t + \ln\Gamma(1 - \sigma t)$
Gamma	$f(x) = \beta^\alpha\big/\Gamma(\alpha)x^{\alpha-1}e^{-\beta x}$	$K(t) = \alpha\left[\ln\beta - n(\beta - t)\right]$
χ^2	$f(x) = \left[1\big/\Gamma(n/2)\,2^{n/2}\right]x^{n/2-1}e\left(-\frac{1}{2}x\right)$	$K(t) = -\frac{1}{2}n\ln(1-2t)$

and

$$K_{f_L^{\max}}\left(\mathbf{X}^{IR},t\right)=\left(a_0+|a_{i+1}|\,X_{i+1}^r+\cdots+|a_n|\,X_n^r\right)t+\sum_{j=1}^{i}K\left(X_j,a_jt\right), \tag{28}$$

respectively.

In order to calculate the lower and upper bounds of probabilities of failure, the saddlepoints t^*_{worst} and t^*_{best} need to be determined for the system under both the worst case and the best case. These two saddlepoints are calculated by solving the following equations

$$\frac{d\left[K_{f_L^{\min}}\left(\mathbf{X}^{IR},t\right)\right]}{dt}-y=0 \tag{29}$$

and

$$\frac{d\left[K_{f_L^{\max}}\left(\mathbf{X}^{IR},t\right)\right]}{dt}-y=0, \tag{30}$$

respectively. The lower and upper bounds of the system probabilities of failure, $P_f^{\min}$ and $P_f^{\max}$, can thus be calculated by

$$P_f^{\min}=P\left[f_L^{\max}\left(\mathbf{X}^{IR}\right)\le y\right]=\Phi\left(w^*_{best}\right)+\phi\left(w^*_{best}\right)\left(\frac{1}{w^*_{best}}-\frac{1}{v^*_{best}}\right) \tag{31}$$

or

$$P_f^{\min}=P\left[f_L^{\max}\left(\mathbf{X}^{IR}\right)\le y\right]=\Phi\left[w^*_{best}+\frac{1}{w^*_{best}}\log\left(\frac{v^*_{best}}{w^*_{best}}\right)\right] \tag{32}$$

and

$$P_f^{\max}=P\left[f_L^{\min}\left(\mathbf{X}^{IR}\right)\le y\right]=\Phi\left(w^*_{worst}\right)+\phi\left(w^*_{worst}\right)\left(\frac{1}{w^*_{worst}}-\frac{1}{v^*_{worst}}\right) \tag{33}$$

or

$$P_f^{\max}=P\left[f_L^{\min}\left(\mathbf{X}^{IR}\right)\le y\right]=\Phi\left[w^*_{worst}+\frac{1}{w^*_{worst}}\log\left(\frac{v^*_{worst}}{w^*_{worst}}\right)\right], \tag{34}$$

where

$$w^*_{best}=\operatorname{sign}(t)\left\{2\left[ty-K_{f_L^{\max}}\left(\mathbf{X}^{IR},t\right)\right]\right\}^{\frac{1}{2}}\Big|_{t^*_{best}} \tag{35}$$

and

$$w^*_{worst}=\operatorname{sign}(t)\left\{2\left[ty-K_{f_L^{\min}}\left(\mathbf{X}^{IR},t\right)\right]\right\}^{\frac{1}{2}}\Big|_{t^*_{worst}} \tag{36}$$

and

$$v^*_{best}=t\left\{\frac{d^2\left[K_{f_L^{\max}}\left(\mathbf{X}^{IR},t\right)\right]}{dt^2}\right\}^{\frac{1}{2}}\Bigg|_{t^*_{best}} \tag{37}$$

and

$$v^*_{worst}=t\left\{\frac{d^2\left[K_{f_L^{\min}}\left(\mathbf{X}^{IR},t\right)\right]}{dt^2}\right\}^{\frac{1}{2}}\Bigg|_{t^*_{worst}} \tag{38}$$

Generally, if all the random variables are tractable, that is, all the random variables have a closed form of CGF, the bounds of system probability of failure can be obtained easily. However, in some cases, some random variables may not have a closed form. There are two ways (Du and Sudjianto 2004) to handle intractable random variables:

1. Transform the random variable into another random variable with tractable CGF.
2. Approximate the CGF using polynomial expansions.

To sum up, the MVFOSPA-UUA involves the following steps to analyze the system probability of failure when both

Table 2 Details of both random and interval variables

Variable	Parameter 1	Parameter 2	Type
X_1	4.5	0.0833	Uniform
X_2	1.0	1.0	Exponential
X_3	2.0	2.5	Interval

Table 3 Probability of failure calculated by different methods

Values z	Probability of failure	MVFOSPA-UUA	FORM-based	MCS
0	$P_f^{\min}$	0.1084	0.0531	0.1064
	$P_f^{\max}$	0.3824	0.3951	0.3680
1.5	$P_f^{\min}$	0.7673	0.7760	0.7670
	$P_f^{\max}$	0.8586	0.8641	0.8588
2.5	$P_f^{\min}$	0.9141	0.9175	0.9145
	$P_f^{\max}$	0.9478	0.9499	0.9480

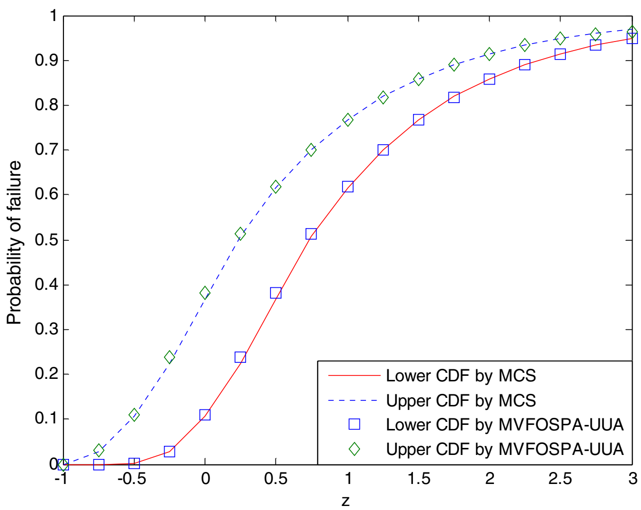

Fig. 1 Lower and upper bounds CDFs comparison between MCS and MVFOSPA-UUA

random variables and interval variables are present in the system simultaneously:

1. Transform the intractable random variables into another random variable with tractable CGF or approximate the CGF by using polynomial expansions.
2. Expand the performance function by Taylor series at the mean values for random variables and the midpoints for interval variables. Only linear terms are kept in the expansion, such as (13).
3. Find the global minimum and maximum values of the linearized function over interval variables using (17) and (18).
4. Calculate both the worst case saddlepoint t^*_{worst} using (29) and best case saddlepoint t^*_{best} using (30).
5. Calculate the lower bound of system probability of failure using (31) or (32), and the upper bound of system probability of failure using (33) or (34).

4 Illustrate examples and discussions

In this section, three examples are used to demonstrate the accuracy and effectiveness of the proposed method. The first example is used to deal with a linear performance function, while the second example is employed to handle non-linear performance function with many non-normal random variables. The third example is analyzed to demonstrate the error of the proposed method. A comparative study is also provided among FORM-based method, MVFOSM-UUA and MCS. The results calculated using MCS are used as reference for the accuracy comparisons.

4.1 Mathematical problem I

A performance function with random interval variables is given by

$$Z = f(\mathbf{X}) = \sum_{i=1}^{3} X_i - 7$$

Details of both random and interval variables are given in Table 2.

In Table 2, parameters 1 and 2 are mean value and standard deviation for uniform and exponential distributions, respectively. For the interval variable, parameters 1 and 2 are lower and upper bounds, respectively.

In this example, since both random and interval variables are present in system, the system probability of failure is an interval rather a precise value. The lower and upper bounds of the system probability of failure $P[Z \leq z]$ calculated using different methods are given in Table 3.

From Table 3, there is a conclusion that the results obtained using MVFOSPA-UUA are almost identical to the results calculated using MCS. In this paper, the error calculated using the FORM-based method is larger than the MVFOSPA-UUA. The reason is that both uniform distribution and exponential distribution are transformed into equivalent standard normal variable by Rosenblatt transformation. This transformation increases the nonlinearity of the performance function largely, and it makes the original linear performance function to be a non-linear performance function. Another main calculation error of the FORM-based method comes from using a linearized function to approximate its non-linear performance function. Furthermore, parameter X_3 belongs to the interval [2, 2.5], which does not necessarily mean that X_3 is uniformly distributed with the lower and upper bounds being 2 and 2.5, respectively. If X_3 is uniformly distributed, the system probability of failure calculated using MCS is 0.2278 under $z = 0$, rather than an interval.

The lower and upper bounds of $F_Z = P\left[f(\mathbf{X}) \leq z\right]$ calculated using MCS and MVFOSPA -UUA are given in

Fig. 2 A beam

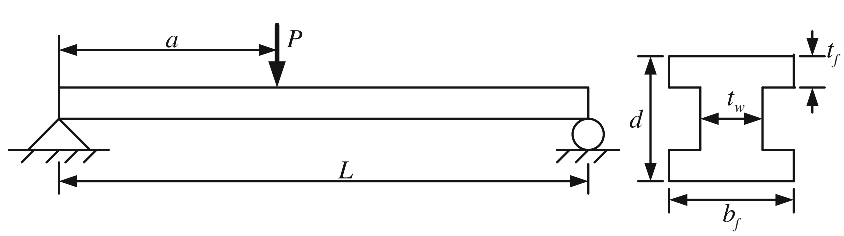

Fig. 1. It shows that the CDFs are almost identical to each other over the entire distribution range.

4.2 A beam example

Consider a beam, shown in Fig. 2, which is used to demonstrate the effectiveness and accuracy of the proposed method. The performance function is given by (Huang and Du 2006; Xiao et al. 2011)

$$Z = f\left(P, L, a, S, d, b_f, t_w, t_f\right) = \sigma_{\max} - S$$

where

$$\sigma_{\max} = \frac{Pa\,(L-a)\,d}{2LI}$$

and

$$I = \frac{b_f d^3 - \left(b_f - t_w\right)\left(d - 2t_f\right)^3}{12}$$

The details of both random and interval variables are given in Table 4. In order to demonstrate the proposed method, we assume the parameter b_f is an interval variable.

In Table 4, parameters 1 and 2 are mean values and standard deviations for normal and uniform distributions, respectively. For interval variable, parameters 1 and 2 are lower and upper bounds, respectively.

In this example, we only consider the case that $z < 0$, the bounds of the system probability of failure calculated by using different methods are given in Table 5.

From Table 5, it shows that the results obtained using MVFOSPA-UUA is more accurate than the FORM-based method when compared to the MCS. The proposed method has the highest efficiency as it requires the least function evaluations when compared to both the FORM-based method and MCS. In this example, we are unable to estimate the system probability of failure by using MCS directly, because the distribution type of the interval variable is unknown. However, a feasible method is that we can evenly divide interval variable into many subintervals, the distribution type in each subinterval can be approximated by the uniform distribution. The more numbers of the subintervals we divide, the more accurate the results will be. In this example, interval variable b_f is evenly divided into 1000 subintervals. For each subinterval, MCS-based methods with 10^5 samples are used, the total numbers of function evaluations are $10^3 \cdot 10^5 = 10^8$. The numbers of system probability of failure are 1000, and the bounds of system probability of failure can be determined from these 1000 results. This example shows that when interval variable exist in the system, the computational burden using MCS is extremely huge. In this example, there are many non-normal random variables in the system. In order to calculate the bounds of system probability of failure, the non-normal variables need to be transformed into equivalent standard normal variables for the FORM-based method by using Rosenblatt transformation. After several transformations, the performance function becomes a highly non-linear function. The bounds of system probability of failure estimated using the FORM-based method become large.

Table 4 Details of both random and interval variables

Variable	Parameter 1	Parameter 2	Distribution type
P	6070	200	Uniform
L	120	6	Uniform
A	72	6	Uniform
S	170000	4760	Uniform
D	2.3	1/24	Normal
t_w	0.16	1/48	Normal
t_f	0.26	1/48	Normal
b_f	2.2	2.4	Interval

Table 5 Probabilities of failure calculated by different methods

Probability of failure	MVFOSPA-UUA	FORM-based	MCS
$P_f^{\min}$	0.8787	0.7044	0.8536
$P_f^{\max}$	0.9914	0.8932	0.9847
Function evaluations	9	74	10^8

4.3 Burst margin of disk

The burst margin M_b of a disk is defined as (Huang and Du 2006)

$$Z = M_b = g\left(f, S, \delta, N, R, R_0\right) = \sqrt{\frac{fS}{3\cdot 385.82\delta\left(N\frac{2\pi}{60}\right)^2\left(R^3 - R_0^3\right)\left(R - R_0\right)}}$$

where f is the material utilization factor, S is the ultimate tensile strength, δ is the density, N is the rotor speed, R is

Table 6 Details of both random and interval variables

Variable	Parameter 1	Parameter 2	Distribution type
F	0.9	1.0	Interval
S	220000 lb/in^2	5000 lb/in^2	Normal
δ	0.28 lb/in^3	0.3 lb/in^3	Interval
N	21000 rpm	1000 rpm	Normal
R	24 in	0.5 in	Normal
R_0	8 in	0.3 in	Normal

Table 7 Probabilities of failure calculated by different methods

Probability of failure	MVFOSPA-UUA	FORM-based	MCS
$P_f^{\min}$	0.5795	0.5724	0.5675
$P_f^{\max}$	0.9268	0.9207	0.9181
Function evaluations	7	37	$5{\cdot}10^{11}$

the outer radius, and R_0 is the inner radius. Details of both random and interval variables are given in Table 6.

In this example, we only consider the case where $z < 2.6 \times 10^{-5}$. The bounds of system probability of failure calculated using different methods are given in Table 7.

From Table 7 it shows that the results obtained using the FORM-based method is more accurate than the MVFOSPA-UUA when compared to the MCS. The proposed method inherently contains two kinds of error, namely, error due to the linearization and error due to the approximation at the mean value points. In this example, since the FORM-based method only has the error due to the linearization, the results calculated by using the FORM-based method is more accurate than the MVFOSPA-UUA. However, the MVFOSPA-UUA is more robust than the FORM-based method, because it does not need MPP search. Furthermore, the proposed method has the highest efficiency when compared to both the FORM-based method and MCS. In this example, both interval variables f and δ are divided into 1000 subintervals, respectively. The total numbers of function evaluations is $10^3 \cdot 10^3 \cdot \left(5 \cdot 10^5\right) = 5 \cdot 10^{11}$ by using MCS, which shows that when more than one interval variables exist in system, the computational burden using MCS is extremely huge.

5 Conclusions

Based on the interval algorithm and MVFOSPA, a novel UUA method has been proposed for the reliability analysis of structural systems with both epistemic and aleatory uncertainties. The method uses a mixture of both random and interval variables rather than only random variables for considering both types of uncertainties that exist widely in the engineering practice. Results of the three examples show that the proposed method is effective and it is generally more robust than the FORM-based method because it does not requires the MPP search, which is an optimization process. In some cases, there are more than one MPPs or the MPP search process does not converge. Furthermore, the proposed method is superior to both the FORM-based method and MCS in terms of the computational efficiency. The examples in the paper also show that MVFOSPA-UUA is more accurate than the FORM-based method when many non-normal variables exist in the system.

It should be noted that the proposed method has some limitations. It is an approximate method that uses the first order Taylor series surrogate for its original performance function. Such surrogating can cause calculation errors in the reliability estimation, especially for highly nonlinear performance function. Therefore, the interval bounds calculated by using the proposed method are approximate solution rather than exact solution. Extension of the method for handling multiple correlated performance functions and improving the computational accuracy will be considered in our future work.

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